St George Girls High School

Year 12

Mid-HSC Course Examination

2014

Mathematics Extension 1

General Instructions

- Working time – 90 minutes
- Reading time – 5 minutes
- Write using blue or black pen.
- Board-approved calculators may be used.
- A table of standard integrals is provided.

Total marks – 61

Section I: 7 marks

- Attempt Questions 1 –7
- All questions are of equal value
- Use the multiple choice answer sheet provided

Section II : 54 marks

- Attempt Questions 8 –13
- All questions are of equal value
- In Questions 8 – 13, show relevant mathematical reasoning and/or calculations

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1} + C, \quad n \neq -1;\quad x \neq 0 \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x + C, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax} + C, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax + C, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax + C, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax + C, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax + C, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + C, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a} + C, \quad a > 0, -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x + \sqrt{x^2-a^2}\right) + C, \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x + \sqrt{x^2+a^2}\right) + C$$

NOTE: $\ln x = \log_e x, \; x > 0$

Section I:

7 Marks
Attempt Questions 1 – 7

Use the multiple choice answer sheet provided for Questions 1-7.

1. $\log_3 5 =$

(A) $\frac{\ln 5}{\ln 3}$

(B) $\frac{\ln 3}{\ln 5}$

(C) $3 \ln 5$

(D) $5 \ln 3$

2. Which is the correct value of $\lim\limits_{x \to 0} \frac{4x}{\sin 5x}$?

(A) 0

(B) $\frac{4}{5}$

(C) $\frac{5}{4}$

(D) 4

3. What is the period of the curve whose equation is $y = \frac{1}{4}\ (cos^2 x - sin^2 x)$?

(A) $\frac{\pi}{4}$

(B) $\frac{\pi}{2}$

(C) π

(D) 2π

Section I (cont'd)

4. What is another expression for $\sin(x + y)$?

(A) $\cos x \cos y + \sin x \sin y$

(B) $\cos x \cos y - \sin x \sin y$

(C) $\sin x \cos y + \sin y \cos x$

(D) $\sin x \cos y - \sin y \cos x$

5. What is the domain of $y = 3\cos^{-1}\frac{x}{2}$?

(A) $-\pi \le x \le \pi$

(B) $-2\pi \le x \le 2\pi$

(C) $-1 \le x \le 1$

(D) $-2 \le x \le 2$

6. Which of the following is the exact value of $\int_{\frac{3}{\sqrt{2}}}^{3} \frac{2}{\sqrt{9 - x^2}}\, dx$?

(A) $-\frac{\pi}{2}$

(B) $\frac{\pi}{2}$

(C) $-\frac{\pi}{6}$

(D) $\frac{\pi}{6}$

7. If $f(x) = e^{x+4}$ then its inverse function $f^{-1}(x)$ is

(A) $f^{-1}(x) = \log_e x - 4$

(B) $f^{-1}(x) = \log_e x + 4$

(C) $f^{-1}(x) = e^{y-4}$

(D) $f^{-1}(x) = e^{y+4}$

End of Section I

Section II

54 marks
Attempt Questions 8 – 13
Start each question in a new booklet.

In Questions 8, 9, 10, 11, 12 and 13 your responses should include relevant mathematical reasoning and/or calculations.

Question 8 (9 Marks) Use a SEPARATE writing booklet. Marks

a) Find $\frac{dy}{dx}$ if

(i) $y = \frac{e^{2x}}{\sin 3x}$.

(ii) $y = e^{1-x}\cos^2 x$. 2

b) Express $\frac{dy}{dx}$ in simplest form, in terms of the sine ratio only 2

$$y = \log_e(\tan 3x) \ .$$

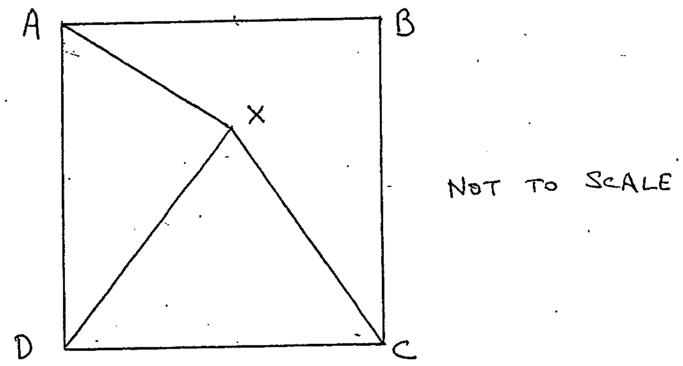
c) Differentiate $\tan^3 x$ and hence find $\int_0^{\frac{\pi}{4}} \tan^2 x \sec^2 x \, dx$. 3

Question 9 (9 Marks) Use a SEPARATE writing booklet. Marks

a) Find the acute angle, to the nearest minute, between the curve $y = x^2$ and the line $5x - y - 6 = 0$ at the point of intersection which is furthest from the origin. 4

b) If $\tan(A+B) = x$ and $\tan B = \frac{1}{2}$, express $\tan A$ in terms of x . 2

c) 3



$ABCD$ is a square and CDX is an equilateral triangle.
AX has been joined.

Find, stating all reasons, the size of $\angle XAD$.

Question 10 (9 Marks) Use a SEPARATE writing booklet. **Marks**

a) Find $\frac{dy}{dx}$ as a single fraction if $y = \ln\left(\frac{1+e^x}{1-e^x}\right)$. **3**

b) Find

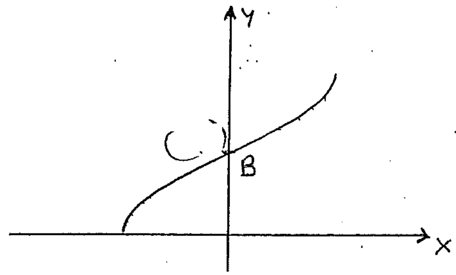
(i) $\int 4e^{1-2x}\, dx$. **2**

(ii) $\int_0^{\frac{1}{2}} \frac{2x}{1-3x^2}\, dx$ [leave your answer in exact form]. **2**

c) State the domain and range of $y = 4\sin^{-1} 6x$. **2**

Question 11 (9 Marks) Use a SEPARATE writing booklet. **Marks**

a)



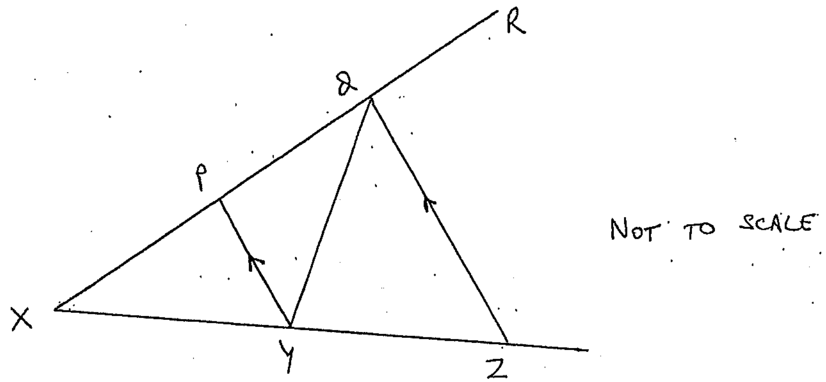
The diagram shows the graph of $y = \pi + 3\sin^{-1} 2x$

(i) What are the co-ordinates of B which lies on the y axis. **1**

(ii) Find the equation of the tangent to $y = \pi + 3\sin^{-1} 2x$ at the point B. **3**

b) Prove the identity $\frac{2\tan A}{1+\tan^2 A} = \sin 2A$. **3**

c) **2**



In the diagram PY and QZ are parallel. The line QZ bisects $\angle RQY$.

Prove $PQ = QY$.

Question 12 (9 Marks) Use a SEPARATE writing booklet. **Marks**

a) The curve $y = 2^x$ is rotated about the x axis between $x = 1$ and $x = 3$.

(i) Write the volume as an integral. **1**

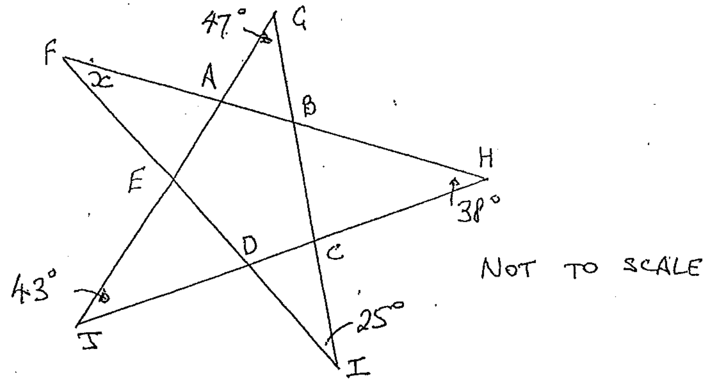
(ii) Use Simpson's Rule with 3 function values to find an approximation of the volume formed [leave your answer in terms of π]. **3**

b) (i) Prove $\frac{1}{2} + \frac{1}{2}\cos 2\theta = \cos^2\theta$. **2**

(ii) Hence, or otherwise, evaluate $\int_0^{\frac{\pi}{4}} \cos^2\theta \; d\theta$. **3**

Question 13 (9 Marks) Use a SEPARATE writing booklet. **Marks**

a) **3**



Find the value of the pronumeral, stating all reasons.

b) Find the exact value of $\sin\left(2\cos^{-1}\frac{12}{13}\right)$. **3**

c) Graph $y = 3\sec^2 2x$ for $-\pi < x < \pi$, use at least $\frac{1}{2}$ a page. **3**

End of Paper

Student Name: ____

Class Teacher: ____________

ST GEORGE G.H.S - YR12 - MID-HSC - EXT 1 SOLUTIONS

Section 1

Multiple-choice Answer Sheet - Questions 1 – 7

Select the alternative A, B, C or D that best answers the question. Fill in the response oval completely.

Sample 2 + 4 = (A) 2 (B) 6 (C) 8 (D) 9

A ○ B ● C ○ D ○

If you think you have made a mistake, put a cross through the incorrect answer and fill in the new answer.

A ● B ~~●~~ C ○ D ○

If you change your mind and have crossed out what you consider to be the correct answer, then indicate this by writing the word *correct* and drawing an arrow as follows:

A ~~●~~ B ~~●~~ *correct* C ○ D ○

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1. A ● B ○ C ○ D ○
2. A ○ B ○ C ● D ○
3. A ○ B ● C ○ D ○
4. A ○ B ○ C ● D ○
5. A ○ B ○ C ○ D ●
6. A ○ B ● C ○ D ○
7. A ● B ○ C ○ D ○

5

SECTION II

(8) a) (i) $\frac{dy}{dx} = \frac{vu' - uv'}{v^2}$ (using quotient rule)

$v = \sin 3x \qquad u = e^{2x}$

$v' = 3\cos 3x \qquad u' = 2e^{2x}$ (1)

$\frac{dy}{dx} = \frac{2e^{2x}\sin 3x - 3e^{2x}\cos 3x}{(\sin 3x)^2}$ (1)

$= \frac{e^{2x}(2\sin 3x - 3\cos 3x)}{(\sin 3x)^2}$

(ii) $\frac{dy}{dx} = vu' + uv'$ (using product rule)

careful (1)

$= -e^{1-x}\cos^2 x \; - \; 2e^{1-x}\cos x \sin x$

~~$= \cos x e^{1-x}(2 - \cos x)$~~

$= -e^{1-x}\cos x(\cos x + 2\sin x)$ (1)

b) $\frac{dy}{dx} = \frac{f'(x)}{f(x)}$

$= \frac{3\sec^2 3x}{\tan 3x}$ (1)

$= \frac{3}{\cos^2 3x} \div \frac{\sin 3x}{\cos 3x}$

~~$= \frac{3\sin 3x}{\cos 3x(1 - \sin^2 3x)}$~~

$= \frac{3}{\cos^{\cancel{2}} 3x} \times \frac{\cancel{\cos} 3x}{\sin 3x}$

$= \frac{3}{\cos 3x \sin 3x}$

$\frac{1}{2}\sin 2x = \cancel{2}\sin x\cos x$

$$= \frac{3}{\frac{1}{2}\sin 6x}$$

$$= 3 \div \frac{\sin 6x}{2}$$

$$= 3 \times \frac{2}{\sin 6x}$$

$$= \frac{6}{\sin 6x}$$

(1)

c) $\cancel{\frac{dy}{dx}}\ \cancel{\tan^3 x}$

let $y = \tan^3 x$

let $u = \tan x \rightarrow \therefore y = u^3$

$$\frac{dy}{du} = 3u^2$$

$$\frac{du}{dx} = \sec^2 x$$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$= 3u^2 \cdot \sec^2 x$$

$$\therefore \frac{dy}{dx} = 3\tan^2 x \sec^2 x$$

(1)

$$\int_0^{\frac{\pi}{4}} \tan^2 x \sec^2 x \; dx$$

$$= \frac{1}{3}\int_0^{\frac{\pi}{4}} 3\tan^2 x \sec^2 x \; dx$$

(1)

$$= \frac{1}{3}\left[\tan^3 x\right]_0^{\frac{\pi}{4}}$$

$$= \frac{1}{3}(1-0)$$

$$= \frac{1}{3}$$

(1)

$\frac{9}{9}$

(9) a) $y = x^2$ | $y = 5x - 6$

$\therefore y' = 2x = m_1$ | $\therefore y' = 5 = m_2$

$\cancel{\tan\theta = \left|\frac{m_1 - m_2}{1+m_1m_2}\right|}$

$\cancel{=}$

$y = x^2$ — (1)

$y = 5x - 6$ — (2)

$$x^2 = 5x - 6$$

$$x^2 - 5x + 6 = 0$$

$$(x-3)(x-2) = 0$$

$\therefore x = 3 \boxed{\text{OR}}\ 2$ *compare actual points*

$x = 3$ is furthest from $(0,0)$

$\therefore$ point of intersection occurs at $(3, 9)$

$\tan\theta = \left|\frac{m_1 - m_2}{1+m_1m_2}\right|$ where $m_1 = 2x = 6$

where $m_2 = 5$

$$= \left|\frac{6-5}{1+(6\times5)}\right|$$

$$\tan\theta = \left|\frac{1}{31}\right|$$

$\therefore \theta = 1°51'$ (nst minute)

$3\frac{1}{2}$

(9) b) $x = \tan(A+B)$

$$= \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

$$x = \frac{\tan A + \frac{1}{2}}{1 - \frac{1}{2}\tan A} \quad \begin{matrix}\times 2\\ \times 2\end{matrix}$$

$$x = \frac{2\tan A + 1}{2 - \tan A}$$

$2x - x\tan A = 2\tan A + 1$

$2\tan A + x\tan A = 2x - 1$

$\tan A(2 + x) = 2x - 1$

$$\therefore \tan A = \frac{2x-1}{2+x}$$

2

c) ~~XDC~~ $AB = BC = CD = AD$ (sides of square are equal)

$XC = CD = XD$ (equal sides of equilateral triangle)

$\therefore \triangle XAD$ is isosceles ($AD = XD$; two equal sides)

let $\angle XDC = \alpha$

$3\alpha = 180°$ (angle sum of triangle)

$\therefore \alpha = 60°$

$\angle ADX = 90 - \alpha$ (angle sum of a right angle; corners of a square are 90°)

$= 30°$

$\angle XAD = \angle AXD$ (~~base~~ angles opposite equal sides ~~angles~~ of isosceles triangle)

$2\angle XAD + 30° = 180°$ (angle sum of a triangle)

$2\angle XAD = 150°$

$\therefore \angle XAD = 75°$

3

$8\frac{1}{2}$

(10)

a) $y = \ln\left(\frac{1+e^x}{1-e^x}\right)$

$= \ln(1+e^x) - \ln(1-e^x)$

$$\frac{dy}{dx} = \frac{e^x}{1+e^x} + \frac{e^x}{1-e^x}$$

$$= \frac{e^x(1-e^x) + e^x(1+e^x)}{1-e^{2x}}$$

$$= \frac{e^x(1-\cancel{e^x} + 1 + \cancel{e^x})}{1-e^{2x}}$$

$$\therefore \frac{dy}{dx} = \frac{2e^x}{1-e^{2x}}$$

3

b) (i) $\int 4e^{1-2x}\,dx$

$$= \frac{-4}{2}\int -2e^{1-2x}\,dx$$

$\therefore \int 4e^{1-2x}\,dx = -2e^{1-2x} + C$

2

(ii) $$= -\frac{1}{3}\int_0^{1/2} \frac{-6x}{1-3x^2}\,dx$$

$$= -\frac{1}{3}\left[\ln(1-3x^2)\right]_0^{1/2}$$

$$= -\frac{1}{3}\left(\ln\frac{1}{4} - \ln 1\right)$$

$$= \frac{1}{3}(\cancel{\ln 1} + \ln 4 - \cancel{\ln 1})$$

$$\therefore \int_0^{\frac{1}{2}} \frac{2x}{1-3x^2}\,dx = \frac{1}{3}\ln 4$$

2

(10)

c) Domain:

$-1 \leq 6x \leq 1$

$\therefore$ Domain: $\left\{ -\frac{1}{6} \leq x \leq \frac{1}{6} \right\}$ ✓

Range:

$-\frac{\pi}{2} \leq \frac{y}{4} \leq \frac{\pi}{2}$ 2 ✓

$\therefore$ Range: $\{ -2\pi \leq y \leq 2\pi \}$ ✓

(11)

a) B is a y intercept

y intercepts occur when $x=0$

$y = 3\sin^{-1}(2 \times 0) + \pi$

$= 0 + \pi$

$\therefore y = \pi$

$\therefore$ B is at $(0, \pi)$ ✓ 1

(ii) $f(x) = \pi + 3\sin^{-1} 2x$

$f'(x) = 3 \cdot \frac{2}{\sqrt{1-4x^2}}$

$f'(x) = \frac{6}{\sqrt{1-4x^2}}$ ✓

$f'(0) = \frac{6}{\sqrt{1-4(0)^2}}$

$\therefore f'(0) = 6$ ✓ 3

$y - y_1 = m(x - x_1)$

$y - \pi = 6(x - 0)$ ✓

$\therefore y = 6x + \pi$ is the equation of the tangent at B

b) LHS $= \frac{2\tan A}{1+\tan^2 A}$

$= \frac{2\tan A}{\sec^2 A}$ ✓

$= \frac{2\sin A}{\cos A} \times \frac{1}{\frac{1}{\cos^2 A}}$

(11)

$= \frac{2\sin A}{\cos A} \div \frac{1}{\cos^2 A}$

$= \frac{2\sin A}{\cancel{\cos A}} \times \cos^{\cancel{2}} A$

3

$= 2\sin A\cos A$

$= \sin 2A$

$= RHS$

c) let $\angle RQZ = \alpha$

$\angle RQZ = \angle YQZ$ (QZ bisects $\angle RQY$ [given] $\therefore$ they are equal)

$= \alpha$

$\angle PQY = 180 - (\alpha + \alpha)$ (angle sum of triangle)

$= 180 - 2\alpha$

$\angle QPY + \angle QYP = 180 - (180 - 2\alpha)$ (angle sum of triangle)

$= 2\alpha$

But $\angle ZQY = \angle QYP$ (alternate angles on parallel lines are equal)

$= \alpha$

2

$\therefore \angle QPY = \angle QYP = \alpha$ ($\angle QPY + \angle QYP = 2\alpha$)

$\therefore \triangle PQY$ is isosceles (equal base angles)

$\therefore PQ = QY$ (equal sides of an isosceles triangle)

$\frac{1}{\ln 4}{}^4 \quad a^x = e^{x\log a}$

(12) a) (i) $V = \pi \int_1^3 y^2 \, dx$

$= \pi \int_1^3 (2^x)^2 \, dx$

$= \pi \int_1^3 2^{2x} \, dx$ (1) OR $\pi \int_1^3 e^{2x\log 2}$

$\therefore V = \pi \int_1^3 4^x \, dx$ OR

~~$\therefore V = \pi \int_1^3 e^{x\log 4} \, dx$~~

(ii) ~~$V = \pi \int_1^3 e^{x\log 4} \, dx$~~ $\quad \boxed{V = \pi \int_1^3 4^x \, dx}$

$= \pi \left(\frac{3-1}{6}\left(f(1) + 4f(2) + f(3)\right)\right)$

$= \frac{\pi}{3}\left(4 + (4 \times 4^2) + 4^3\right)$

$= \frac{132\pi}{3}$

$\therefore$ Volume $= 44\pi \, u^3$

3

b) (i) LHS $= \frac{1}{2} + \frac{1}{2}\cos 2\theta$

$= \frac{1}{2}(\cos 2\theta + 1)$

$= \frac{1}{2}(2\cos^2\theta - 1 + 1)$

$= \cos^2\theta$

$= RHS$

2

(ii) $\int_0^{\frac{\pi}{4}} \cos^2\theta \, d\theta$

$= \frac{1}{2}\int_0^{\frac{\pi}{4}} 1 + \cos 2\theta \, d\theta$

$\sin \to \cos$
$\cos \to -\sin$

(12)

$$= \frac{1}{2}\left[\theta + \frac{1}{2}\sin 2\theta\right]_0^{\frac{\pi}{4}}$$

$$= \frac{1}{4}\left[2\theta + \sin 2\theta\right]_0^{\frac{\pi}{4}}$$

$$= \frac{1}{4}\left(\left(\frac{\pi}{2} + 1\right) - \cancel{(0 + 0)}\right)$$

$$= \frac{\pi}{8} + \frac{1}{4}$$ ✓ 3

$$\therefore \int_0^{\frac{\pi}{4}} \cos^2\theta \, d\theta = \frac{\pi + 2}{8}$$

$$\cancel{= \frac{1}{2}\left(\left(\frac{\pi}{4} + \frac{1}{2}\right) - (0 + 0)\right)}$$

$$\cancel{= \frac{\pi}{8} + \frac{1}{4}}$$

(13) a) $\angle GEI = 180 - (\angle JGI + \angle GIF)$ (angle sum of triangle)

$= 180 - (47 + 25)$

$= 108°$

$\angle FEA = 180 - \angle GEI$ (angle sum of a straight line)

$= 180 - 108°$

$\therefore \angle FEA = 72°$ ✓

$\angle FAE = \angle FHJ + \angle GJH$ (external angle of a triangle equals sum of opposite interior angles)

$\angle FAE = 38 + 43$

$\therefore \angle FAE = 81°$ 3

$x = 180 - (\angle FAE + \angle FEA)$ (angle sum of a triangle)

$= 180 - (81 + 72)$

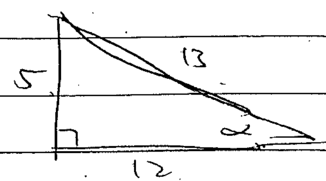
$= 180 - 153$

$\therefore x = 27°$ ✓

b) ~~sin(2~~ let $y = \sin\left(2\cos^{-1}\frac{12}{13}\right)$

let $\alpha = \cos^{-1}\frac{12}{13}$

$\cos\alpha = \frac{12}{13}$



$\sin\alpha = \frac{5}{13}$

$\cos\alpha = \frac{12}{13}$

$\sin 2\alpha = 2\sin\alpha\cos\alpha$

$= 2 \cdot \frac{5}{13} \cdot \frac{12}{13}$

$= \frac{120}{169}$

$\therefore \sin 2\cos^{-1}\frac{12}{13} = \frac{120}{169}$

3

c) $y = 3\sec^2 2x \qquad -\pi < x < \pi$

x intercepts ~~occur~~ when $y = 0$

$\sec^2 2x = 0$

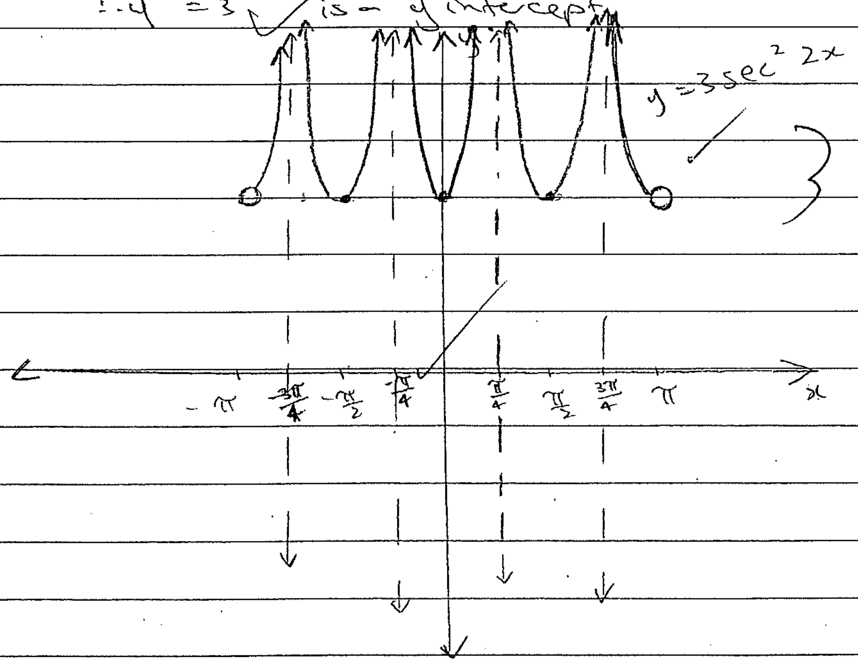
$\frac{1}{\cos^2 2x} = 0$

$\therefore$ no x intercepts

y intercepts occur when $x = 0$

$y = 3\sec^2 0$

$\therefore y = 3$ is a y intercept



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