Name: .... Maths Class: ....

SYDNEY TECHNICAL HIGH SCHOOL

Year 12

Mathematics Extension 1

HSC Course

Assessment 2

March, 2015

Time allowed: 70 minutes

General Instructions:

- Marks for each question are indicated on the question.
- Approved calculators may be used
- All necessary working should be shown.
- Full marks may not be awarded for careless work or illegible writing
- ***Begin each question on a new page***
- Write using black or blue pen
- All answers are to be in the writing booklet provided
- A set of Standard Integrals is provided at the rear of this Question Booklet, and may be removed at any time.

Section 1	Multiple Choice Questions 1-5 5 Marks	
Section 2	Questions 6-11 48 Marks	

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1},\quad n\neq -1;\quad x\neq 0,\text{ if } n<0$$

$$\int \frac{1}{x}\,dx = \ln x,\quad x>0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax},\quad a\neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax,\quad a\neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax,\quad a\neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax,\quad a\neq 0$$

$$\int \sec ax\tan ax\,dx = \frac{1}{a}\sec ax,\quad a\neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a},\quad a\neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a},\quad a>0,\ -a<x<a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right),\quad x>a>0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE: $\ln x = \log_e x,\quad x>0$

SECTION 1: MULTIPLE CHOICE (5 Marks)

Write your answers on the Multiple Choice Answer Sheet, included in your answer booklet.
All questions are worth 1 mark

1 Joan pays back her bank loan of \$105 000 in 10 years with equal monthly payments of \$1 500.

Her equivalent simple interest charge would be:

A. 4.17% p.a. B. 5.83% p.a. C. 7.14% p.a. D. 17.14% p.a.

2 The curve $y = x^4 - 2x^3 - 12x^2 + 12x - 2$ is concave up for:

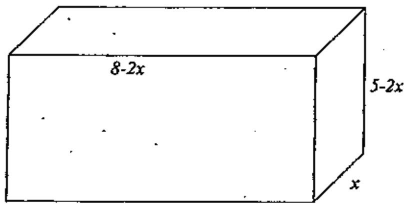
A. $-1 < x < 2$ B. $x < -1$ C. $x > 2$ D. $x < -1$ or $x > 2$

3 When the area between the curve $y = \sqrt{x(x^2 - 9)}$ and the x-axis is revolved about the x-axis, the volume of the solid formed is given by:

A. $\pi \int_0^3 x(x^2 - 9)dx$ B. $\pi \int_0^3 x^2(x^2 - 9)^2 dx$

C. $\pi \int_{-3}^3 x(x^2 - 9)dx$ D. $\pi \int_{-3}^3 x^2(x^2 - 9)^2 dx$

4



A box measures x cm by (5-2x) cm by (8 – 2x) cm.
The maximum volume of this box occurs when

A. $x = 1$ B. $x = {}^{10}/_3$ C. $x = 1$ or $x = {}^{10}/_3$ D. $x = 2.5$

5 The value of $\int_{-4}^{4} \sqrt{16 - x^2}\,dx$ is:

A. 8π B. 16π C. $2\sqrt{2}\pi$ D. 0

SECTION 2

(START EACH QUESTION ON A NEW PAGE OF YOUR ANSWER BOOKLET)

QUESTION 6: (8 Marks)

Marks

(a) Find indefinite integrals of: 2

(i) $(x + {}^1/_x)^2$ (ii) $\frac{5}{\sqrt{x}}$

(b) Find the value of $\int_1^{32} \frac{4}{x^{1.4}}\,dx$ 2

(c) Prove, by the method of Mathematical Induction, that 4

$$1^3 + 2^3 + 3^3 + \dots + n^3 = \frac{n^2}{4}(n+1)^2 \quad \text{for all values of } n > 0.$$

QUESTION 7: (8 Marks) Start a New Page

Marks

(a) Differentiate $(1 + x^2)^3$ and hence find $\int x(1 + x^2)^2\,dx$ 2

(b) Using the Trapezoidal Rule with 3 function values, find an approximation, to 2 decimal places, for: 3

$$\int_1^5 \sqrt{25 - x^2}\,dx$$

(c) The area enclosed by the curves $y = x^3$ and $y^2 = 32x$ is rotated about the x-axis. 3

What is the exact volume of the solid formed?

QUESTION 8: (8 Marks) Start a New Page

Marks

(a) The power loss in a length of electrical wiring, in watts pere km, is given by the formula

$$L = C^2 r + {}^{5}/_{r}$$

Where C is the current (in amps) and r is the resistance (in ohms) .

(i) For a given current, C, what is the resistance required to give a minimum loss of power per kilometre? 3

(ii) What is the value of this loss? (in watts) 1

(b) Jeffrey would like to save $60 000 for a deposit on his first home. He decides to invest $3 000 from his monthly salary into a bank account which earns interest of 6% per annum, compounded monthly. Jeffrey intends to withdraw $M from this account at the end of each month, straight after the interest has been paid, for living expenses.

(i) Show that the amount in the account, following the withdrawal of the second set of living expenses is given by $3 030.08 – 2.005M 1

(ii) Calculate, showing all working, the value of M, to the nearest dollar, if Jeffrey is to reach his goal after 5 years. 3

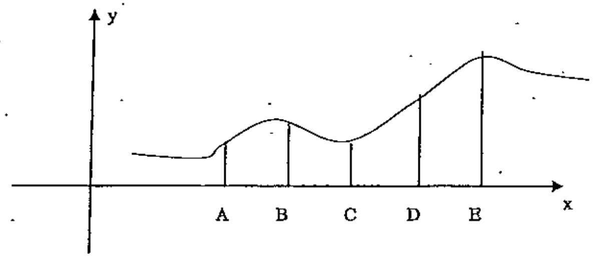
QUESTION 9: (8 Marks) Start a New Page

Marks

(a) Using the substitution $u = 2x^2 - 1$, or otherwise, find the value of 4

$$\int_1^3 \frac{x}{\sqrt{2x^2-1}}\,dx$$ correct to 2 decimal places.

(b) A vase is formed by rotating the area between the curve $y = g(x)$, shown below, the x-axis, and the lines $x = 1$ and $x = 5$. 4



A table of values for the curve, at the points where $x = 1, 2, 3, 4$ and 5 is given below

Point	A	B	C	D	E
x	1	2	3	4	5
$g(x)$	3	5	4	6	8

Using Simpson's Rule, with 5 function values, find the volume of the vase. (Give your answer in terms of π)

QUESTION 10: (8 Marks) Start a New Page

Marks

For the curve $y = \sqrt{x}(4 - x)$,

(i) Give any restrictions on the Domain of x — 1

(ii) Find $^{dy}/_{dx}$ — 1

(iii) Find all turning point(s) and their nature. — 3

(You may assume the result $\frac{d^2y}{dx^2} = \frac{-3x-4}{4x\sqrt{x}}$)

(iv) There are no inflexion points for this graph. *Explain why not.* — 1

(v) Using all of the information above, sketch $y = \sqrt{x}(4 - x)$ — 2

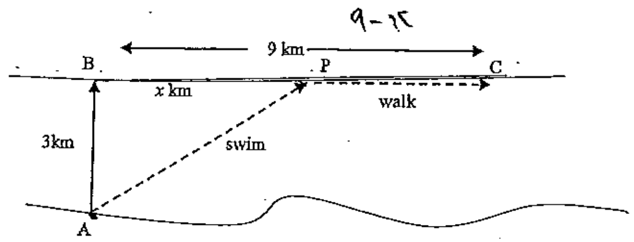
QUESTION 11: (8 Marks) Start a New Page

Marks

(a) B is a point across a river from a man at A, and is 3 km due North of A.

C is a position on the same side of the river as B, and 9 km due East of it.

The man at A intends to swim from A to a certain spot, P, on the opposite bank, where P is in a direct line from B to C, and x km from B.



(i) He can swim at 4 kph. Show that the time taken for him to swim to P from A is — 1

$$\frac{1}{4}\sqrt{x^2 + 9} \text{ hours.}$$

(ii) The man walks at a speed of 5 kph from P to C. — 1

Find the total time (T) for him to get from A to C, via P, by a combination of swimming and walking.

(iii) Calculate the value of x which will minimize the time it takes him to get from A to C — 3

(You must show all working)

(b) (i) Show that $1 - t + t^2 - \frac{t^3}{1+t} = \frac{1}{1+t}$ — 1

(ii) Prove that, for $t > 0$ and $x > 0$, — 2

$$\int_0^x \frac{dt}{1+t} < x - \frac{x^2}{2} + \frac{x^3}{3}$$

END OF THE EXAMINATION

MATHEMATICS EXTENSION 1

YEAR 12 2015

MULTIPLE CHOICE

1. C 2. D 3. A 4 A 5 A.

SECTION 2

6

(a)(i) $\int \left(x^2 + 1/x^2 + 2\right) dx = \frac{1}{3}x^3 - 1/x + 2x + k$

(ii) $\int 5/\sqrt{x}\, dx = \begin{cases} 10x^{1/2} + k \\ 10\sqrt{x} + k \end{cases}$

(b) $\int_1^{32} 4x^{-14/10}\, dx = -10x^{-4/10}\Big]_1^{32}$

$= -10x^{-2/5}\Big]_1^{32}$

$= -10\left(\frac{1}{4}\right) + 10(1)$

$= 7\frac{1}{2}$

(c) For $n=1$, LHS $= 1$, RHS $= 1$

∴ true for $n=1$

Assume the formula is true for $n=k$

ie $1^3 + 2^3 + \dots + k^3 = \frac{k^2}{4}(k+1)^2$

For $n=k+1$

$1^3 + 2^3 + \dots + k^3 + (k+1)^3 = \frac{k^2}{4}(k+1)^2 + (k+1)^3$

$= \frac{(k+1)^2}{4}\left[k^2 + 4(k+1)\right]$

$= \frac{(k+1)^2}{4}(k+2)^2$

which is of the same form.

∴ If the formula is true for $n=k$, it is true for $n=k+1$.

BUT it is true for $n=1$

∴ " " " " $n=2$ and so on

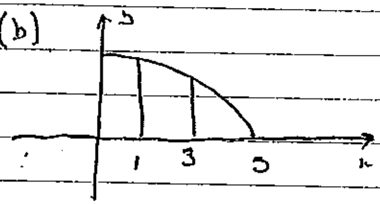
ie true $\forall\, n$

7

(a) $\frac{d}{dx}\left(1+x^2\right)^3 = 6x\left(1+x^2\right)^2$

∴ $\int x\left(1+x^2\right)^2 dx = \frac{1}{6}\left(1+x^2\right)^3 + k$.

(b)



$T_1 = \frac{1}{2} \times 2 \times \left[\sqrt{24} + 4\right]$

$T_2 = \frac{1}{2} \times 2 \times \left[4 + 0\right]$

∴ $A = \begin{cases} \sqrt{24} + 8 \\ 2\sqrt{6} + 8 \end{cases}$ OR $\begin{cases} 12.90 \\ 12.89 \text{ (to 2dp)} \end{cases}$

(c) The curves intersect at $x^6 = 32x$

∴ $x(x^5 - 32) = 0$

$\Rightarrow \begin{cases} x = 0 \\ y = 0 \end{cases}$ OR $\begin{cases} x = 2 \\ y = 8 \end{cases}$

∴ $Vol_1 = \int_0^2 x^6\, dx$ $= \frac{1}{7}x^7\Big]_0^2$ $= \frac{128}{7}$

$Vol_2 = \int_0^2 32x\, dx$ $= 16x^2\Big]_0^2$ $= 64$

∴ $Vol_{req} = \frac{448 - 128}{7}$

$= \begin{cases} \frac{320}{7} \text{ cu units} \\ 45\frac{5}{7} \text{ cu units} \\ 45.71 \text{ cu units} \end{cases}$

(8) (a) $dL/dr = c^2 - 5/r^2$

(i) $d^2L/dr^2 = 10/r^3$

At MIN $dL/dr = 0$

$\therefore c^2 = 5/r^2$

$\therefore \begin{cases} r = \sqrt{5}/c \\ L'' > 0 \Rightarrow \text{MIN.} \end{cases}$

$\therefore$ MIN loss occurs when $r = \sqrt{5}/c$ ohms

(ii) $L = c^2\left(\sqrt{5}/c\right) + 5/(\sqrt{5}/c) = 2c\sqrt{5}$ watts/km.

THIS NEXT QUESTION CONTAINED SERIOUS ERRORS. IT COULD HAVE BEEN INTERPRETED IN 2 WAYS, WITH ONE WAY LEADING TO THE ANSWER REQUIRED IN PART (i). THIS WAY WAS MARKED AS CORRECT. THIS THEN LED TO A PROBLEM IN PART (ii). THE CORRECT QUESTION AND ANSWERS FOR BOTH PARTS IS ATTACHED. HOWEVER, FOR PART (ii), HAVING BEEN "LED ASTRAY" BY PART (i), MARKING WAS BASED ON BEING CONSISTENT WITH THE WORKING, FINDING THE CORRECT G.P.'s and USING $S_{60} = 60000$. This method led to a strange answer of $-\$801$ which was paid full marks. (3)

The other way, using $S_{60} = 0$ is clearly wrong, but led to an answer of \$58 which earned 2 MARKS

[No person was penalised if they showed consistency - except for 1 mark for $A_{60} = 0$]

(9)

(a) $u = 2x^2 - 1$

x	1	3
u	1	17.

$\frac{du}{dx} = 4x$

$du = 4x\,dx$

(1) $\longrightarrow$ $dx = du/4x$.

$$\therefore \int_1^3 \frac{x}{\sqrt{2x^2-1}}\,dx = \int_1^{17} \frac{x}{\sqrt{u}} \cdot \frac{1}{4x}\,du$$

$$= \int_1^{17} \frac{du}{4\sqrt{u}}$$

$$= \left[\tfrac{1}{4} \cdot 2u^{1/2}\right]_1^{17}$$

$$= \tfrac{1}{2}\left(\sqrt{17} - 1\right)$$

(1) ≈ 1.56

(b)	A	B	C	D	E
x	1	2	3	4	5
$g(x)$	3	5	4	6	8
$g(x)^2$	9	25	16	36	64

$VOL_1 = \pi/3 \cdot 1\,[9 + 16 + 100]$ $\quad VOL_2 = \pi/3 \times 1\,[16 + 64 + 144]$

$VOL = \pi/3\,[349]$

$= 349\pi/3$

(10)

(i) $x \geq 0$

(ii) $\frac{dy}{dx} = 2x^{-1/2} - \frac{3}{2}x^{1/2}$

$= \frac{1}{2}x^{-1/2}\left[4 - 3x\right]$

$= \frac{4-3x}{2\sqrt{x}}$

(iii) At S.P.s $\frac{dy}{dx} = 0$

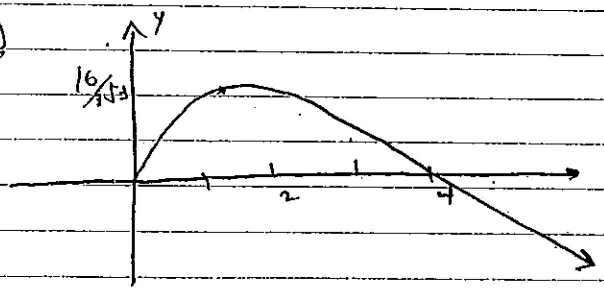
$\therefore \begin{cases} x = 4/3 \\ y = \sqrt{4/3}\left(4 - 4/3\right) \end{cases}$

$= \frac{16}{3\sqrt{3}}$.

est $y'' < 0 \Rightarrow$ max T.P. at $\left(\frac{4}{3}, \frac{16}{3\sqrt{3}}\right)$.

(iv) Because for $y'' = 0$, $x = -4/3$ which is outside the Domain

(v)



(11) (a)(i) $AP = \sqrt{x^2+9}$

Time = Distance/speed

$= \frac{\sqrt{x^2+9}}{4}$

(ii) from P to C, time $= \frac{9-x}{5}$

$\therefore$ Time taken $= \frac{1}{4}\sqrt{x^2+9} + \frac{9-x}{5}$.

(iii) $\frac{dT}{dx} = \frac{1}{4}\cdot\frac{1}{2}\left(x^2+9\right)^{-1/2}\cdot 2x - \frac{1}{5}$

$= \frac{x}{4\sqrt{x^2+9}} - \frac{1}{5}$

$\frac{d^2T}{dx^2} = \frac{1}{4}\left[\frac{(x^2+9)^{1/2}\cdot 1 - x\cdot\frac{1}{2}\cdot 2x\left(x^2+9\right)^{-1/2}}{(x^2+9)}\right]$

$= \frac{\frac{1}{4}(x^2+9)^{-1/2}}{x^2+9}\left[x^2+9-x\right]$

$= \frac{x^2-x+9}{4(x^2+9)\sqrt{x^2+9}}$

At S.P.s $\frac{dT}{dx} = 0$

$\therefore 5x = 4\sqrt{x^2+9}$

$25x^2 = 16x^2 + 144$

$9x^2 = 144$

$x^2 = 16$

$\therefore \begin{cases} x = 4 \quad \text{or} \quad x = -4 \\ T'' > 0 \end{cases}$ NOT A SOLUTION

$\Rightarrow$ MIN

$\therefore x$ is 4 km.

(b) (i) $1 - t + t^2 - \frac{t^3}{1+t}$

$= \frac{1+t - t - t^2 + t^2 + t^3 - t^3}{1+t}$

$= \frac{1}{1+t}$

(ii) $\int_0^x \frac{dt}{1+t} = \int_0^x dt - \int_0^x t\,dt + \int_0^x t^2\,dt$

$- \int_0^x \frac{t^3}{1+t}\,dt.$

$< \int_0^x dt - \int_0^x t\,dt + \int_0^x t^2\,dt$

$= \left[t - \frac{1}{2}t^2 + \frac{1}{3}t^3 \right]_0^x$

$= x - \frac{1}{2}x^2 + \frac{1}{3}x^3.$

(b) Jeffrey would like to save \$60 000 for a deposit on his first home. He decides to invest \$3 000 each month from his monthly salary into a bank account which earns interest of 6% per annum, compounded monthly. Jeffrey intends to withdraw \$M from this account at the end of each month, straight after the interest has been paid, for living expenses.

(i) Show that the amount in the account, following the withdrawal of the second set of living expenses is given by \$6 045.08 – 2.005M — 1

(ii) Calculate, showing all working, the value of M, to the nearest dollar, if Jeffrey is to reach his goal after 5 years. — 3

Correct working for Q(8) Part (b)

(i) $A_1 = 3000(1.005) - M.$

$A_2 = [3000(1.005) - M]1.005 + 3000(1.005) - M$

$= 3000(1.005)^2 + 3000(1.005) - M(1.005 + 1)$

$= 6{,}045.08 - 2.005M$

(ii) $A_n = 3000(1.005)^n + 3000(1.005)^{n-1} + \ldots - M(1 + 1.005 + \ldots + 1.005^{n-1})$

$= 3000[1.005^n + 1.005^{n-1} + \ldots + 1.005] - M[1 + 1.005 + \ldots + 1.005^{n-1}]$

And $A_{60} = 60000$

$\therefore 60000 = 3000(1.005)[1.005^{59} + \ldots + 1] - m[1 + 1.005 + \ldots + 1.005^{59}]$

$m\left[\frac{1.005^{60} - 1}{0.005}\right] = 3000(1.005)\left[\frac{1.005^{60} - 1}{0.005}\right] - 60000$

$m = 3000(1.005) - \frac{60000}{\frac{1.005^{60} - 1}{0.005}}$

$\approx 3015 - \frac{60000}{69.77}$

$= 3015 - 859.97$

≈ 2155.03