Name: .... Maths Class: ....

SYDNEY TECHNICAL HIGH SCHOOL

Year 12

Mathematics Extension 2

HSC Course

Assessment 2

March, 2015

Time allowed: 70 minutes

General Instructions:

- Marks for each question are indicated on the question.
- Approved calculators may be used
- All necessary working should be shown
- Full marks may not be awarded for careless work or illegible writing
- *Begin each question on a new page*
- Write using black or blue pen
- All answers are to be in the writing booklet provided
- A set of Standard Integrals is provided at the rear of this Question Booklet, and may be removed at any time.

Section 1	Multiple Choice Questions 1-5 5 Marks	
Section II	Questions 6-9 40 Marks	

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \ -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE: $\ln x = \log_e x, \quad x > 0$

Section I
5 marks
Attempt Questions 1-5

Use the multiple choice answer sheet for Questions 1 – 5.

1. A square root of $8+6i$ is :

(A) $3-i$ (B) $5-3i$

(C) $-3-i$ (D) $-3+i$

2. The equation of a curve is given by $x^2+xy+y^2=9$. Which of the following expressions will provide the value of $\frac{dy}{dx}$ at any point on the curve?

(A) $\frac{-2x-y}{2y}$ (B) $\frac{-2x-y}{x+2y}$

(C) $\frac{-2x+y}{2y}$ (D) $\frac{-2x+y}{x+2y}$

3. The equation of an hyperbola is given by $9x^2-4y^2=36$. The foci and the directrices of this hyperbola are:

(A) $(\pm\sqrt{13},0)$ and $x=\pm\frac{4\sqrt{13}}{13}$.

(B) $(0,\pm\sqrt{13})$ and $x=\pm\frac{4\sqrt{13}}{13}$.

(C) $(\pm\sqrt{13},0)$ and $y=\pm\frac{4\sqrt{13}}{13}$.

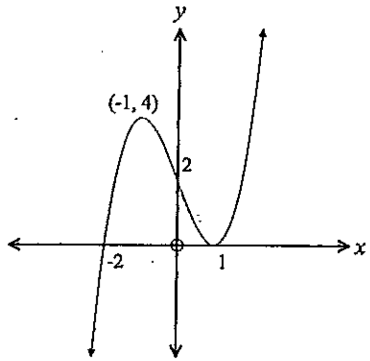
(D) $(0,\pm\sqrt{13})$ and $y=\pm\frac{4\sqrt{13}}{13}$.

4. The area bounded by the curves $y=x^2$ and $x=y^2$ is rotated about the x – axis. The volume of the solid of revolution formed in cubic units is:

(A) $\frac{9\pi}{70}$ (B) $\frac{3\pi}{10}$

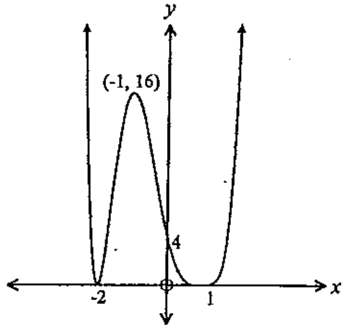
(C) $\frac{7\pi}{10}$ (D) $\frac{3\pi}{2}$

5. The graph of the function $y=f(x)$ is drawn below:

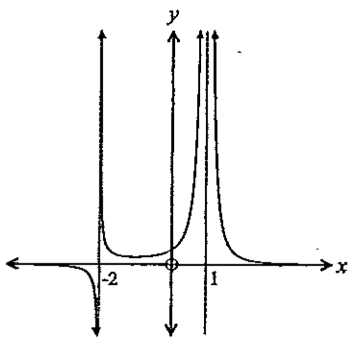


Which of the following graphs best represents the graph $y=\sqrt{f(x)}$?

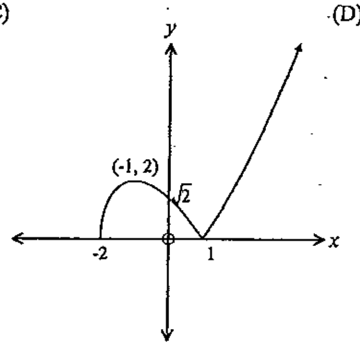
(A)



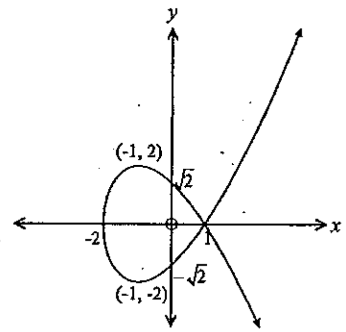
(B)



(C)



(D)



End of Section I

Section II

Total marks (40)
Attempt Questions 6 - 9

Question 6 (10 marks) **Marks**

a) An ellipse E has equation $\frac{x^2}{4} + \frac{y^2}{2} = 1$

(i) Show that the equation of E can be written in the parametric form 2

$$x = 2cos\theta, y = \sqrt{2}sin\theta$$

(ii) Assuming the perimeter of E is given by the formula 2

$$p = 2\int_0^{\pi} \sqrt{(\frac{dx}{d\theta})^2 + (\frac{dy}{d\theta})^2}\ d\theta\ ,$$

show that $p = 2\sqrt{2}\ \int_0^{\pi} \sqrt{2 - cos^2\theta}\ d\theta$

b) (i) If $w = \frac{1+i\sqrt{3}}{2}$ show that $w^3 = -1$ 1

(ii) Hence calculate w^{12} 1

(iii) Find all the cube roots of -1, both Real and Complex. 2

c) Given that one root of the equation $x^4 - 5x^3 + 5x^2 + 25x - 26 = 0$ is $3 + 2i$, solve the equation. 2

Question 7 (10 marks) Start a new page

a) If $f(x) = -x^2 + 7x - 10$, on separate diagrams and without using calculus, sketch the following graphs, indicating the intercepts with the axes and any asymptotes for each sketch:

(i) $y = f(x)$ 1

(ii) $y = |f(x)|$ 2

(iii) $y = \frac{1}{f(x)}$ 2

(iv) $y = -f(x+2)$ 2

b) Find all the roots of $18x^3 + 3x^2 - 28x + 12 = 0$, given that two roots are equal. 3

Question 8 (10 marks) Start a new page

a) $z_1 = 1 + i\sqrt{3}$ and $z_2 = 1 - i$ are two complex numbers 2

find $\frac{z_1}{z_2}$ in modulus-argument form

b) Given that the Argand Diagram for $|z - 2| + |z - 4| = 10$ is an ellipse,

(i) Find the co-ordinates of the centre of this ellipse and the lengths of the major and minor axes 3

(ii) On an Argand Diagram, show the region for which z satisfies the inequalities 3

$$z + \bar{z} \le 6 \text{ and } |z - 2| + |z - 4| \le 10$$

c) Find the perimeter of the shape in the Argand Diagram described by 2

$$|z - 1| \le 1 \quad \text{and} \quad 0 \le \arg z \le \frac{\pi}{6}$$

Question 9 (10 marks) Start a new page

a) Find the equation of the tangent to $\frac{x^2}{16}+\frac{y^2}{25}=1$ at the point $P(4\cos\theta, 5\sin\theta)$. 2

b) P $(2p, \frac{2}{p})$ is a variable point on the hyperbola $xy=4$.

The normal to the hyperbola at P meets the hyperbola again at Q $(2q, \frac{2}{q})$.

M is the midpoint of PQ.

(i) Show that the equation of the normal at P is given by $p^3x-py=2(p^4-1)$ 2

(ii) Show that $q=-\frac{1}{p^3}$ 1

(iii) Show that M has coordinates $[\frac{1}{p}\left(p^2-\frac{1}{p^2}\right),\ p\left(\frac{1}{p^2}-p^2\right)]$ 2

(iv) Show that, as P moves on the curve $xy=4$, the locus of M is given by 3

$$(x^2-y^2)^2=-x^3y^3$$

End of Examination

SOLUTIONS S.T.H.S. YR12 EXT 2. ASS. 2 MAR 2015

SECTION I

1. C 2. B 3. A 4. B 5. C

5×1 = 5 MARKS

SECTION II

6 (a) (i) $x = 2\cos\theta$ & $y = \sqrt{2}\sin\theta$

$x^2 = 4\cos^2\theta$ $y^2 = 2\sin^2\theta$

$\therefore \frac{x^2}{4} + \frac{y^2}{2} = \cos^2\theta + \sin^2\theta$

$= 1$ as req'd

$\therefore$ E can be written parametrically as

$x = 2\cos\theta$ $y = \sqrt{2}\sin\theta$ (2)

(ii) If $x = 2\cos\theta$ & if $y = \sqrt{2}\sin\theta$

$\frac{dx}{d\theta} = -2\sin\theta$ $\frac{dy}{d\theta} = \sqrt{2}\cos\theta$

$\therefore p = 2\int_0^{\pi}\sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\,d\theta$ becomes

$= 2\int_0^{\pi}\sqrt{4\sin^2\theta + 2\cos^2\theta}\,d\theta$

$= 2\int_0^{\pi}\sqrt{4(1-\cos^2\theta) + 2\cos^2\theta}\,d\theta$

$= 2\int_0^{\pi}\sqrt{4 - 2\cos^2\theta}\,d\theta$ (2)

$= 2\sqrt{2}\int_0^{\pi}\sqrt{2-\cos^2\theta}\,d\theta$ as req'd

(b) (i) $w = \frac{1}{2}(1+i\sqrt{3})$

$\therefore w^3 = \frac{1}{8}(1+i\sqrt{3})^3$

$= \frac{1}{8}(1+i\sqrt{3})(1+i\sqrt{3})^2$

$= \frac{1}{8}(1+i\sqrt{3})(-2+2\sqrt{3}i)$

$= -\frac{2}{8}(1+i\sqrt{3})(1-i\sqrt{3})$

$= -\frac{1}{4}\times 4$ (1)

$= -1$ as req'd

(ii) $w^{12} = (w^3)^4$

$= (-1)^4$ (1)

$= 1$

(iii) Cube roots of -1 are solutions of $w^3 = -1$

Let $w = \cos\theta + i\sin\theta$

$\therefore w^3 = \cos 3\theta + i\sin 3\theta$

Thus $\cos 3\theta + i\sin 3\theta = -1$ w $0 \le 3\theta \le 6\pi$

$\therefore 3\theta = \pi, 3\pi, 5\pi$

$\theta = \frac{\pi}{3}, \pi, \frac{5\pi}{3}$

Thus $w_1 = \text{cis}\,\frac{\pi}{3}$ or $\frac{1}{2} + \frac{i\sqrt{3}}{2}$ (as given)

$w_2 = \text{cis}\,\pi$ or -1 (real)

$w_3 = \text{cis}\,\frac{5\pi}{3}$ or $\frac{1}{2} - \frac{i\sqrt{3}}{2}$ (2)

(2)

c) If $3+2i$ is a root, then $3-2i$ is also a root

$\therefore (x-3-2i)(x-3+2i)$ is a factor

$\therefore (x-3)^2+4$ " " "

$\therefore x^2-6x+13$ " " "

Using long division

$$
\begin{array}{r}
x^2 + x - 2 \\
x^2-6x+13 \;\big)\; x^4-5x^3+5x^2+25x-26 \\
\underline{x^4-6x^3+13x^2} \\
x^3-8x^2+25x \\
\underline{x^3-6x^2+13x} \\
-2x^2+12x-26 \\
\underline{-2x^2+12x-26} \\
\cdot \quad \cdot \quad \cdot
\end{array}
$$

$\therefore P(x) = (x^2-6x+13)(x^2+x-2)$

$= (x-3-2i)(x-3+2i)\,(x+2)(x-1)$

$\therefore P(x) = 0$ has solutions

$3+2i,\ 3-2i,\ -1,\ 1$

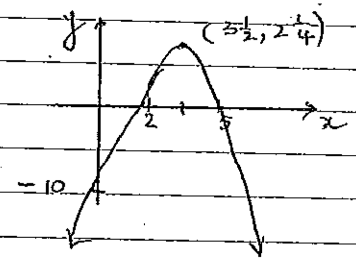
(2)

7. a) $f(x) = -x^2+7x+10$

$= -(x^2-7x+10)$

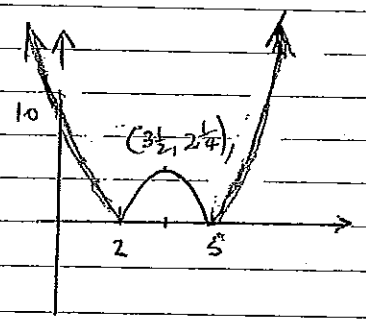
$= -(x-2)(x-5)$

(i) $y = f(x)$



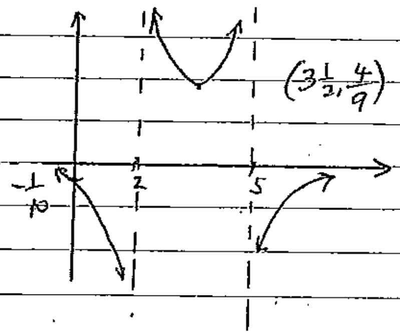
(1)

(ii) $y = |f(x)|$



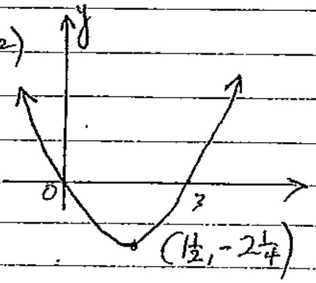
(2)

(iii) $y = \dfrac{1}{f(x)}$



(2)

(iv) $y = -f(x+2)$



(2)

(b) $P(x) = 18x^3 + 3x^2 - 28x + 12$

Solve $P'(x) = 54x^2 + 6x - 28 = 0$

$\therefore \quad 27x^2 + 3x - 14 = 0$

$\therefore x = \dfrac{-3 \pm \sqrt{9 + 4 \times 27 \times 14}}{54}$

$= \dfrac{-3 \pm 39}{54}$

$= \dfrac{36}{54}$ & $\dfrac{-42}{54}$

$= \dfrac{2}{3}$ & $\dfrac{-7}{9}$

So one of these is a repeated root of $P(x)$

$P\left(\frac{2}{3}\right) = 0$

$\therefore \quad x = \frac{2}{3}$ is a double root of $P(x)$

$\therefore (3x-2)^2$ is a factor

$\therefore 9x^2 - 12x + 4$ is a factor

$$
\begin{array}{r}
2x + 3 \\
9x^2 - 12x + 4 \overline{\big)\, 18x^3 + 3x^2 - 28x + 12} \\
18x^3 - 24x^2 + 8x \\
\hline
27x^2 - 36x + 12 \\
27x^2 - 36x + 12 \\
\hline
\end{array}
$$

$\therefore P(x) = (3x-2)^2(2x+3)$

which has roots

$\dfrac{2}{3}$ & $-\dfrac{3}{2}$ ONLY

3

Q8

a) $z_1 = 1 + i\sqrt{3}$ & $z_2 = 1 - i$

$= 2\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)$ $\qquad = \sqrt{2}\left(\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\right)$

$= 2\,\text{cis}\,\frac{\pi}{3}$ $\qquad = \sqrt{2}\,\text{cis}\left(-\frac{\pi}{4}\right)$

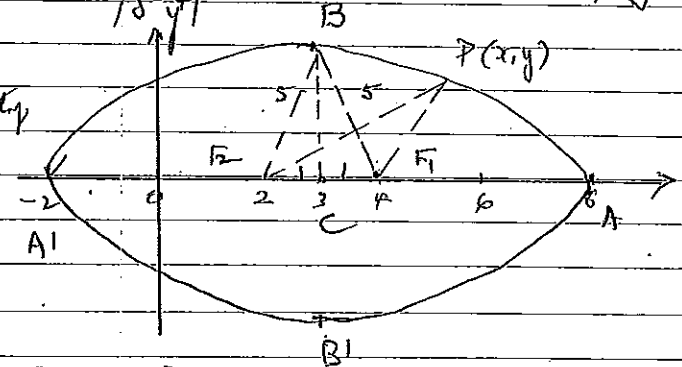
$\therefore \dfrac{z_1}{z_2} = \dfrac{2\,\text{cis}\,\frac{\pi}{3}}{\sqrt{2}\,\text{cis}\left(-\frac{\pi}{4}\right)}$

$= \sqrt{2}\,\text{cis}\left(\frac{7\pi}{12}\right)$ IN MOD-ARG FORM

b)(i) Given $|z-2|$ is distance from z to $x=2$ } in Argand

$|z-4|$ " " " " to $x=4$ } diagram

Using symmetry



Since P is on curve such that $F_1P + F_2P =$

$\therefore$ A must be $(8,0)$ A' must be $(-2,0)$

Length of major axis is $AA' = 10$ units

Centre is at C which must be $(5,0)$

$BC^2 + 1^2 = 5^2$

$\therefore BC = \sqrt{24}$

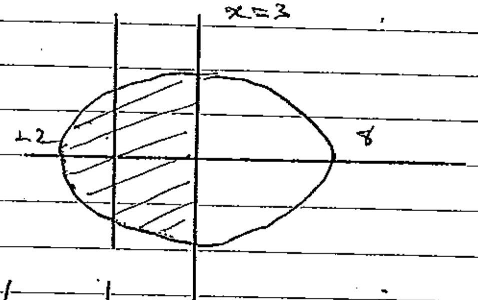
$= 2\sqrt{6}$

$\therefore$ Length of minor axis is $BB' = 4\sqrt{6}$

(ii) $z+\bar{z} \le 6$ & $|z-2|+|z-4| \le 10$

is $2x \le 6$ is region inside ellipse

$x \le 3$ above.

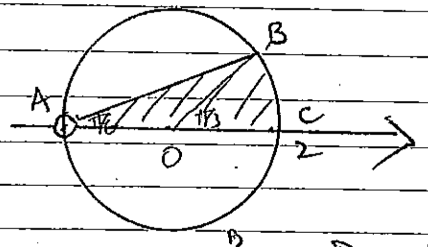


(3)

(c) $|z-1| \le 1$ is region inside circle centre $(1,0)$ radius 1

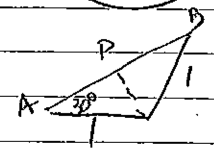
& $0 \le \arg z \le \pi/6$ as shown

[Actually $0 < \arg z \le \pi/6$]



Shape ABC

$\triangle ABD$



P is midpoint of AD

$AP = \cos 30° = \frac{\sqrt{3}}{2}$

$\therefore AB = 2 \times \frac{\sqrt{3}}{2} = \sqrt{3}$

(2)

Arc $BC = l = r\theta$

$= 1 \times \pi/3$

$= \frac{\pi}{3}$ $\therefore P = AB + \text{arc } BC + 2$

$= \sqrt{3} + \frac{\pi}{3} + 2$ units

P4.

9. a) $\frac{x^2}{16} + \frac{y^2}{25} = 1$ Diff. implicitly

$\frac{2x}{16} + \frac{2y}{25}\frac{dy}{dx} = 0$ $\therefore \frac{dy}{dx} = -\frac{25x}{16y}$

$\therefore m_T = \frac{-25}{16} \cdot \frac{4\cos\theta}{5\sin\theta}$

$= -\frac{5}{4}\frac{\cos\theta}{\sin\theta}$

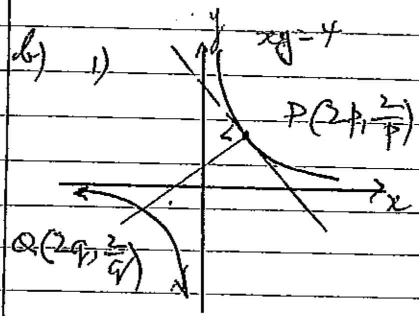
Eqn of tang is $y - 5\sin\theta = -\frac{5\cos\theta}{4\sin\theta}(x - 4\cos\theta)$

$4\sin\theta . y - 20\sin^2\theta = -5\cos\theta x + 20\cos^2\theta$

OR $\frac{\cos\theta}{4}x + \frac{\sin\theta}{5}y = 1$

(2)

b) 1)



$xy = 4$

$y = 4x^{-1}$

$y' = -\frac{4}{x^2}$

at P $m_T = -\frac{4}{(2p)^2} = -\frac{1}{p^2}$

$\therefore m_N = +p^2$

Eqn of NORMAL is

$y - \frac{2}{p} = p^2(x - 2p)$ — (1)

(1) × p $py - 2 = p^3x - 2p^4$

OR $p^3x - py = 2(p^4 - 1)$ as req'd

(2)

PS1

(ii) Reverting to ① & substituting $Q\left(2q, \frac{2}{q}\right)$

$$\frac{2}{q} - \frac{2}{p} = p^2(2q - 2p)$$

OR/ ALSO by $m_{PQ} = p^2$

$$\frac{1}{q} - \frac{1}{p} = p^2(q - p)$$

$$\frac{\cancel{(p-q)}}{pq} = -p^2\cancel{(p-q)}$$

& Noting $p \neq q$

$$\frac{1}{pq} = -p^2 \qquad (1)$$

OR $q = \frac{-1}{p^3}$ as req'd

(iii) Midpt of PQ is $M\left(\frac{2p+2q}{2}, \frac{\frac{2}{p}+\frac{2}{q}}{2}\right)$

or $\left(\frac{p+q}{1}, \frac{1}{p} + \frac{1}{q}\right)$

But $q = \frac{-1}{p^3}$ $\therefore M\left(p - \frac{1}{p^3}, \frac{1}{p} - p^3\right) \qquad (2)$

M is $\left[\frac{1}{p}\left(p^2 - \frac{1}{p^2}\right), p\left(\frac{1}{p^2} - p^2\right)\right]$

(iv) Checking $(x^2 - y^2)^2 = -x^3y^3$

$$LHS = \left[\frac{1}{p^2}\left(p^2 - \frac{1}{p^2}\right)^2 - p^2\left(\frac{1}{p^2} - p^2\right)^2\right]^2$$

$$= \left[\frac{1}{p^2}\left(\frac{1}{p^2} - p^2\right)^2 - p^2\left(\frac{1}{p^2} - p^2\right)^2\right]^2$$

$$= \left[\left(\frac{1}{p^2} - p^2\right)^2\left(\frac{1}{p^2} - p^2\right)\right]^2$$

$$= \left(\frac{1}{p^2} - p^2\right)^6$$

$$RHS = -\frac{1}{\cancel{p^3}}\left(p^2 - \frac{1}{p^2}\right)^3 \cdot \cancel{p^3}\left(\frac{1}{p^2} - p^2\right)^3$$

$$= +\left(\frac{1}{p^2} - p^2\right)^3\left(\frac{1}{p^2} - p^2\right)^3$$

$$= \left(\frac{1}{p^2} - p^2\right)^6$$

As $LHS = RHS$ we have confirmed Locus of M is

$$(x^2 - y^2)^2 = -x^3y^3 \qquad (3)$$