Name:____________________ Class:____________

SYDNEY TECHNICAL HIGH SCHOOL

TRIAL HIGHER SCHOOL CERTIFICATE

2007

EXTENSION 1 MATHEMATICS

Instructions:

General Instructions

- Reading time – 5 minutes
- Working time – 2 hours
- Write using black or blue pen
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- All necessary working should be shown in every question
- Start each question on a new page

Total Marks – 84

- Attempt Questions 1-7
- All questions are of equal value

(For markers use only)

Q1	Q2	Q3	Q4	Q5	Q6	Q7	Total

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1},\quad n \neq -1;\quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x,\quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax},\quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax,\quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax,\quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax,\quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax,\quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a},\quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}}\,dx = \sin^{-1}\frac{x}{a},\quad a > 0,\quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}}\,dx = \ln\left(x + \sqrt{x^2 - a^2}\right),\quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}}\,dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE : $\ln x = \log_e x,\quad x > 0$

Question 1 (12 marks)

a) Find $\log_2 3$ correct to 3 decimal places. 1

b) i) Sketch $y = |2x|$ 1

ii) By drawing suitable lines on your sketch above, determine that one of the following equations $A: |2x| = x - 1$ and $B: |2x| = 1 - x$ has no solutions and solve the other. 3

c) Find $\lim_{x \to 0} \frac{\sin 2x}{\frac{1}{2}x}$ 1

d) If α, β and δ are the roots of $2x^3 + 12x^2 - 6x + 1 = 0$ find the values of

i) $\alpha + \beta + \delta$ 1

ii) $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\delta}$ 2

e) Use the substitution $u = 4 - x^2$ or otherwise to find $\int x\sqrt{4 - x^2}\,dx$ 3

Question 2 (12 marks) (start a new page)

a) Given that $\log_x 2 = a$ and $\log_x 3 = b$ find $\log_x 2.25$ in terms of a and b. 2

b) Evaluate $\int_{-1}^{2} |1 - 2x|\,dx$ by considering a graph or otherwise. 2

c) Find i) $\int \frac{3x}{x^2 + 1}\,dx$ 2

ii) $\int \frac{3}{x^2 + 1}\,dx$ 2

d) Solve $\sin 2\theta = \sin\theta$ for $0 \le \theta \le 2\pi$ 2

e) An area of 1 unit^2 is bounded by the curve $y = \frac{1}{x}$, the x axis and the lines $x = e$ and $x = k$ 2

Find the value of k (in exact form), if $k > e$.

Question 3 (12 marks) (start a new page)

a) Find $\int \cos^2 3x\,dx$ 2

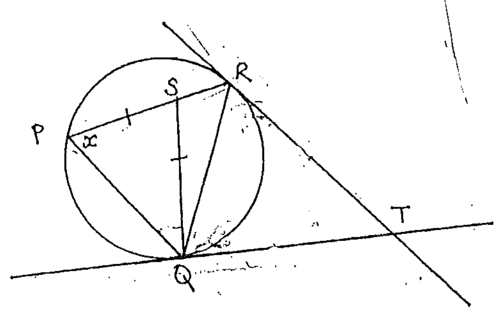
b) i) Show that $\tan 75° = \sqrt{3} + 2$ 2

ii) The lines $y = mx$ and $x = y\sqrt{3}$ meet at an angle of $75°$. Find only one value of m. 2

c) PQR is a triangle inscribed in a circle. S is a point on PR, chosen so that $QS = SP$. Tangents from an external point T touch the circle at Q and R.

Copy the diagram onto your page and prove that the quadrilateral $QTRS$ is cyclic. 3

Let $\angle SPQ = x$



d) i) Show that $\frac{d}{dx}\left[\tan^{-1}(e^x) + \tan^{-1}(e^{-x})\right] = 0$ 2

ii) Hence evaluate $\tan^{-1}(e^x) + \tan^{-1}(e^{-x})$ for all values of x. 1

Question 4 **(12 marks)** **(start a new page)**

a) If $y = xe^x$

i) Prove $\frac{dy}{dx} = e^x(x+1)$ and $\frac{d^2y}{dx^2} = e^x(x+2)$ 2

ii) Hence prove by mathematical induction for all positive integers n, that

$\frac{d^n y}{dx^n} = e^x(x+n)$ 3

b) For the curve $y = 2\sin^{-1}(1-4x)$, state the domain and range and sketch the graph. 3

c) The point $P(2ap, ap^2)$ lies on the parabola $x^2 = 4ay$. The line ℓ is a tangent at P

i) Write the equation of ℓ 1

ii) If ℓ meets the y axis at A, show that $SP=SA$ where S *is* the focus of the parabola. 2

iii) Hence show that ℓ is equally inclined to SP and the axis of the parabola. 1

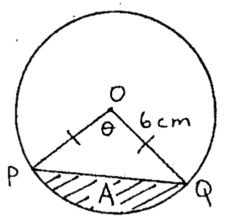
Question 5 **(12 marks)** **(start a new page)**

a) i) The polynomial equation $P(x) = 0$ has a double root at $x = a$. By putting $P(x) = (x-a)^2.Q(x)$ show that $P'(a) = 0$ 2

ii) You are told the polynomial $P(x) = mx^4 + nx^3 - 6x^2 + 22x - 12$ has a double root at $x = 1$. Find the value of m and n. 3

b) O is the centre of a circle with radius 6cm.

$\angle POQ = \theta$ radians



i) Find an expression for A, the area of the minor segment, cut off by the chord PQ, in terms of θ. 1

ii) If θ is increasing at 0.75 radians/second, what is the rate of change of A when $\theta = \frac{\pi}{3}$? 2

c) Katrina, a sky-diver, opens her parachute when falling at 30m/s. Thereafter her acceleration is given by $\frac{dv}{dt} = k(6-v)$ where k is a constant.

i) Show that this condition is satisfied when $v = 6 + Ae^{-kt}$ and find the constant A. 2

ii) One second after opening her chute, her velocity has fallen to 10.7 m/s. Find the value of k correct to 2 decimal places. 1

iii) Find her velocity, correct to 1 decimal place, two seconds after her chute has opened. 1

Question 6 **(12 marks)** **(start a new page)**

a) i) Show that $\frac{d}{dx}(\frac{1}{2}v^2) = \frac{d^2x}{dt^2}$ 1

ii) An object is falling through a fluid in such a way that its acceleration is given by, $\frac{d^2x}{dt^2} = \frac{4}{\sqrt{x}}$ were x is the distance the object has fallen in metres and t is time in seconds. 3.

If the object started from rest, how fast would it be travelling after falling through a distance of 7 metres. (to 1 decimal place)?

b) i) Sketch the function $f(x) = x + \frac{1}{x}$ for $x > 0$ showing the stationary point and asymptotes. 2

ii) State the largest possible domain containing $x = 2$ for which $f(x)$ has an inverse $f^{-1}(x)$. 1

iii) Sketch $y = f^{-1}(x)$ on the diagram above. 1

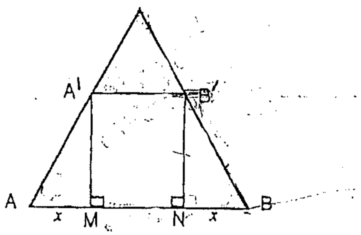
iv) Show that $f^{-1}(x) = \frac{x}{2} + \frac{1}{2}\sqrt{x^2 - 4}$ 2

v) Assume $x = N$, when N is not in the domain chosen for part ii) but still in the domain for $f(x)$.

Find $f^{-1}[f(N)]$ 2

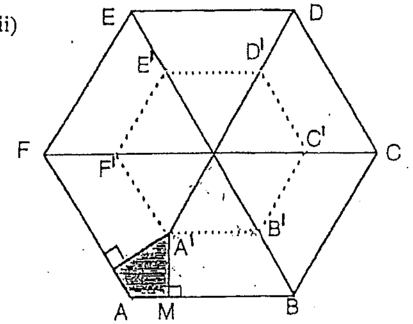
Question 7 (12 marks) (start a new page)

a) i) OAB is an equilateral triangle side m units. $A^1B^1 \,//\, AB$ and $AM = NB = x$ units.



Show that the area of ΔOA^1B^1 is given by $\dfrac{\sqrt{3}(m-2x)^2}{4}$ 2

ii)



Use part i) as you answer part ii)

$ABCDEF$ is a regular hexagon, side m units. The sides of $A^1B^1C^1D^1E^1F^1$ are parallel to those of $ABCDEF$. From each vertex, portions such as the one shaded are removed. The remainder is folded along the dotted lines to form a hexagonal prism

α) If $AM = x$ units prove the volume of the prism is given by

$$V = \frac{9x(m-2x)^2}{2} \text{ units}^3$$ 2

β) Prove that the maximum volume of such a prism is $\dfrac{m^3}{3}$ units^3 3

b) If $\tan 2x = \dfrac{\tan x}{a\tan x + b}$ and $\tan x \neq 0$

i) Find a condition in terms of a and b, for the equation above, to have two different roots $\tan\alpha$ and $\tan\beta$ 2

ii) Assuming this condition to be satisfied prove $\tan^2(\alpha-\beta) = \dfrac{a^2-2b+1}{b^2}$ 3

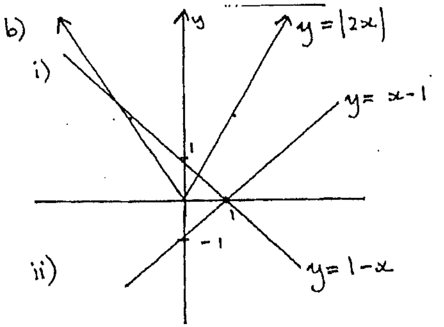
Question 1

a)

$$\log_2 3 = \frac{\log_e 3}{\log_e 2}$$

$= \underline{\underline{1.585}}$ (3 dec. pl.)

b)



i) ∴ $|2x| = x-1$ no solutions; no pts of intersection

ii) $|2x| = 1-x$ 2 solutions

$2x = 1-x$ $\quad 2x = -(1-x)$

$x = \frac{1}{3}$ and $x = -1$

c) $\lim\limits_{x\to 0} \dfrac{\sin 2x}{\frac{1}{2}x}$

$= \lim\limits_{x\to 0} 2 \times \dfrac{\sin 2x}{2x} \cdot 2 = 4$

d) $a=2 \quad b=12 \quad c=-6 \quad d=1$

i) $\alpha+\beta+\gamma = \dfrac{-b}{a} = \dfrac{-12}{2} = \underline{\underline{-6}}$

ii) $\dfrac{1}{\alpha}+\dfrac{1}{\beta}+\dfrac{1}{\gamma} = \dfrac{\beta\gamma+\alpha\gamma+\alpha\beta}{\alpha\beta\gamma}$

$= \dfrac{\frac{c}{a}}{-\frac{d}{a}} = \dfrac{-3}{-1/2}$

$= \underline{\underline{6}}$

e) $u = 4-x^2$

$\dfrac{du}{dx} = -2x$

$du = -2x\,dx$

$\therefore dx = \dfrac{du}{-2x}$

$\displaystyle\int x\sqrt{4-x^2}\,dx = \int x\cdot\sqrt{u}\cdot\frac{du}{-2x}$

$= -\dfrac{1}{2}\displaystyle\int u^{1/2}\,du$

$= -\dfrac{1}{2}\left[\dfrac{2u^{3/2}}{3}\right] + c$

$= \underline{\underline{-\dfrac{1}{3}\sqrt{(4-x^2)^3} + c}}$

Question 2

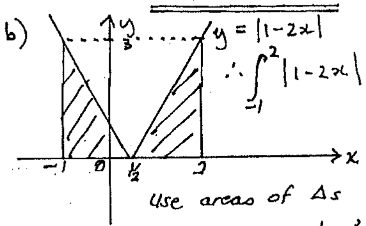
a) $\log_x 2\cdot25 = \log_x \dfrac{9}{4}$

$= \log_x 9 - \log_x 4$

$= 2\log_x 3 - 2\log_x 2$

$= \underline{\underline{2b - 2a}}$

b)



$\therefore \displaystyle\int_{-1}^{2} |1-2x|\,dx$

Use areas of Δs

$= \frac{1}{2}\times\frac{3}{2}\times 3 + \frac{1}{2}\times\frac{3}{2}\times 3$

$= \frac{9}{2} = 4\frac{1}{2}$ units²

c) i) $\displaystyle\int \frac{3x}{x^2+1}\,dx = \frac{3}{2}\int\frac{2x}{x^2+1}\,dx$

$= \underline{\underline{\dfrac{3}{2}\ln(x^2+1) + c}}$

ii) $\displaystyle\int \frac{3}{x^2+1}\,dx = 3\int\frac{1}{x^2+1}\,dx$

$= \underline{\underline{3\tan^{-1}x + c}}$

d) $\sin 2\theta = \sin\theta$

$2\sin\theta.\cos\theta = \sin\theta$

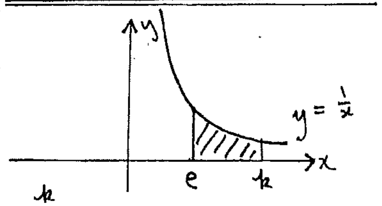
$2\sin\theta.\cos\theta - \sin\theta = 0$

$\sin\theta(2\cos\theta - 1) = 0$

$\sin\theta = 0 \qquad \cos\theta = \frac{1}{2}$

$\theta = 0, \pi, 2\pi \qquad \theta = \dfrac{\pi}{3}, \dfrac{5\pi}{3}$

e)



$\displaystyle\int_e^k \frac{1}{x}\,dx = 1$

$\left[\ln x\right]_e^k = 1$

$\ln k - \ln e = 1$

$\ln k - 1 = 1$

$\log_e k = 2$

$\underline{\underline{k = e^2}}$

∴ $\displaystyle\int \cos^2 3x\,dx = \frac{1}{2}\int(\cos 6x + 1)\,dx$

$= \frac{1}{2}\left[\frac{1}{6}\sin 6x + x\right] + c$

$= \underline{\underline{\dfrac{1}{12}\sin 6x + \dfrac{x}{2} + c}}$

b) i) $\tan 75° = \tan(30+45)°$

$= \dfrac{\tan 30° + \tan 45°}{1-\tan 30°.\tan 45°}$

$= \left(\dfrac{1}{\sqrt{3}}+1\right) \div \left(1-\dfrac{1}{\sqrt{3}}\right)$

$= \dfrac{1+\sqrt{3}}{\sqrt{3}} \times \dfrac{\sqrt{3}}{\sqrt{3}-1}$

$= \dfrac{1+\sqrt{3}}{\sqrt{3}-1} \times \dfrac{\sqrt{3}+1}{\sqrt{3}+1}$

$= \dfrac{2\sqrt{3}+4}{2}$

$= \underline{\underline{\sqrt{3}+2}}$

ii) gradients are m and $\frac{1}{\sqrt{3}}$

$\tan 75 = \dfrac{m - \frac{1}{\sqrt{3}}}{1+\frac{m}{\sqrt{3}}}$ (take +ve case only, one solution required)

$\sqrt{3}+2 = \dfrac{m-\frac{1}{\sqrt{3}}}{1+\frac{m}{\sqrt{3}}}$

$(\sqrt{3}+2)\left(1+\dfrac{m}{\sqrt{3}}\right) = m - \dfrac{1}{\sqrt{3}}$

$\sqrt{3}+m+2+\dfrac{2m}{\sqrt{3}} = m - \dfrac{1}{\sqrt{3}}$

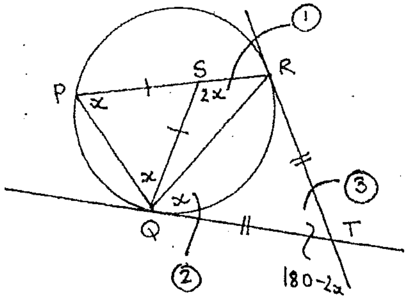
$\dfrac{2m}{\sqrt{3}} = -\dfrac{1}{\sqrt{3}} - \sqrt{3} - 2$

$2m = -1-3-2\sqrt{3}$

$\underline{\underline{m = -2-\sqrt{3}}}$

Question 3

c)



$R\hat{S}Q = 2x$ (ext. angle of isosceles triangle)

$R\hat{Q}T = x$ (alt. segment theorem)

since $QT = TR$ (tangents from a external pt are =)

$R\hat{T}Q = 180 - 2x$ (angle sum of isosceles triangle)

$R\hat{S}Q + R\hat{T}Q = 180°$

∴ QTRS is cyclic since opposite angles are supp.

d) i) $\frac{d}{dx}\left(\tan^{-1} e^{x} + \tan^{-1} e^{-x}\right)$

$= \frac{e^{x}}{e^{2x}+1} - \frac{e^{-x}}{e^{-2x}+1}$

$= \frac{e^{x}}{e^{2x}+1} - \left[\frac{1}{e^{x}} \div \left(\frac{1}{e^{2x}}+1\right)\right]$

$= \frac{e^{x}}{e^{2x}+1} - \left[\frac{1}{e^{x}} \times \frac{e^{2x}}{1+e^{2x}}\right]$

$= \frac{e^{x} - e^{x}}{1+e^{2x}} = 0 = RHS.$

ii) subst. $x = 0$ since true for all x

$\tan^{-1}(e^{0}) + \tan^{-1}(e^{0})$

$2\tan^{-1} 1 = 2 \times \frac{\pi}{4}$

$= \pi/2$

Question 4

a) $y = xe^{x}$ | $u = x$, $v = e^{x}$; $u' = 1$, $v' = e^{x}$

i) $\frac{dy}{dx} = e^{x} + xe^{x}$

$\frac{dy}{dx} = e^{x}(1+x)$

$\frac{d^2y}{dx^2} = e^{x} + e^{x}(1+x)$

$= e^{x}(1+1+x)$

$\therefore \frac{d^2y}{dx^2} = e^{x}(2+x)$

ii) Step 1: Show true for $n=1$

$\therefore \frac{dy}{dx} = e^{x}(x+1)$ from above

Step 2: Assume true for $n=k$ some +ve integer

$\frac{d^ky}{dx^k} = e^{x}(x+k)$

Step 3: Prove true for $n=k+1$

{ie show that $\frac{d^{(k+1)}y}{dx^{(k+1)}} = e^{x}(x+k+1)$}

$LHS = \frac{d^{(k+1)}y}{dx^{(k+1)}}$

$= \frac{d}{dx}\left(e^{x}(x+k)\right)$ from Step 2

$= \frac{d}{dx}\left(x.e^{x} + ke^{x}\right)$

$= e^{x} + xe^{x} + ke^{x}$

$= e^{x}(1+x+k)$

$= RHS$

Step 4: Since true for $n=1$ and if assumed true for $n=k$ (some +ve integer) we have shown true for $n=k+1$

∴ true for all +ve integers

b) $y = 2\sin^{-1}(1-4x)$

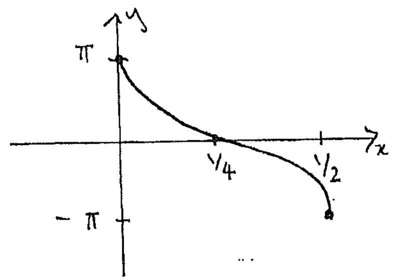
$\frac{y}{2} = \sin^{-1}(1-4x)$

$-\frac{\pi}{2} \leq \frac{y}{2} \leq \frac{\pi}{2}$

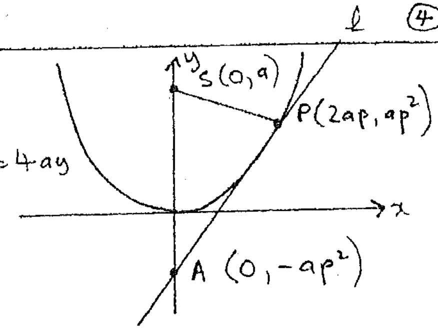
∴ Range: $-\pi \leq y \leq \pi$

$-1 \leq 1-4x \leq 1$

Domain: $0 \leq x \leq \frac{1}{2}$



c)



i) $y = \frac{x^2}{4a}$

$\frac{dy}{dx} = \frac{2x}{4a} = \frac{x}{2a}$

$m_p = \frac{2ap}{2a} = p$

tang: $y - ap^2 = p(x - 2ap)$

$y - ap^2 = px - 2ap$

$y = px - ap^2$

ii) $A(0, -ap^2)$

$SA = a + ap^2 = a(1+p^2)$

$SP = \sqrt{(2ap-0)^2 + (ap^2-a)^2}$

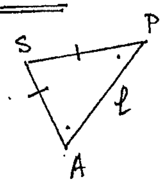
$= \sqrt{4a^2p^2 + a^2p^4 - 2a^2p^2 + a^2}$

$= a\sqrt{p^4 + 2p^2 + 1}$

$= a\sqrt{(p^2+1)^2}$

$SP = a(p^2+1)$

iii) ∴ $SP = SA$



△SPA is isosceles

∴ $S\hat{A}P = S\hat{P}A$

Question 5

a) i) $P(x) = (x-a)^2 \cdot Q(x)$

$u = (x-a)^2 \qquad v = Q(x)$

$u' = 2(x-a) \qquad v' = Q'(x)$

$P'(x) = 2(x-a) \cdot Q(x) + (x-a)^2 \cdot Q'(x)$

$P'(a) = 2(a-a)Q(a) + (a-a)^2 \cdot Q'(\ )$

$\therefore P'(a) = 0$

ii) $P(1) = 0$ and $P'(1) = 0$

$P(x) = mx^4 + nx^3 - 6x^2 + 22x - 12$

$P(1) = m + n - 6 + 22 - 12 = 0$

$m + n = -4$ —— ①

$P'(x) = 4mx^3 + 3nx^2 - 12x + 22$

$P'(1) = 4m + 3n - 12 + 22 = 0$

$4m + 3n = -10$ —— ②

① × 4 : $4m + 4m - 16$

② $4m + 3n = -10$

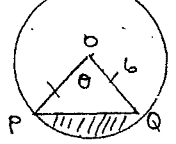
$n = -6$

$m - 6 = -4$

$m = 2$

$\therefore m = 2,\ n = -6$

b)



i) $A = \frac{1}{2} \cdot 6^2 (\theta - \sin\theta)$

$A = 18\theta - 18\sin\theta$

ii) $\frac{d\theta}{dt} = .75$ require $\frac{dA}{dt}$ when $\theta = \pi/2$

$\frac{dA}{d\theta} = 18 - 18\cos\theta$

$\therefore \frac{dA}{dt} = .75\left(18 - 18\cos\frac{\pi}{3}\right)$

$= .75\left(18 - 18 \cdot \frac{1}{2}\right)$

$\frac{dA}{dt} = 6.75\ \text{cm}^2/\text{second}$

c) i) $V = 6 + Ae^{-kt}$

$\therefore \frac{dv}{dt} = -k(Ae^{-kt})$

$= -k(v - 6)$

$= -kv + 6k$

$\therefore \frac{dv}{dt} = k(6 - v)$ as required

$V = 30$ m/s $t = 0$ subs.

$\therefore 30 = 6 + Ae^0$

$\therefore A = 24$

ii) $t = 1 \quad v = 10.7$

$V = 6 + 24e^{-kt}$

$10.7 = 6 + 24e^{-k}$

$4.7 = 24e^{-k}$

$\ln\left(\frac{4.7}{24}\right) = -k$

$k = 1.63$ (2 dec. pl)

iii) $t = 2$

$V = 6 + 24e^{-1.63 \times 2}$

$V = 6.9$ m/s (1 dec. pl)

Question 6

a) i) $\frac{d}{dx}\left(\frac{1}{2}v^2\right) = \frac{d}{dv}\left(\frac{1}{2}v^2\right) \cdot \frac{dv}{dx}$

$= v \cdot \frac{dv}{dx}$

$= \frac{dx}{dt} \cdot \frac{dv}{dx}$

$= \frac{dv}{dt}$

$= \ddot{x}$

ii) $\ddot{x} = \frac{4}{\sqrt{x}}$

$t = 0,\ \dot{x} = 0,\ x = 0$

$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = 4x^{-1/2}$

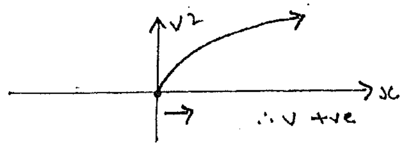
$\frac{1}{2}v^2 = \frac{4x^{1/2}}{1/2} + c_1$

$= 8\sqrt{x} + c_1$

$v^2 = 16\sqrt{x} + c$

sub $x = 0 \quad v = 0 \quad \therefore c = 0$

$v^2 = 16\sqrt{x}$



sub. $x = 7$

$v^2 = 16\sqrt{7}$

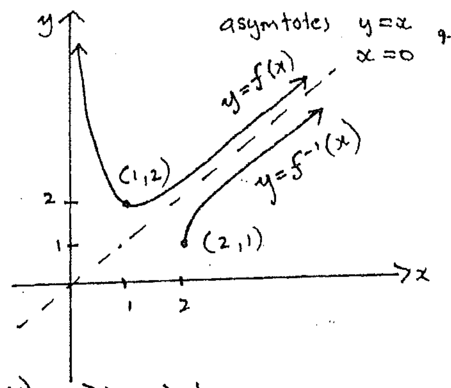
$v = 6.5$ m/s

b) i) $f(x) = x + x^{-1}$

$f'(x) = 1 - x^{-2}$

since $x > 0$ at $(1, 2)$ $f''(x) > 0$ min

asymptotes $y = x$, $x = 0$



ii) D: $x \geq 1$

iii) see diagram

iv) $x = y + \frac{1}{y}$

$xy = y^2 + 1$

$-1 = y^2 - xy$

$-4 = 4y^2 - 4xy$

$-4 + x^2 = 4y^2 - 4xy + x^2$

$-4 + x^2 = (2y - x)^2$

$\pm\sqrt{x^2 - 4} = 2y - x$

from domain above take $+\sqrt{\ }$

$2y = x + \sqrt{x^2 - 4}$

$\therefore y = \frac{x}{2} + \frac{1}{2}\sqrt{x^2 - 4}$

v) $f^{-1}[f(N)]$

N not in above domain

$\therefore$ use $f^{-1}(x) = \frac{x}{2} - \frac{1}{2}\sqrt{x^2-4}$

$f(N) = N + \frac{1}{N}$

$f^{-1}(f(N)) = \frac{1}{2}\left(N+\frac{1}{N}\right) - \frac{1}{2}\sqrt{\left(N+\frac{1}{N}\right)^2 - 4}$

$= \frac{N}{2} + \frac{1}{2N} - \frac{1}{2}\sqrt{N^2+2+\frac{1}{N^2}-4}$

$= \frac{N}{2} + \frac{1}{2N} - \frac{1}{2}\sqrt{N^2-2+\frac{1}{N^2}}$

$= \frac{N}{2} + \frac{1}{2N} - \frac{1}{2}\sqrt{\left(N-\frac{1}{N}\right)^2}$

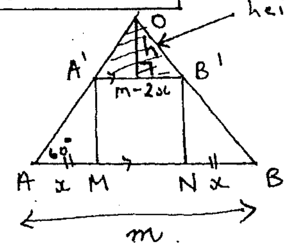
N.B. maintain the -ve here

$= \frac{N}{2} + \frac{1}{2N} - \frac{1}{2}\left(N - \frac{1}{N}\right)$

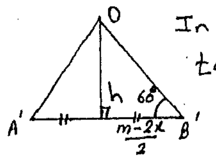
$= \frac{N}{2} + \frac{1}{2N} - \frac{N}{2} + \frac{1}{2N}$

$f^{-1}(f(N)) = \frac{1}{N}$

Question 7



height of $\Delta OA'B'$



In Δ OA'B'

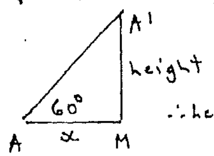
$\tan 60° = \frac{h}{\frac{m-2x}{2}}$

$\frac{\sqrt{3}(m-2x)}{2} = h$

$\therefore$ Area of $OA'B' = \frac{1}{2}(m-2x)\frac{\sqrt{3}}{2}(m-2x)$

$A = \frac{\sqrt{3}}{4}(m-2x)^2$

ii) V = Abase × height



$\therefore$ height $= x\tan 60°$

$= x\sqrt{3}$

$V = 6\left(\frac{\sqrt{3}}{4}(m-2x)^2\right) \cdot x\sqrt{3}$

$V = \frac{9}{2}x(m-2x)^2$

iii) α) $u = \frac{9x}{2}$ $\quad v = (m-2x)^2$

$u' = \frac{9}{2}$ $\quad v' = 2 \cdot -2(m-2x)$

$v' = -4(m-2x)$

using product rule

$V' = \frac{9}{2}(m-2x)^2 - 18x(m-2x)$

$V' = 9(m-2x)\left[\frac{1}{2}(m-2x) - 2x\right]$

$V' = 9(m-2x)\left(\frac{m}{2} - 3x\right)$

st pt $V' = 0$

$\therefore$ $x = \frac{m}{2}$ OR $x = \frac{m}{6}$

test max/min

x	m/8	m/6	m/4	m/2	m
V'	+	0	−	0	+

+/ max \ − _o_/ +

β) sub $x = \frac{m}{6}$ into V

$V = \frac{9}{2} \cdot \frac{m}{6} \cdot \left(m - \frac{2m}{6}\right)^2$

$= \frac{3m}{4}\left(\frac{2m}{3}\right)^2$

$= \frac{3m}{4} \cdot \frac{4m^2}{9}$

max. $V = \frac{m^3}{3}$ unit^3

b) i) different roots if $\Delta > 0$

$\tan 2x = \frac{\tan x}{a\tan x + b}$

$\frac{2\tan x}{1-\tan^2 x} = \frac{\tan x}{a\tan x + b}$

$2\tan x(a\tan x + b) = \tan x(1-\tan^2 x)$

since $\tan x \neq 0$ given

$2(a\tan x + b) = 1 - \tan^2 x$

$2a\tan x + 2b = 1 - \tan^2 x$

$\tan^2 x + 2a\tan x + 2b - 1 = 0$

$\Delta = (2a)^2 - 4 \cdot 1 \cdot (2b-1)$

require $\Delta > 0$

$4a^2 - 8b + 4 > 0$

$a^2 - 2b + 1 > 0$

ii) LHS $= \tan^2(\alpha - \beta)$

$= [\tan(\alpha-\beta)]^2$

$= \left(\frac{\tan\alpha - \tan\beta}{1 + \tan\alpha \cdot \tan\beta}\right)^2$

$= \frac{\tan^2\alpha + \tan^2\beta - 2\tan\alpha \cdot \tan\beta}{(1 + \tan\alpha \cdot \tan\beta)^2}$

(since $A^2 + B^2 = (A+B)^2 - 2AB$)

$= \frac{(\tan\alpha + \tan\beta)^2 - 4\tan\alpha \cdot \tan\beta}{(1 + \tan\alpha \cdot \tan\beta)^2}$

$= \frac{(\text{sum of roots})^2 - 4 \times \text{product of roots}}{(1 + \text{product of roots})^2}$

$= \frac{(-2a)^2 - 4(2b-1)}{(1 + (2b-1))^2}$

$= \frac{4a^2 - 8b + 4}{4b^2}$

$= \frac{a^2 - 2b + 1}{b^2}$