St George Girls High School

Year 12

Mid-HSC Course Examination

2014

Mathematics

General Instructions

- Reading Time: 5 minutes
- Working Time: 1 hour 30 minutes
- Write using blue or black pen
- Board-approved calculators may be used
- A table of standard integrals is provided

Section I
Total marks (5)
Attempt Questions 1 – 5
Use the Multiple-choice Answer Sheet provided at the end of the paper

Section II
Total marks (60)
Attempt Questions 6 – 10
Start each question in a new booklet
All necessary working should be shown

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1} + C, \quad n \neq -1; \quad x \neq 0 \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x + C, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax} + C, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax + C, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax + C, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax + C, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax + C, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + C, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}}\,dx = \sin^{-1}\frac{x}{a} + C, \quad a > 0, -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}}\,dx = \ln\left(x + \sqrt{x^2 - a^2}\right) + C, \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}}\,dx = \ln\left(x + \sqrt{x^2 + a^2}\right) + C$$

NOTE: $\ln x = \log_e x, \; x > 0$

Section I:

5 Marks
Attempt Questions 1 – 5

Use the multiple choice answer sheet provided for Questions 1 – 5.

1. The solution to the equation $3^x = \frac{1}{81}$ is

(A) 0.4
(B) $\frac{1}{4}$
(C) 4
(D) -4

2. If $y = \ln(x^2 - 1)$, $\frac{dy}{dx}$ is

(A) $\frac{1}{x^2-1}$
(B) $\frac{2}{x^2-1}$
(C) $\frac{2x}{x^2-1}$
(D) $\frac{x^2-1}{2x}$

3. The derivative of $\frac{x^3}{e^x}$ is

(A) $\frac{x^2(3-x)}{e^x}$
(B) $\frac{e^x}{3x^2}$
(C) $\frac{e^x(3-x)}{x^2}$
(D) $\frac{3x^2}{e^x}$

Section I (cont'd)

4. The semicircle $y = \sqrt{4 - x^2}$ is rotated about y-axis. The correct expression for the volume of the solid of revolution is

(A) $V = 2\pi \int_0^2 (4 - x^2)\, dx$
(B) $V = \pi \int_0^2 (4 - y^2)\, dy$
(C) $V = \pi \int_0^2 (4 - x^2)\, dx$
(D) $V = 2\pi \int_0^2 (4 - y^2)\, dy$

5. The correct expression for the area enclosed between the curves $y = x^3$ and $y = 4x$ is

(A) $A = \int_{-2}^{2} (4x - x^3)\, dx$
(B) $A = 2\int_0^2 (4x - x^3)\, dx$
(C) $A = 2\int_0^2 (x^3 - 4x)\, dx$
(D) $A = \int_{-2}^{2} (x^3 - 4x)\, dx$

Section II

60 marks
Attempt Questions 6 – 10
Start each question in a new booklet

All necessary working should be shown in every question.

Question 6 (12 Marks) Use a SEPARATE writing booklet. **Marks**

a) Simplify:

(i) $e^{2x} \times e^{5x}$ 1

(ii) $e^{2x} \div e^{5x}$ 1

b) Find:

(i) $\int (7 - x)\ dx$ 1

(ii) $\int 3x^2\ dx$ 1

(iii) $\int \frac{2}{x^3}\ dx$ 2

c) Evaluate the following definite integrals:

(i) $\int_{-2}^{0} \frac{2x^3 - x^5}{x^2}\ dx$ 2

(ii) $\int_{-3}^{3} (2x^3 - x)\ dx$ 2

d) Sketch the graph of $y = e^{x+1}$, showing the y-intercept and one other point on the curve. 2

Question 7 (12 Marks) Use a SEPARATE writing booklet. **Marks**

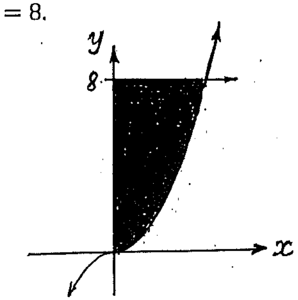
a) Differentiate and simplify the result where possible:

(i) $y = e^{2x+1}$ 1

(ii) $y = x^5 e^x$ 2

(iii) $y = \frac{e^{3x}}{3x+1}$ 2

b) The shaded region in the diagram is bounded by the curve $y = x^3$, y-axis, and the line $y = 8$.



Calculate the area of the shaded region. 3

c) (i) Show that $\int_1^k (2x + 1)\ dx = k^2 + k - 2$ 2

(ii) Hence find the two values of k if $\int_1^k (2x + 1)\ dx = 10$ 2

Question 8 (12 Marks) Use a SEPARATE writing booklet. **Marks**

a) Rewrite the equation $5^x = 817$ in the logarithmic form and solve for x correct to four significant figures. **2**

b) Find the derivative of 3^x **1**

c) Use the log laws to simplify $2e\left(\log_e\sqrt{e} + \log_e \frac{1}{e}\right)$ **2**

d) Consider the curve $y = x - e^x$

(i) Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ **2**

(ii) Show that there is a stationary point at $(0, -1)$ and determine its nature. **2**

(iii) Explain why the curve is concave down for all values of x. **1**

(iv) Sketch the curve and write down its range. **2**

Question 9 (12 Marks) Use a SEPARATE writing booklet. **Marks**

a) The parabola $y = x^2 - 5x + 4$ and the line $y = x - 1$ have two points of intersection $(1, 0)$ and $(5, 4)$.

(i) Sketch the parabola and the line on the same number plane and shade the area enclosed between them. **2**

(ii) Hence, find the shaded area. **3**

b) The region between the curve $y = e^{-x}$ and the x-axis from $x = 0$ to $x = 2$ is rotated about the x-axis.

(i) Sketch the region. **1**

(ii) Find the exact value for the volume of the solid of revolution. **3**

c) Some of the values of the function $y = \frac{\log_e x}{\sqrt{x}}$ (correct to two decimal places) are given in the following table:

x	1	2	3	4	5
y		0.49			0.72

(i) Copy the above table into your booklet. Complete the table by inserting the missing values. **1**

(ii) Give the best possible estimate for $\int_1^5 \frac{\log x}{\sqrt{x}}\,dx$ using the Simpson's rule and the available data. **2**

Question 10 (12 Marks) Use a SEPARATE writing booklet. **Marks**

a) Differentiate with respect to x:

(i) $y = \dfrac{\log_e x}{x^3}$ 2

(ii) $y = x\ln(x^2+1)$ 2

b) Evaluate the following definite integral $\displaystyle\int_0^1 \frac{dx}{3+4x}$ 2

c) (i) Using the log laws expand $\ln\sqrt{\dfrac{3+x}{3-x}}$ fully 1

(ii) Hence show that $\dfrac{d}{dx}\ln\sqrt{\dfrac{3+x}{3-x}} = \dfrac{3}{9-x^2}$ 3

(iii) Hence evaluate $\displaystyle\int_0^1 \frac{3dx}{9-x^2}$ 2

End of Paper

a) (i) $= e^{2x+5x}$

$\therefore e^{2x}\times e^{5x} = e^{7x}$ 1

(ii) $= e^{2x-5x}$

$\therefore e^{2x}\div e^{5x} = e^{-3x}$ $\boxed{\text{or}}$ $\dfrac{1}{e^{3x}}$ 1

b) (i) $\int (7-x)\,dx$

$\therefore \int(7-x)\,dx = 7x - \dfrac{x^2}{2} + C$ 1

(ii) $= \dfrac{\cancel{3}x^3}{\cancel{3}} + C$

$\therefore \int 3x^2\,dx = x^3 + C$ 1

(iii) $= \int 2x^{-3}\,dx$

$= \dfrac{\cancel{2}x^{-2}}{-\cancel{2}} + C$

$\therefore \int \dfrac{2}{x^3}\,dx = \dfrac{-1}{x^2} + C$ 2

c) (i) $= \int_{-2}^{0} 2x - x^3\,dx$

$= \left[\dfrac{\cancel{2}x^2}{\cancel{2}} - \dfrac{x^4}{4}\right]_{-2}^{0}$

$= 0 - (4-4)$

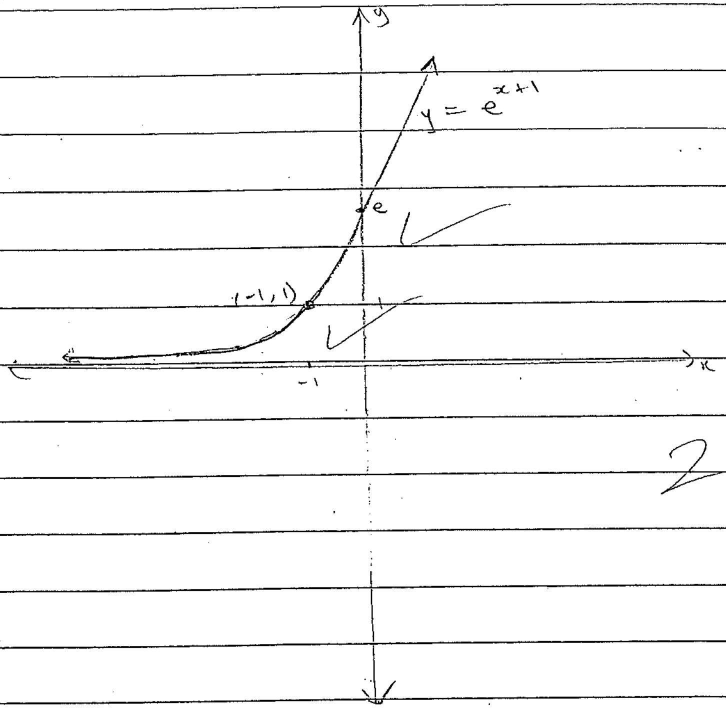
$\therefore \displaystyle\int_{-2}^{0}\left(\frac{2x^3-x^5}{x^2}\right)dx = 0$ 2

(ii) $= \left[\dfrac{\cancel{2}x^4}{\cancel{4}\,2} - \dfrac{x^2}{2}\right]_{-3}^{3}$

$= \left[\dfrac{x^4}{2} - \dfrac{x^2}{2}\right]_{-3}^{3}$

$= 36 - \left(\dfrac{81}{2} - \dfrac{9}{2}\right)$

$\therefore \displaystyle\int_{-3}^{3}(2x^3 - x)\,dx = 0$ 2 d)



2

y intercept occurs when $x = 0$

$y = e^{0+1}$

$\therefore y = e$

(7)

a) (i) $\frac{dy}{dx} = 2e^{2x+1}$ 1

(ii) $\frac{dy}{dx} = vu' + uv'$ (using product rule)

$= 5x^4 e^x + x^5 e^x$

$\therefore \frac{dy}{dx} = x^4 e^x (5 + x)$ 2

(iii) $\frac{dy}{dx} = \frac{vu' - uv'}{v^2}$ (using quotient rule)

$= \frac{3e^{3x}(3x+1) - 3e^{3x}}{(3x+1)^2}$ 1

$= \frac{3e^{3x}(3x\cancel{+1}\cancel{-1})}{(3x+1)^2}$

$\therefore \frac{dy}{dx} = \frac{9xe^{3x}}{(3x+1)^2}$ 2

b) ~~Area~~ $y = x^3$ —— ①

$y = 8$ —— ②

$x^3 = 8$

$\therefore x = 2$

$\text{Area} = (8\times 2) - \int_0^2 x^3 \, dx$

$= 16 - \left[\frac{x^4}{4}\right]_0^2$

$= 16 - \frac{1}{4}\left[x^4\right]_0^2$

$= 16 - \frac{1}{4}(16 - 0)$ 3

$= 16 - 4$

$\therefore \text{Area} = 12 \, u^2$

(7) c) (i) $\int_1^k (2x+1)\,dx$

$= \left[\frac{\cancel{2}x^2}{\cancel{2}} + x\right]_1^k$

$= k^2 + k - (1+1)$

$\therefore \int_1^k (2x+1)\,dx = k^2 + k - 2$, as required

(ii) $\int_1^k (2x+1)\,dx = 10$

$k^2 + k - 2 = 10$

$k^2 + k - 12 = 0$

$\cancel{k+4}$

$(k+4)(k-3) = 0$

$\therefore k = 3$ $\boxed{\text{OR}}$ -4

(8) a) $5^x = 817$

$x = \log_5 817$

$= \frac{\ln 817}{\ln 5}$ (using change of base rule)

$\simeq 4.166447828\ldots$

$\therefore x \simeq 4.166$ (4sf)

b) $\frac{d}{dx} 3^x = \frac{d}{dx} e^{x\log 3}$

$= \log 3\, e^{x\log 3}$

$\therefore \frac{d}{dx} 3x = 3^x \log 3$

c) $= 2e\log_e e^{1/2} + 2e\log_e e^{-1}$

$= (\cancel{2}e \cdot \frac{1}{\cancel{2}}) + (2e \cdot -1)$

$= e - 2e$

$2e(\log_e\sqrt{e} + \log_e \frac{1}{e}) = -e$

d) (i) $y = x - e^x$

$\frac{dy}{dx} = 1 - e^x$

$\frac{d^2y}{dx^2} = -e^x$

(ii) stationary points occur when $\frac{dy}{dx} = 0$

$1 - e^x = 0$

$e^x = 1$

$x = \ln 1$

$\therefore x = 0$

when $x = 0$

$y = 0 - e^0$

$\therefore y = 1$

$\therefore$ a stationary point occurs at $(0,-1)$

$f''(x) = -e^x$

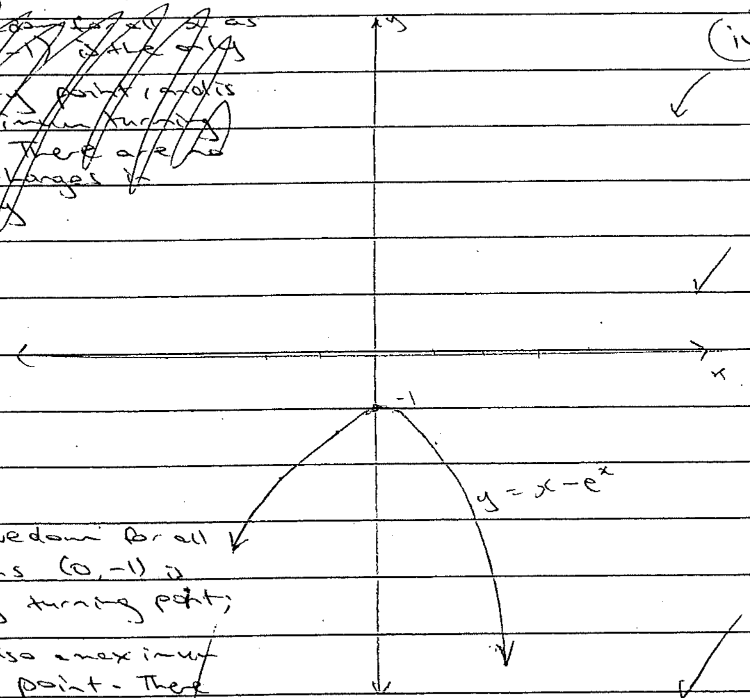
$f''(0) = -e^0$

$= -1$

< 0

$\therefore (0,-1)$ is a maximum turning point

(iv)



(iii) concave down for all x as $(0,-1)$ is the only turning point; it is also a maximum turning point. There are no changes in concavity as $f''(x) \neq 0$

$\therefore$ concave down for all x

Range: $\{y \leq -1\}$

(9)

a) (i) $y = x^2 - 5x + 4$

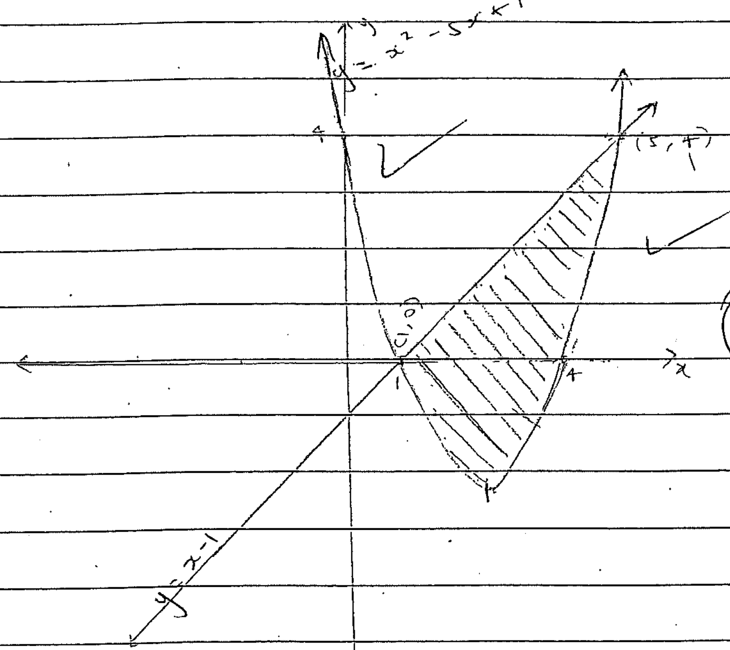
$0 = (x-4)(x-1)$ (x intercepts occur when $y=0$)

$\therefore$ x-intercepts are at $x = 4, 1$

$y = x - 1$ (x intercepts occur when $y=0$)

$0 = x - 1$

$\therefore x = 1$ is an intercept



(ii) Area enclosed $= \int_1^5 (x-1) - (x^2 - 5x + 4)\,dx$

$= \int_1^5 -x^2 + 6x - 5 \,dx$

$= \left[-\frac{x^3}{3} + \frac{6x^2}{2} - 5x\right]_1^5$

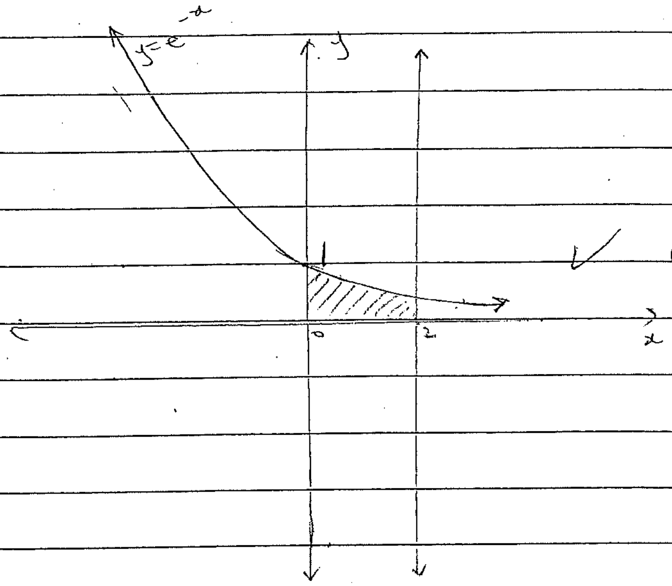
$= \left(-\frac{5^3}{3} + 3.25 - 25\right) - \left(-\frac{1}{3} + 3 - 5\right)$

$8\frac{1}{3} + 2\frac{1}{3} = \underline{10\frac{2}{3}}$

$\therefore$ ~~Area between the~~

$\therefore$ ~~Shaded Area $= 9u^2$~~

(9) b) (i)



1/2

(ii) $V = \pi \int_0^2 y^2 \, dx$

$= \pi \int_0^2 e^{-2x} \, dx$

$= \pi \left[\frac{-1}{2} e^{-2x} \right]_0^2$

$= -\frac{\pi}{2} \left[e^{-2x} \right]_0^2$

$= -\frac{\pi}{2} \left(e^{-4} - 1 \right)$

$= \frac{\pi}{2} \left(1 - e^{-4} \right)$

3

$\therefore$ Volume $= \frac{\pi}{2} \left(1 - \frac{1}{e^4} \right) u^3$

c)

x	1	2	3	4	5
y	0	0.49	0.63	0.69	0.72

(9)(c) (ii) $\int_1^5 \frac{\log x}{\sqrt{x}} \, dx \doteqdot \frac{3-1}{6} \left(0 + (0.49 \times 4) + 0.63 \right)$

$+ \frac{5-3}{6} \left(0.63 + (4 \times 0.69) + 0.72 \right)$

$\doteqdot \cancel{\frac{1}{3}(2.59} \left(\frac{1}{3} \times 2.59 \right)$

$+ \left(\frac{1}{3} \times 4.11 \right)$

~~$= \frac{67}{30}$ or $2.2\dot{3}$~~

$= \frac{67}{30}$ or $2.2\dot{3}$

3

(10) a) (i) $y = \frac{\ln x}{x^3}$

$\cancel{= x^{-3}\ln x}$

$\cancel{\frac{dy}{dx} = vu' + uv'}$ ~~(using product rule)~~

$\cancel{= -3x^{-2}\ln x + \left(x^{-3}\cdot\frac{1}{x}\right)}$

$\cancel{= \frac{-3\ln x}{x^2} + \left(\frac{1}{x^3}\cdot\frac{1}{x}\right)}$

$\cancel{= \frac{-3\ln x}{x^2} + \frac{1}{x^4}}$

$\cancel{= \frac{-3x^2\ln x}{x^4} + \frac{1}{x^4}}$

$\cancel{= \frac{1-3x^2\ln x}{x^4}}$

$\frac{dy}{dx} = \frac{vu' - uv'}{v^2}$ (using quotient rule) 1

$u = \ln x \qquad v = x^3$

$u' = \frac{1}{x} \qquad v' = 3x^2$

$\frac{dy}{dx} = \frac{\frac{x^3}{x} - 3x^2\ln x}{x^6}$ 2

$= \frac{\cancel{x^2} - 3\cancel{x^2}\ln x}{\cancel{x^6}\ x^4}$

$\therefore \frac{dy}{dx} = \frac{1 - 3\ln x}{x^4}$

(ii) $y = x\ln(x^2+1)$

$\frac{dy}{dx} = vu' + uv'$ (using product rule)

$= \ln(x^2+1) + x\cdot\frac{2x}{x^2+1}$

$\therefore \frac{dy}{dx} = \ln(x^2+1) + \frac{2x^2}{x^2+1}$ ✓ 2

(10) b) $\int_0^1 \frac{1}{3+4x}\,dx$

$= \frac{1}{4}\int_0^1 \frac{4}{3+4x}$

$= \frac{1}{4}\left[\ln(3+4x)\right]_0^1$

$= \frac{1}{4}\cdot(\ln 7 - \ln 3)$

$= \frac{1}{4}\times\ln\frac{7}{3}$

$= \frac{1}{4}\ln\frac{7}{3}$ ✓ 2

c) (i) $= \ln\left(\frac{3+x}{3-x}\right)^{1/2}$

$= \frac{1}{2}\left(\ln(3+x) - \ln(3-x)\right)$

$= \frac{1}{2}\ln(3+x) - \frac{1}{2}\ln(3-x)$ ✓ 1

(ii) $\frac{d}{dx}\ln\sqrt{\frac{3+x}{3-x}} = \frac{d}{dx}\ \frac{1}{2}\ln(3+x) - \frac{1}{2}\ln(3-x)$

$= \left(\frac{1}{2}\cdot\frac{1}{3+x}\right) - \left(\frac{1}{2}\cdot\frac{-1}{3-x}\right)$

$= \frac{1}{6+2x} + \frac{1}{6-2x}$

$=$

$= \frac{1}{2}\left(\frac{1}{3+x} + \frac{1}{3-x}\right)$

$= \frac{1}{2}\left(\frac{3\cancel{-x}+3\cancel{+x}}{9-x^2}\right)$

$= \frac{\cancel{6}\,3}{\cancel{2}(9-x^2)}$ ✓ 3

$\therefore \frac{dy}{dx} = \frac{3}{9-x^2}$

(iii) From (ii) $\frac{d}{dx}\left(\ln\sqrt{\frac{3+x}{3-x}}\right) = \frac{3}{9-x^2}$

$\therefore \int_0^1 \frac{3}{9-x^2}\,dx = \left[\ln\sqrt{\frac{3+x}{3-x}}\right]_0^1$

$= \ln\sqrt{\frac{4}{2}} - \ln\sqrt{\frac{3}{3}}$

$= \ln\sqrt{2}$