Name: .... Maths Class: ....

SYDNEY TECHNICAL HIGH SCHOOL

Year 11

Extension 2 Mathematics

Assessment 1
HSC Course

December, 2014

Time allowed: 70 minutes

General Instructions:

- Questions are not of equal value
- Approved calculators may be used
- All necessary working should be shown
- Begin each question on a new page
- Write using black or blue pen
- Full marks may not be awarded for careless work or illegible writing

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE: $\ln x = \log_e x, \quad x > 0$

QUESTION 1: (12 Marks)

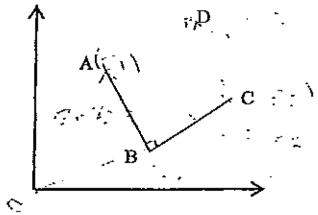
(a) Given that $z = 2 + i$ and $w = 1 - i$ find, in the form $a + ib$ **2**

(i) $z\overline{w}$ (ii) z/w

(b) Shade the area where $|z - 1 - i| < 2$ and $0 < \arg(z - 1 - i) < \pi/4$ hold simultaneously. **3**

(c) In the diagram below, A, B and C are the points representing the complex numbers z_1, z_2, and z_3 respectively.

<ABC is a right angle, and AB = BC



(i) Explain why $(z_1 - z_2)^2 = -(z_3 - z_2)^2$ **2**

(ii) If ABCD is a square, find, in terms of z_1, z_2, and z_3 the complex number represented by the point D **2**

(d) Find the values of x and y if $(x + iy)^2 = 24 + 10i$ **3**

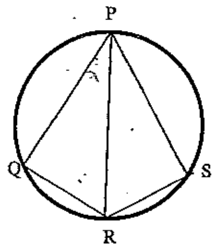
QUESTION 2: (11 Marks)

(a) (i) Show that $z\bar{z} = |z|^2$ **1**

(ii) If $z = 2(\cos\theta + i\sin\theta)$ find $\overline{1 - z}$ in terms of θ **1**

(iii) Show that $Re\left(\frac{1}{1-z}\right) = \frac{1-2\cos\theta}{5-4\cos\theta}$ **3**

(b) In the circle below, <QPR = < SPR and <QRP = <SRP.
It is NOT <u>given</u> that PR is a diameter.



Showing all working and reasoning, prove that PR is a diameter. **3**

(c) If $x = 1 + i\sqrt{3}$ find the value of x^{11} in the form $a + ib$ **3**

QUESTION 3: (10 Marks)

(a) (i) If $\alpha = -1 + i$, express α in mod-arg form. **2**

(ii) Show that α satisfies the complex equation $z^4 + 4 = 0$ **2**

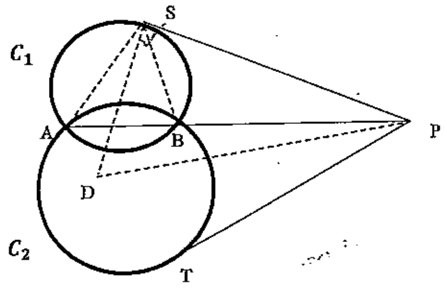
(iii) Hence, or otherwise, factorise $z^4 + 4$ into two Real quadratic factors **2**

(b) If n is an even positive integer, show, without multiplying out, that **4**

$$(1 + x + x^2 + \cdots x^n) \times (1 - x + x^2 - \cdots x^n) = (1 + x^2 + \cdots x^{2n})$$

QUESTION 4: (11 Marks)

(a) Two circles C_1 and C_2 intersect in the points A and B



AB is produced to the point P.

From the point P the tangents PT and PS are drawn as shown.

Redraw the diagram onto your answer sheet **(no marks)**

(i) Prove that ΔASP is similar to ΔSBP **2**

(ii) Hence prove that $SP^2 = AP \times PB$ **(DO NOT USE THE INTERCEPT THEOREM)** **1**

(iii) Deduce that $PT = PS$ **1**

(iv) The perpendicular from S to SP meets the bisector of $< SPT$ at D. **3**

Prove that DT passes through the centre of the circle C_2

(b) (i) How many terms are there in the geometric series **1**

$2^N + 2^{N-1} + 2^{N-2} \ldots + 2^{1-N} + 2^{-N}$, if N is a positive integer?

(ii) Prove that **2**

$2^N + 2^{N-1} + 2^{N-2} \ldots + 2^{1-N} + 2^{-N} = 2^{N+1} - 2^{-N}$

(iii) As $N \to \infty$, is there a limiting sum to this series? Justify your answer. **1**

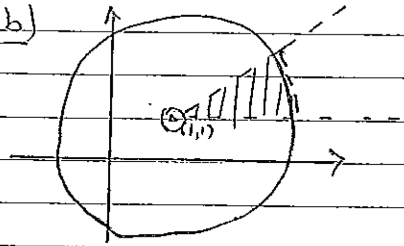
4 UNIT SOLUTIONS

1 (a)(i) $(2+i)(1+i) = 1+3i$

(ii) $z/w = \dfrac{z\bar{w}}{|w|^2}$

$= \dfrac{1+3i}{2}$ ← from part (i)

(b)



(c)(i) CB is $z_3 - z_2$

AB is $z_1 - z_2$

AB is CB rotated by $90°$

$\therefore z_1 - z_2 = i(z_3 - z_2)$

$\therefore (z_1 - z_2)^2 = -(z_3 - z_2)^2$

(ii) D is $z_1 + (z_3 - z_2)$

(d) $x^2 - y^2 = 24$ and $2xy = 10$

$\Rightarrow y = 5/x.$

$\therefore x^2 - 25/x^2 = 24$

$\therefore x^4 - 24x^2 - 25 = 0$

$(x^2 - 25)(x^2 + 1) = 0$

Taking only the result $x = 5,\ y = 1$

$\therefore \begin{cases} x = 5 \\ y = 1 \end{cases}$ OR $\begin{cases} x = -5 \\ y = -1 \end{cases}$

2 (i) Let $z = x + iy$, $z\bar{z} = (x+iy)(x-iy)$

$= x^2 + y^2$

$= |z|^2$

(ii) $z = 2\cos\theta + 2i\sin\theta$

$\therefore 1 - z = 1 - 2\cos\theta - 2i\sin\theta$

and $\overline{1-z} = 1 - 2\cos\theta + 2i\sin\theta$.

(iii) $Re\left(\dfrac{1}{1-z}\right) = Re\left[\dfrac{\overline{1-z}}{|1-z|^2}\right]$

$= \dfrac{1 - 2\cos\theta}{(1-2\cos\theta)^2 + 4\sin^2\theta}$

$= \dfrac{1 - 2\cos\theta}{5 - 4\cos\theta}$

(b) ONE METHOD: Let $\angle QPR = \angle SPR = x$

and $\angle QRP = \angle SRP = y$

$\therefore \angle PQR = 180° - (x+y)$ (angle sum of $\triangle PQR$)

and $\angle PSR = 180 - (x+y)$ " " " $\triangle PSR$)

ALSO $\angle PQR + \angle PSR = 180°$ (opposite angles in a cyclic quadrilateral)

$\therefore 180° - (x+y) + 180 - (x+y) = 180$

$\Rightarrow x + y = 90°$

$\therefore \angle PQR = 90°$ (angle sum of $\triangle PQR$)

$\therefore$ PR is a diameter (angle in a s/circle is $90°$)

(c) $x = 2\,\text{cis}\,\pi/3$

$x^{11} = 2^{11}\,\text{cis}\,11\pi/3$ and $11\pi/3 = -\pi/3$

$\therefore x^{11} = 2^{11}[\cos(-\pi/3) + i\sin(-\pi/3)]$

$= 2^{11}(1/2 - \sqrt{3}/2\, i)$

$= 1024 - 1024\sqrt{3}\, i$

3 (a)(i) $\alpha = \sqrt{2}\,\text{cis}(3\pi/4)$

(ii) $\alpha^4 + 4 = 4\,\text{cis}\,3\pi + 4$

$= -4 + 4$

$= 0$

(iii) If $-1+i$ is a factor, so is $-1-i$

$\therefore$ one factor is $(z+1-i)(z+1+i) = z^2 + 2z + 2$

∴ the other is z^2-2z+2 (inspection)

(b) $S_1 = \frac{1(1-x^{n+1})}{1-x}$ $\quad S_2 = \frac{1-(-x)^{n+1}}{1+x}$

$= \frac{1-x^{n+1}}{1-x}$ $\quad = \frac{1+x^{n+1}}{1+x}$ since n is even

∴ and $S_3 = \frac{1-(x^2)^{n+1}}{1-x^2}$

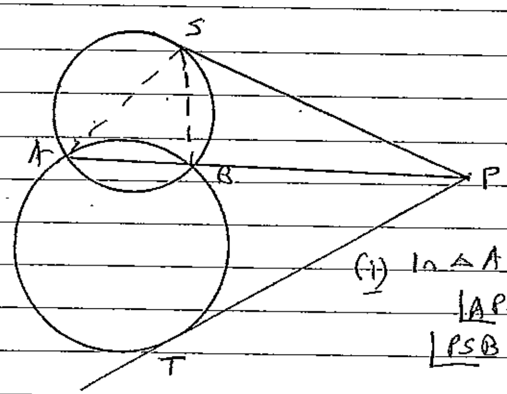
$= \frac{1-x^{2n+2}}{1-x^2}$

$S_1 \times S_2 = \frac{1-(x^{n+1})^2}{1-x^2}$

$= \frac{1-x^{2n+2}}{1-x^2} = S_3$.

4

(a)



(i) In $\triangle ASP$ and $\triangle SBP$

$\angle APS$ is common

$\angle PSB = \angle SAP$ (alternate angle theorem)

∴ $\triangle ASP \mathrel{|||} \triangle SBP$ (equiangular)

(ii) ∴ Corresponding sides are in ratio

∴ $SP/BP = AP/SP$

$\Rightarrow SP^2 = BP.AP$.

(iii) Similarly in $\triangle ATP$ and $\triangle TBP$

$PT^2 = AP.TB$.

OR By intercept theorem in bottom circle,

$PT^2 = AP.PB$.

Since $PS^2 = AP.TB$ (part (ii))

∴ $PT = PS$

(iv) In $\triangle SPD$ and $\triangle TPD$,

$PS = PT$ (proven above)

$\angle SPD = \angle TPD$ (PD bisects $\angle SPT$)

PD is common,

∴ $\triangle SPD \equiv \triangle TPD$ (SAS)

∴ $\angle PSD = \angle PTD$ (corresponding angles in congruent tria

$= 90°$

∴ DT strikes a tangent at $90°$, So DT is part of a

∴ D is the centre.

(b) (i) $a = 2^N$, $r = 2^{-1}$, $ar^{n-1} = 2^N(2^{-1})^{n-1}$

∴ $2^{-N} = 2^{N-n+1}$

∴ $-2N = -n+1$

∴ $n = 2N+1$.

(ii) $S_N = \frac{2^N\left[1-(2^{-1})^{2N+1}\right]}{1/2}$

$= 2^{N+1}\left[1-2^{-1-2N}\right]$

$= 2^{N+1} - 2^{-N}$.

(iii) as $N \to \infty$ $2^{N+1} \to \infty$ (while $2^{-N} \to 0$)

So there is no limiting sum (goes to infinity)