St George Girls High School

Year 12

Mid-HSC Course Examination

2013

Mathematics Extension 1

General Instructions

- Working time – 90 minutes
- Reading time – 5 minutes
- Write using blue or black pen.
- Board-approved calculators may be used.
- A table of standard integrals is provided.

Total marks – 68

Section 1 :8 marks

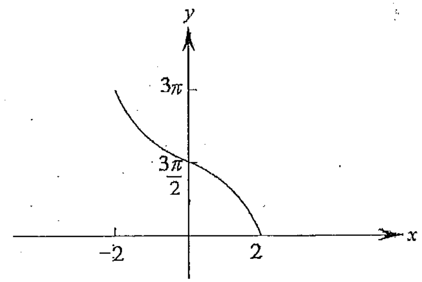
- Attempt Questions 1 –8
- All questions are of equal value
- Use the multiple choice answer sheet provided

Section 2 : 60 marks

- Attempt Questions 9 –12
- All questions are of equal value
- In Questions 9–12, show relevant mathematical reasoning

Section 1
8 marks Attempt Questions 1–8
Allow about 11 minutes for this section
Use the multiple-choice answer sheet for Questions1-8

1. Which function best describes the graph above?



(A) $y = 3\cos^{-1} 2x$

(B) $y = \frac{3}{2}\cos^{-1} 2x$

(C) $y = 3\cos^{-1} \frac{x}{2}$

(D) $y = \frac{3}{2}\cos^{-1} \frac{x}{2}$

2 What is the derivative of $y = \cos^{-1}\left(\frac{1}{x}\right)$ with respect to x

(A) $\frac{-1}{\sqrt{x^2-1}}$

(B) $\frac{-1}{x\sqrt{x^2-1}}$

(C) $\frac{1}{\sqrt{x^2-1}}$

(D) $\frac{1}{x\sqrt{x^2-1}}$

3. Which of the following is the correct expression for $\int \frac{dx}{\sqrt{36-x^2}}$?

(A) $\cos^{-1}\frac{x}{6}+c$

(B) $\cos^{-1}6x+c$

(C) $\sin^{-1}\frac{x}{6}+c$

(D) $\sin^{-1}6x+c$

4. Which of the following is an expression for $\int \sin^2 2x\,dx$?

(A) $\frac{1}{2}x-\frac{1}{8}\sin 4x+c$

(B) $\frac{1}{2}x-\frac{1}{4}\sin 4x+c$

(C) $\frac{1}{2}x+\frac{1}{8}\sin 4x+c$

(D) $\frac{1}{2}x+\frac{1}{4}\sin 4x+c$

5. The equation $x^3+2x^2-3x-6=0$ has roots $\alpha, -\alpha$ and β.
What is the value of β ?

(A) -6

(B) -2

(C) 2

(D) 6

6. Which of the following is a monic polynomial of degree 4?

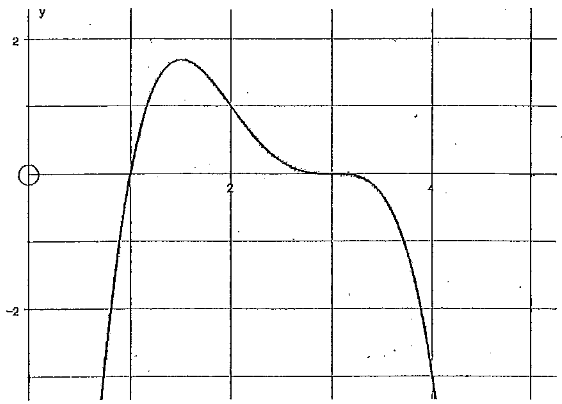
(A) $1-x+x^2-x^4$

(B) $1+3x+2x^2+4x^3$

(C) $5x+x^4$

(D) $4x$

7. Which of the polynomials below could be represented by this graph?



(A) $P(x)=(x-1)(x-3)^3$

(B) $P(x)=(1-x)(x-3)^3$

(C) $P(x)=(x-3)(x-1)^3$

(D) $P(x)=(3-x)(x-1)^3$

8. If $t=\tan\frac{x}{2}$ which of the following is an expression for $\frac{dx}{dt}$?

(A) $\frac{1}{2}(1+t^2)$

(B) $(1+t^2)$

(C) $\frac{2}{1+t^2}$

(D) $\frac{1}{1+t^2}$

End of Section 1

Section 2

Answer each question in a SEPARATE writing booklet.
Extra writing booklets are available.
In Questions 9–12, your responses should include relevant mathematical reasoning and/or calculations

Question 9 (15 marks) Use a SEPARATE writing booklet. **Marks**

a) Give values for a and b such that the graph of **2**

$$f(x) = (ax - 4)(x - b)^2$$

cuts the x - axis at $x = 1.5$ and touches the x axis at $x = 7$

b) α, β, γ, are the roots of the equation $x^3 + 6x - 2 = 0$. Find **3**

(i) $\alpha + \beta + \gamma$

(ii) $\alpha\beta\gamma$

(iii) $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}$

c) Sketch the following polynomials showing the x and y intercepts without the use of calculus. Diagrams should each be about one third of a page.

(i) $P(x) = x(x - 4)(x - 5)^2$ **2**

(ii) $P(x) = (1 - 2x)(x + 3)(4 - x)^3$ **2**

d) When the polynomial $P(x)$ is divided by $(x + 1)(x - 4)$, the remainder is $3x + 7$. What is the remainder when $P(x)$ is divided by $(x - 4)$? **2**

e) $P(x) = x^3 + 5x^2 + 2x - 8$

(i) Evaluate $P(-2)$ **1**

(ii) Factorise $P(x)$ fully. **2**

f) For what value of n is the polynomial **1**

$P(x) = x^3 + 3x^2 - 2x + n$ exactly divisible by $(x + 3)$

Question 10 (15 marks) Use a SEPARATE writing booklet. **Marks**

a) Solve the equation **3**

$2\sin^2\theta = \sin 2\theta$ for $0 \leq \theta \leq 2\pi$

b) Angle A is acute and angle B is obtuse. **3**

Sin $A = \frac{1}{\sqrt{3}}$ and Sin $B = \frac{1}{\sqrt{2}}$

Find the exact value of $\sin(A + B)$

c) (i) Express $\sin x - \cos x$ in the form $R\sin(x - \alpha)$ where α is in radians. **2**

(ii) Hence, or otherwise solve the equation **2**

$\sin x - \cos x = 1$ for $0 \leq x \leq 2\pi$

d) If $t = tan\frac{\theta}{2}$,

(i) Write $sin\,\theta$ in terms of t, **1**

(ii) Write $cos\,\theta$ in terms of t **1**

(iii) Solve the equation $3cos\theta + 4sin\theta = 4$ **3**

for $0^0 \leq \theta \leq 360^0$ using the 't' results.

(answer to the nearest minute)

Question 11 (15 marks) Use a SEPARATE writing booklet. Marks

a) Differentiate with respect to x

(i) $3\cos^{-1}\sqrt{x}$ 2

(ii) $\tan^{-1}\left(\frac{x+1}{x}\right)$ 2

b) For the function $y = 2\sin^{-1}(x+1)$

(i) state the domain 1

(ii) state the range 1

(iii) graph $y = 2\sin^{-1}(x+1)$ 1

(iv) find the gradient of $y = 2\sin^{-1}(x+1)$ at the point where it cuts the x axis. 2

c) Evaluate 2

$$\int_0^1 \frac{1}{\sqrt{16-x^2}}\,dx$$

d) Find

(i) $\int \frac{-1}{\sqrt{1-25x^2}}\,dx$ 2

(ii) $\int \frac{1}{100+9x^2}\,dx$ 2

Question 12 (15 marks) Use a SEPARATE writing booklet. Marks

a) (i) Given that $x^2 + 6x + 34 \equiv (x+a)^2 + b$, find a and b where $a > 0$, $b > 0$ 2

(ii) Using the result in (i) find 2

$$\int \frac{1}{x^2 + 6x + 34}\,dx$$

b) Find the exact value of $\sin\left[\tan^{-1}\left(\frac{-3}{5}\right)\right]$ 1

c) (i) Show that $\frac{1+\cos 2x}{\sin 2x} = \cot x$ 2

(ii) Hence, find the exact value of $\cot 15°$ 2

d) (i) Find $\frac{d}{dx}(x\tan^{-1}x)$ 2

(ii) Hence or otherwise find $\int \tan^{-1}x\,dx$ 1

e) Consider $\tan^{-1}y = 2\tan^{-1}x$. 3

Express y as a function of x, independent of any trigonometric ratio.

End of Paper

Section 1

1. Considering domain of $y = a\cos^{-1} b \quad -1 \le b \le 1$

if $b = 2x$

$-1 \le 2x \le 1$

$-\frac{1}{2} \le x \le \frac{1}{2}$, which is not the domain of the graph

If $b = \frac{x}{2}$

$-1 \le \frac{x}{2} \le 1$

$-2 \le x \le 2$, which is the domain of the graph ☺

∴ Check the range of part (c) and (d) only.

Considering the range of

(c) $\frac{y}{3} = \cos^{-1}\frac{x}{2}$

$0 \le \frac{y}{3} \le \pi$

$0 \le y \le 3\pi$, which is the range of the graph.

∴ (C) is the correct answer.

2. $y = \cos^{-1}\left(\frac{1}{x}\right)$

$\frac{dy}{dx} = \frac{-1}{\sqrt{1-\frac{1}{x^2}}} \times \frac{-1}{x^2}$ (chain rule)

$= \frac{1}{x^2\sqrt{\frac{x^2-1}{x^2}}}$

$= \frac{1}{x\sqrt{x^2-1}}$

∴ (D) is the correct answer

3. $\int \frac{dx}{\sqrt{6^2-x^2}} = \sin^{-1}\frac{x}{6} + C$

∴ (C) is the correct answer.

4. $\int \sin^2 2x\, dx$

$= \frac{1}{2}\int (1-\cos 4x)\,dx$

$= \frac{1}{2}\left[x - \frac{\sin 4x}{4}\right] + C$

$= \frac{x}{2} - \frac{\sin 4x}{8} + C$

∴ (A) is the correct answer

5. $\alpha + (-\alpha) + \beta = \frac{-2}{1}$

∴ $\beta = -2$

∴ (B) is the correct answer

6. Monic + degree 4:

(C) is the correct answ.

7.

As $x \to -\infty$ $P(x) \to -\infty$

As $x \to +\infty$ $P(x) \to -\infty$

Since $\deg(P(x))$ is even

The leading coefficient must be < 0. Also $P(x)$ must have triple root at $x = 3$ since there is a horizontal inflexion point there.

∴ (B) is the correct answer

8. If $\tan\frac{x}{2} = t$

$\tan^{-1} t = \frac{x}{2}$

$x = 2\tan^{-1} t$

$\frac{dx}{dt} = \frac{2}{1+t^2}$

(or:)

$\frac{dt}{dx} = \frac{1}{2}\sec^2\frac{x}{2}$

$= \frac{1}{2}\left(1 + \tan^2\frac{x}{2}\right)$

$= \frac{1}{2}\left(1+t^2\right)$

$= \frac{1+t^2}{2}$

∴ $\frac{dx}{dt} = \frac{2}{1+t^2}$

∴ (C) is the correct answer.

Section 2

Q.9

(a) $\frac{4}{a} = \frac{3}{2}$

$a = \frac{2\times 4}{3}$

$a = \frac{8}{3}$

$b = 7$

(b) $x^3 + 6x - 2 = 0$

$a=1,\ b=0,\ c=6,\ d=-2$

(i) $\alpha+\beta+\gamma = 0$

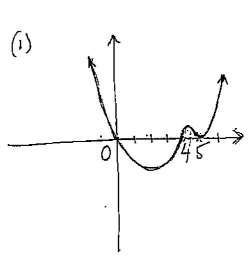
(ii) $\alpha\beta\gamma = 2$

(iii) $\alpha\beta + \alpha\gamma + \beta\gamma = 6$

$\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = \frac{\beta\gamma + \alpha\gamma + \alpha\beta}{\alpha\beta\gamma}$

$= \frac{6}{2} = 3$

(c) (i)



Zeroes: $x = 0, 4, 5, 5$

as $x \to \pm\infty$ $P(x) \to +\infty$

y-intercept is

$y = 0$

(ii)

Zeroes: $x = \frac{1}{2}, -3, 4, 4, 4$.

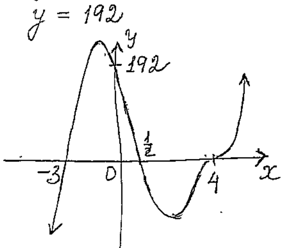
degree is and the leading coefficient > 0

∴ As $x \to \pm\infty$ $P(x) \to +\infty$

y-intercept when $x = 0$:

$y = 1 \times 3 \times 64$

$y = 192$



(d)

$P(x) = (x+1)(x-4)Q(x) + 3x + 7$

$P(4) = (4+1)(4-4)Q(x) + 3\times 4 + 7$

$= 19$ Remainder $= 19$

(e) (i) $P(x) = x^3 + 5x^2 + 2x - 8$

$P(-2) = -8 + 20 + (-4) - 8$

$= 20 - 20$

$= 0$

(ii) $P(x) = (x+2)(x^2+3x-4)$

$= (x+2)(x+4)(x-1)$

(f) $P(-3) = 0$

$(-3)^3 + 3(-3)^2 - 2(-3) + n = 0$

$n = 27 - 27 - 6$

∴ For $n = -6$

Q10

(a)

$2\sin^2\theta = \sin 2\theta, \quad 0 \le \theta \le 2\pi$

$2\sin^2\theta = 2\sin\theta\cos\theta$

$\sin^2\theta = \sin\theta\cos\theta$

$\sin^2\theta - \sin\theta\cos\theta = 0$

$\sin\theta(\sin\theta - \cos\theta) = 0$

$\sin\theta = 0$ or $\sin\theta - \cos\theta = 0$

∴ $\theta = 0, \pi, 2\pi$; $\quad \sin\theta = \cos\theta$

$\frac{\sin\theta}{\cos\theta} = 1$, since $\cos\theta = 0$ is not a solution

$\tan\theta = 1$

∴ $\theta = \frac{\pi}{4}, \frac{5\pi}{4}$

(Note that $\cos\theta = 0$ is not a solution since $\cos\theta$ and $\sin\theta$ can not equal to 0 simultaneously)

Solution:

$\theta = 0, \frac{\pi}{4}, \pi, \frac{5\pi}{4}, 2\pi$

(b) $\sin A = \frac{1}{\sqrt{3}}$ $\quad \sin B = \frac{1}{\sqrt{2}}$

Since B is obtuse

$B = \frac{3\pi}{4}$ (2nd quadrant)

and $\cos B = -\frac{1}{\sqrt{2}}$

Also $\cos A = \frac{\sqrt{2}}{\sqrt{3}}$

$\sin(A+B) = \cos A\sin B + \cos B\sin A$

$= \frac{\sqrt{2}}{\sqrt{3}} \times \frac{1}{\sqrt{2}} + \left(-\frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{3}}\right)$

$= \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{2}\sqrt{3}}$

$= \frac{\sqrt{2}-1}{\sqrt{6}} = \frac{\sqrt{2}\times 6 - \sqrt{6}}{6}$

$= \frac{2\sqrt{3} - \sqrt{6}}{6}$

(c)

(i) $R\sin(x-\alpha) = R(\sin x\cos\alpha - \sin\alpha\cos x)$

$= R\cos\alpha\sin x - R\sin\alpha\cos x$

$R\cos\alpha = 1$

$R\sin\alpha = 1$

$\tan\alpha = 1 \quad ∴ \alpha = \frac{\pi}{4}$

Since $R\cos\frac{\pi}{4} = 1$

$R = \frac{1}{\cos\frac{\pi}{4}} = \sqrt{2}$

∴ $\sin x - \cos x = \sqrt{2}\sin\left(x - \frac{\pi}{4}\right)$

(ii) $\sin x - \cos x = 1$ $\quad \left\{0 \le x \le 2\pi,\ -\frac{\pi}{4} \le x - \frac{\pi}{4} \le \ldots\right.$

$\sqrt{2}\sin\left(x - \frac{\pi}{4}\right) = 1$

$\sin\left(x - \frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$

$x - \frac{\pi}{4} = \frac{\pi}{4}, \frac{3\pi}{4}$,

$x = \frac{\pi}{2}, \pi$

(d) $t = \tan\frac{\theta}{2}$

(i) $\sin\theta = \frac{2t}{1+t^2}$

(ii) $\cos\theta = \frac{1-t^2}{1+t^2}$

(iii) $\frac{3(1-t^2)}{1+t^2} + \frac{4(2t)}{1+t^2} = 4$

$3 - 3t^2 + 8t = 4 + 4t^2$

$7t^2 - 8t + 1 = 0$

$7t^2 - 7t - t + 1 = 0$

$7t(t-1) - 1(t-1) = 0$

$(7t-1)(t-1) = 0$

$t = \frac{1}{7}$ or $t = 1$

$\tan\frac{\theta}{2} = 1$ or $\tan\frac{\theta}{2} = \frac{1}{7}$

$\frac{\theta}{2}$ lies in 1st or 3rd quad.

since $\tan\frac{\theta}{2} > 0$, but

$0 \le \frac{\theta}{2} \le 180^\circ$ $\therefore$ only 1st quadrant

$\frac{\theta}{2} = 45^\circ$ or $\frac{\theta}{2} = \tan^{-1}(\frac{1}{7})$

$\theta = 90^\circ$ or $\theta = 2\tan^{-1}(\frac{1}{7})$

$= 16^\circ 16'$

Question 11

(a)

(i) $\frac{d}{dx}(3\cos^{-1}\sqrt{x})$

$= 3 \times \frac{-1}{\sqrt{1-(\sqrt{x})^2}} \times \frac{1}{2\sqrt{x}}$

$= \frac{-3}{2\sqrt{x(1-x)}}$

$= \frac{-3}{2\sqrt{x-x^2}}$

(ii) $\frac{d}{dx}\left(\tan^{-1}\frac{x+1}{x}\right)$

Let $u = \frac{x+1}{x} = 1 + \frac{1}{x}$

$\frac{du}{dx} = -x^{-2} = -\frac{1}{x^2}$

$\frac{d}{dx}\left(\tan^{-1}\frac{x+1}{x}\right) = \frac{1}{1+\left(\frac{x+1}{x}\right)^2} \times \left(-\frac{1}{x^2}\right)$

$= \frac{-1}{x^2+(x+1)^2}$

$= -\frac{1}{x^2+x^2+2x+1}$

$= -\frac{1}{2x^2+2x+1}$

(b) $y = 2\sin^{-1}(x+1)$

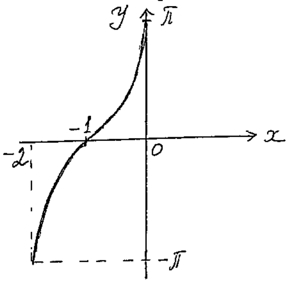
(i) Domain : $-1 \le x+1 \le 1$

$-2 \le x \le 0$

(ii) Range : $-\frac{\pi}{2} \le \frac{y}{2} \le \frac{\pi}{2}$

$-\pi \le y \le \pi$

(iii)



(iv) $\frac{dy}{dx} = \frac{2}{\sqrt{1-(x+1)^2}} \times 1$

$= \frac{2}{\sqrt{1-(x^2+2x+1)}}$

$\frac{2}{\sqrt{-x(x+2)}}$

When $x = -1$

$\frac{dy}{dx} = \frac{2}{\sqrt{1(-1+2)}} = 2$

$\therefore$ The gradient is 2.

(c)

$\int_0^1 \frac{1}{\sqrt{16-x^2}}\,dx$

$= \int_0^1 \frac{1}{\sqrt{4^2-x^2}}\,dx$

$= \left[\sin^{-1}\frac{x}{4}\right]_0^1$

$= \sin^{-1}\frac{1}{4} - \sin^{-1}0$

$= \sin^{-1}\frac{1}{4}$

(d)

(i) $\int \frac{-1}{\sqrt{1-25x^2}}\,dx$

$= \int \frac{-1}{\sqrt{25(\frac{1}{25}-x^2)}}\,dx$

$= \frac{1}{5}\int \frac{-1}{\sqrt{(\frac{1}{5})^2-x^2}}\,dx$

$= \frac{1}{5}\cos^{-1}5x + C$

or

$= -\frac{1}{5}\sin^{-1}5x + C$

(ii) $\int \frac{1}{100+9x^2}\,dx$

$= \int \frac{1}{9(\frac{100}{9}+x^2)}\,dx$

$= \frac{1}{9} \times \frac{3}{10} \int \frac{\frac{10}{3}}{(\frac{10}{3})^2+x^2}\,dx$

$= \frac{1}{30}\tan^{-1}\frac{3x}{10} + C$

Question 12:

(a) $x^2+6x+34 \equiv (x+a)^2+b$

(i) RHS $= x^2+2ax+a^2+b$

Equating coefficients of x and constant terms :

$2a = 6$

$a = 3$

$a^2+b = 34$

$3^2+b = 34$

$b = 25$

(ii) $\int \frac{1}{x^2+6x+34}\,dx$

$= \int \frac{1}{25+(x+3)^2}\,dx$

$= \frac{1}{5}\int \frac{5}{(5)^2+(x+3)^2}\,dx$

$= \frac{1}{5}\tan^{-1}\frac{x+3}{5} + C$

(b) $\sin\left(\tan^{-1}\left(-\frac{3}{5}\right)\right)$

Let $\alpha = \tan^{-1}\left(-\frac{3}{5}\right)$

$-\frac{\pi}{2} < \alpha < \frac{\pi}{2}$

$\sqrt{34}$, 3, 5

α must be in 4th quadr.

$\sin\alpha = -\frac{3}{\sqrt{34}}$

$\therefore \sin\left(\tan^{-1}\left(-\frac{3}{5}\right)\right) = -\frac{3}{\sqrt{34}}$

(c)

(i) LHS $= \frac{1+\cos 2x}{\sin 2x}$

$= \frac{1+\cos^2 x - \sin^2 x}{2\sin x\cos x}$

$= \frac{\cos^2 x + \sin^2 x + \cos^2 x - \sin^2 x}{2\sin x\cos x}$

$= \frac{\cancel{2}\cos^2 x}{\cancel{2}\sin x\cos x}$

$= \frac{\frac{\cos^2 x}{\cos x}}{\frac{\sin x\cos x}{\cos x}}$

$= \frac{\cos x}{\sin x}$

$= \cot x$

$=$ RHS, as required.

(ii) $\cot 15^\circ = \frac{1+\cos 30^\circ}{\sin 30^\circ}$

$= \frac{1+\frac{\sqrt{3}}{2}}{\frac{1}{2}}$

$= 2+\sqrt{3}$

(e) (i) $\frac{d}{dx}(x\tan^{-1}x)$

$= \tan^{-1}x + x \times \frac{1}{1+x^2}$

$= \tan^{-1}x + \frac{x}{1+x^2}$

(ii) From part (i):

$\tan^{-1}x = \frac{d}{dx}(x\tan^{-1}x) - \frac{x}{1+x^2}$

$\therefore \int \tan^{-1}x\,dx = x\tan^{-1}x - \int \frac{x}{1+x^2}\,dx$

$= x\tan^{-1}x - \frac{1}{2}\int \frac{2x}{1+x^2}\,dx$

$= x\tan^{-1}x - \frac{1}{2}\ln(1+x^2) + C$

(f)

$\tan^{-1}y = 2\tan^{-1}x$

Let $\alpha = \tan^{-1}y$

$-\frac{\pi}{2} < \alpha < \frac{\pi}{2}$

$y = \tan\alpha$

Since $\tan^{-1}y = 2\tan^{-1}x$

$\alpha = 2\tan^{-1}x$

$\frac{\alpha}{2} = \tan^{-1}x \Rightarrow x = \tan\left(\frac{\alpha}{2}\right)$

Since $\tan\alpha = \tan\left(\frac{\alpha}{2}+\frac{\alpha}{2}\right)$

$= \frac{2\tan\frac{\alpha}{2}}{1-\tan^2\frac{\alpha}{2}}$

$= \frac{2x}{1-x^2}$

Since $\tan\alpha = y$

$y = \frac{2x}{1-x^2}$