St Catherine's School

Year: 12
Subject: Extension 1I Mathematics
Time allowed: 2 hours plus 5 minutes reading time
Date: April 2006

Student number________________________

Directions to candidates:

- All questions are to be attempted.(Q.1 to Q.5)
- Marks may be deducted for careless or badly arranged work
- All necessary **working** must be shown
- Approved calculators may be used

Marks:

Q 1	
Q 2	
Q 3	
Q 4	
Q.5	
Total	

Extension II

Question 1.

(a) If α, β *and* γ are the roots of the equation $x^3 - 2x^2 - 7 = 0$

(i) Find the equation whose roots are α^2, β^2 *and* γ^2 1 **2**

(ii) Hence or otherwise evaluate $\alpha^3 + \beta^3 + \gamma^3$ 2 **2**

(b) Express $\dfrac{x^2 + x - 3}{x^2 + 3x + 2}$ as sum of partial fractions in the field of real numbers. 3 **3**

(c) Consider the ellipse E: $\dfrac{x^2}{9} + \dfrac{y^2}{4} = 1$.

(i) Show that the eccentricity is $\dfrac{\sqrt{5}}{3}$ 1 **1**

(ii) Find the coordinates of the foci, S and S' 1 **1**

(iii) Find the equations of the directrices 1 **1**

(iv) Sketch this ellipse, showing the above features 1 **1**

(v) Show clearly the position of θ for the point

$(3\cos\theta, 2\sin\theta)$ on the ellipse 0 **1**

(vi) Show that PS+PS$'$=6 for any point P on the ellipse. 2 **2**

(vii) Sketch the ellipse $\dfrac{x^2}{4} + \dfrac{y^2}{9} = 1$ highlighting its main features. 2 **2**

Question 2.

(a) (i) Sketch the locus of z: $|z-i|=\frac{1}{2}$ 1 1

(ii) What is the maximum value of $\arg z$ in this locus? 0 3

(b) (i) Show that eccentricity of the rectangular Hyperbola

$$x^2-y^2=8 \text{ is } \sqrt{2}$$

2 2

(ii) This Hyperbola is rotated by 45^0 and assumes the equation $xy=4$. Sketch this Hyperbola and find the coordinates of its foci

3 3

c) (i) Show that $\cos(p+q)+\cos(p-q)=2\cos p\cos q$, and deduce that

$$\cos\alpha+\cos\beta=2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$$

2 2

(ii) PQ is a focal chord of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, where P is $(a\cos\alpha, b\sin\alpha)$ and Q is $(a\cos\beta, b\sin\beta)$, show that

$$\cos\frac{\alpha-\beta}{2}=\pm\cos\frac{\alpha+\beta}{2}$$

0 2

and also that

$$PQ=2a\left(1-e^2\cos^2\frac{\alpha+\beta}{2}\right)$$

0 3

(You are given the equation of the chord PQ is $\frac{x}{a}\cos\left(\frac{\alpha+\beta}{2}\right)+\frac{y}{b}\cos\left(\frac{\alpha-\beta}{2}\right)=\cos\left(\frac{\alpha-\beta}{2}\right)$)

a ii)

c i)

ii)

Question 3

(a) Show that the sum of the infinite series

$$1+\cos x+\cos^2 x+\cos^3 x+\ldots\ldots\ldots=\frac{1}{2}\operatorname{cosec}^2 x \quad \text{, where } x \neq \pm n\pi$$

/ 1

(b) Given that $\alpha, \alpha^2, \alpha^3, \alpha^4, \alpha^5 \, and \; \alpha^6$ are the complex roots of $z^7=1$,
find the cubic equation whose roots are $(\alpha+\alpha^6),(\alpha^2+\alpha^5) and \; (\alpha^3+\alpha^4)$ 4 4

(c) Consider the Hyperbola $\frac{x^2}{16}-\frac{y^2}{9}=1$

(i) Show that the equation of the normal at P ($4\sec\theta, 3\tan\theta$)
is given by $4x\sin\theta+3y=25\tan\theta$ 3 3

This normal meets the x axis at G. PN is perpendicular to the x axis.

(ii) show that $\frac{OG}{ON}=\frac{25}{16}$ 3 3

(d) A sequence of numbers u_n is defined as follows. 5 5
$u_1=6,\ u_2=11 \ and \ u_{n+2}=2u_{n+1}-u_n+2.$

Show using the principle of Mathematical Induction that
$u_n=n^2+2n+3$ for all $n>0$

9?|

Question 4.

(a) State the domain of $y = \sin^{-1}\sqrt{x+1}$ 1 1

(b) (i) If $z = \cos\theta + i\sin\theta$, show using De Moivre's theorem or otherwise that $\dfrac{1}{z} = \cos\theta - i\sin\theta$ 1 1

(ii) Show that $z^n + \dfrac{1}{z^n} = 2\cos n\theta$ 2 2

(iii) Hence show that $\cos^5\theta = \dfrac{1}{16}(\cos 5\theta + 5\cos 3\theta + 10\cos\theta)$

(Consider the expansion of $(z + \dfrac{1}{z})^5$)

1 3

(c) Consider the hyperbola H: $xy=16$.

P $(4p, \dfrac{4}{p})$ and Q $(4q, \dfrac{4}{q})$ are two arbitrary points in H

(i) Show that the equation of the tangent at P is given by the equation $x + p^2y = 8p$ 2 2

(ii) The tangents at P and Q meet at T. Show that the coordinates of T are $(\dfrac{8pq}{p+q}, \dfrac{8}{p+q})$ 3 3

(iii) If the chord PQ passes through the point (0,8), show that $p+q=2pq$ 1 2

(iv) Find the equation of the locus of T 2

b iii)
c iii), iv)

Question 5.

(a) Given that $y = \sin^{-1}(\sin x)$,

(i) show that $\dfrac{dy}{dx} = \pm 1$ 1 1

(ii) Find the values of x for which $\dfrac{dy}{dx} = -1, 0 \le x \le 2\pi$ 0 2

(b) If $\tan\alpha$ *and* $\tan\beta$ are the roots of the equation $x^2 - ax + (1-a) = 0$,

show that $\alpha + \beta = n\pi + \dfrac{\pi}{4}$ 1/2 3

(c) Consider the polynomial $P(x) = x^3 - 6x^2 + 9x + k$, where k is a constant.

(i) Find the values of x for which $P'(x)=0$ 1 1

(ii) Show that the values of k for which $P(x)=0$ has a repeated root are 0 and -4. 3 3

(iii) Using the fact that, $x^3 - 6x^2 + 9x - 4 = x(x-3)^2 - 4$, sketch the graphs of $y=P(x)$ for these values of k on the same set of axes. 2 3

(iv) Using the sketches, state the values of k for which the polynomial equation $P(x)=0$ has only one root. 0 3

End of Paper

b), c) iv,

Extension II - Half yearly 2006

1) $P(x)$: $x^3 - 2x^2 - 7 = 0$ $P(\alpha) = 0$

(i) $P(\sqrt{x})$: $P(\sqrt{\alpha^2}) = P(\alpha) = 0$

$\alpha^2, \beta^2, \gamma^2$ are roots of $P(\sqrt{x}) = 0$

$(\sqrt{x})^3 - 2(\sqrt{x})^2 - 7 = 0$

$x\sqrt{x} = 2x + 7$

$x^3 = (2x+7)^2$

$\alpha^3 + \beta^3 + \gamma^3 = 2(\alpha^2+\beta^2+\gamma^2) + 21$

$\alpha^2+\beta^2+\gamma^2 = (\alpha+\beta+\gamma)^2 - 2\sum\alpha\beta$

$= 4 - 2.0$

$= 4$

$\therefore \alpha^3+\beta^3+\gamma^3 = 8 + 21$

$= 29$

b)

$$\begin{array}{r} 1 \\ x^2+3x+2 \,\big)\, x^2 + x - 3 \\ x^2 + 3x + 2 \\ \hline -2x - 5 \end{array}$$

$\therefore \dfrac{x^2+x-3}{x^2+3x+2} = 1 - \dfrac{2x+5}{x^2+3x+2}$

$\dfrac{2x+5}{(x+2)(x+1)} = \dfrac{A}{x+2} + \dfrac{B}{x+1}$

$2x + 5 = A(x+1) + B(x+2)$

~~$2 = A + B$~~

$x = -1$: $3 = B$

$x = -2$: $1 = -A$

$\therefore \dfrac{2x+4}{(x+2)(x+1)} = \dfrac{3}{x+1} - \dfrac{1}{x+2}$

$\therefore \dfrac{x^2+x-3}{x^2+3x+2} = 1 - \dfrac{3}{x+1} + \dfrac{1}{x+2}$.

(c) (i) $4 = 9(1-e^2)$

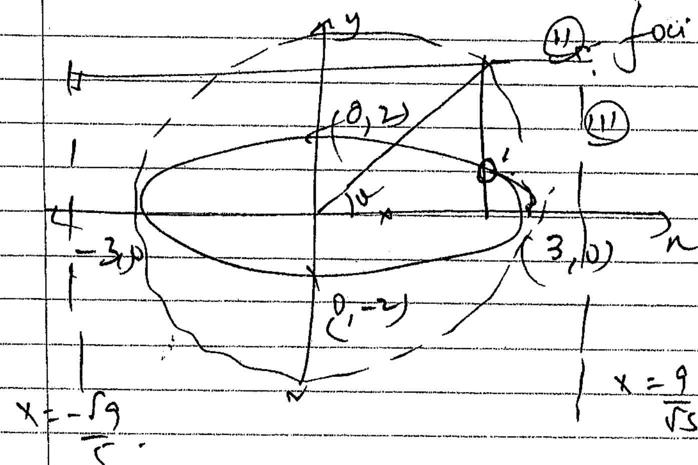
$\dfrac{4}{9} = 1 - e^2$

$e^2 = \dfrac{5}{9}$

$e = \dfrac{\sqrt{5}}{3}$ $(e > 0)$

(ii) foci: $(\pm\sqrt{5}, 0)$

(iii) $x = \pm \dfrac{9}{\sqrt{5}}$

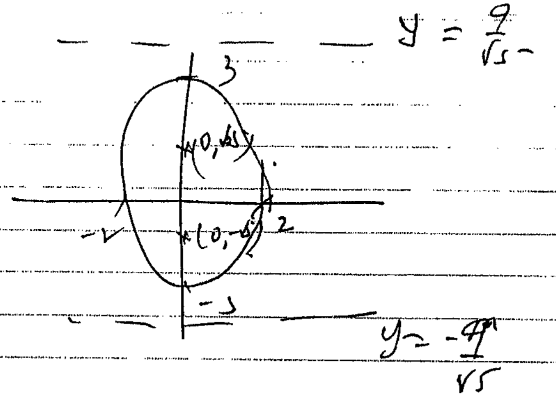


$PS + PS'$

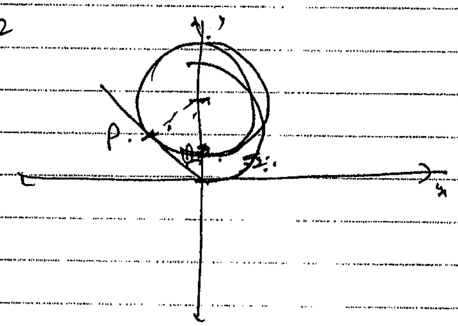
$= e(PM + PM')$

$x = \dfrac{9}{\sqrt{5}}$ $= e.(MM')$

$= \dfrac{\sqrt{5}}{3} \times \dfrac{18}{\sqrt{5}} =$



Q2



$\sin\theta = \frac{1/2}{1}$

$\theta = \frac{\pi}{6}$

$\therefore$ arg of P: $\frac{\pi}{2} + \frac{\pi}{6}$

$= \frac{2\pi}{3}$

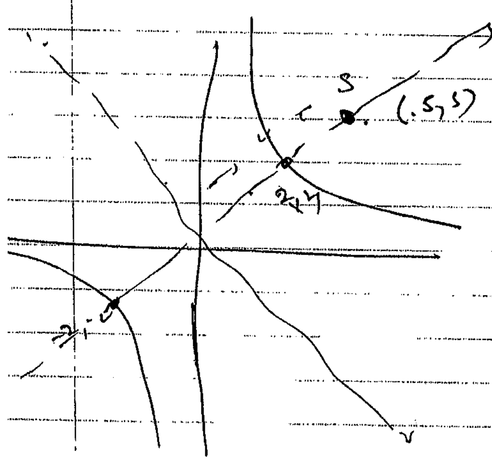
(b) $\frac{x^2}{8} - \frac{y^2}{8} = 1$

$8 = 8(e^2 - 1)$

$e^2 - 1 = 1$

$e^2 = 2$

$e = \sqrt{2}$ $\quad e > 0$



more OS: $2\sqrt{2} \times \sqrt{2}$

$= 4$

S lie on the line $y = x$

$s^2 + s^2 = 4^2$

$2s^2 = 16$

$s^2 = 8$

$s = 2\sqrt{2}$

$\therefore (2\sqrt{2}, 2\sqrt{2})$

$\cos(p+q) = \cos p\cos q - \sin p \sin q$

$\cos(p-q) = \cos p \cos q + \sin p \sin q$

$\therefore \cos(p+q) + \cos(p-q) = 2\cos p \cos q$

$\cos\alpha + \cos\beta = 2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$

$p + q = \alpha$

$p - q = \beta$

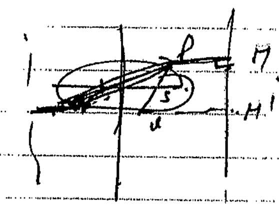
$p = \frac{\alpha+\beta}{2}$

$q = \frac{\alpha-\beta}{2}$

$\frac{x}{a}\cos\left(\frac{\alpha+\beta}{2}\right) + \frac{y}{b}\cos\left(\frac{\alpha-\beta}{2}\right) = \cos\frac{(\alpha}{2}$

PQ is a focal chord, passes through $(\pm ae, 0)$

$\pm e\cos\frac{\alpha+\beta}{2} = \cos\frac{\alpha-\beta}{2}$



$PQ = PS + SQ$

$= e\,PM + e\,QM'$

$= e\left(\frac{a}{e} - a\cos\alpha + \frac{a}{e} - a\cos\beta\right)$

$= 2a - ae(\cos\alpha + \cos\beta)$

$= 2a - ae\left(2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}\right)$

$= 2a - 2ae^2\left(\cos\frac{\alpha+\beta}{2} \cdot \pm\cos\frac{\alpha+\beta}{2}\right)$

$= 2a\left(1 - e^2\cos^2\frac{\alpha+\beta}{2}\right)$

Q.3

$1 + \cos x + \cos^2 x + \cos^3 x + \cdots$

$a = 1 \; ; \; r = \cos x$

nore $|\cos x| < 1$

$$\frac{1}{1-\cos x} = \frac{1}{1-(1-2\sin^2 x/2)} = \frac{1}{2}\operatorname{cosec}^2\frac{x}{2}$$

(5)

$z^7 - 1 = (z-1)(z^6 + z^5 + \cdots)$

The roots are $1 + \alpha + \cdots + \alpha^6 = 0$

$\therefore \Sigma \alpha = -1$ (1)

$(\alpha+\alpha^6)(\alpha^2+\alpha^5) + (\alpha^2+\alpha^5)(\alpha^3+\alpha^4) + (\alpha+\alpha^6)(\alpha^3+\alpha^4)$

$= \alpha^3 + \alpha^8 + \alpha^6 + \alpha^{11} + \alpha^5 + \alpha^8 + \alpha^6 + \alpha^9 + \alpha^4 + \alpha^5 + \alpha^9 + \alpha^{10}$ (1)

$\alpha^7 = 1$

$= \alpha^3 + \alpha + \alpha^6 + \alpha^4 + \alpha^5 + \alpha + \alpha^6 + \alpha^2 + \alpha^4 + \alpha^5 + \alpha^2 + \alpha^3$

$= 2x - 1$

$= -2$ (1)

$(\alpha+\alpha^6)(\alpha^2+\alpha^5)(\alpha^3+\alpha^4) \quad = 1 + 1 - 1$

$= 1$

$= (\alpha^3 + \alpha^6 + \alpha + \alpha^4)(\alpha^3 + \alpha^4)$

eq: $x^3 + x^2 - 2x + 1 = 0$

(C) $\frac{x^2}{16} - \frac{y^2}{9} = 1$

$\frac{2x}{16} - \frac{2y}{9} y' = 0$

$\frac{y y'}{9} = \frac{x}{16}$

$y' = \frac{9}{16} \frac{x}{y}$

$y'_{at (4\sec\theta, 3\tan\theta)} = \frac{9}{16} \times \frac{4\sec\theta}{3\tan\theta}$

$= \frac{3}{4}\frac{\sec\theta}{\tan\theta}$

Grad of normal : $-\frac{4\tan\theta}{3\sec\theta}$

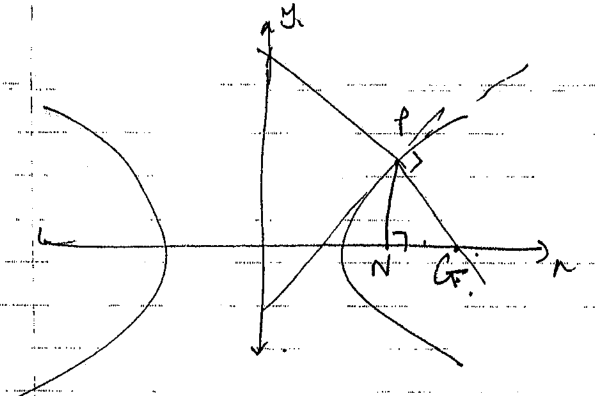
Eqn. of normal :

$y - 3\tan\theta = -\frac{4\tan\theta}{3\sec\theta}(x - 4\sec\theta)$

$3y\sec\theta - 9\tan\theta\sec\theta = -4x\tan\theta + 16\sec\theta\tan\theta$

$4x\tan\theta + 3y\sec\theta = 25\sec\theta\tan\theta$

$4x\frac{\tan\theta}{\sec\theta} + 3y = 25\tan\theta$



G: $4x\sin\theta = 25\tan\theta$

$x = \frac{25}{4}\sec\theta$.

$G: \left(\frac{25}{4}\sec\theta, 0\right)$

$N: (4\sec\theta, 0)$

$\frac{OG}{ON} = \frac{25/4\sec\theta}{4\sec\theta} = \frac{25}{16}$.

d) $u_1 = 6 \quad u_2 = 11$.

$u_{n+2} = 2u_{n+1} - u_n + 2$.

Let $P(n): u_n = n^2 + 2n + 3$;

$P(1)$ $1^2 + 2\times 1 + 3 = 6$ ✓ $P(1)$ is true.

$u_2 = 2^2 + 2\times 2 + 3 = 11$ ✓ $P(2)$ is true

Let $P(n)$ be true for all $n \le k$. — (I)

Consider $P(k+1)$:

$u_{k+1} = 2u_k - u_{k-1} + 2$.

$= 2(k^2+2k+3) - ((k-1)^2 + 2(k-1) + 3) + 2$. — (A).

$= 2(k^2+2k+3) - (k^2 - \cancel{2k} + 1 + \cancel{2k} - 2 + 3) + 2$.

$= k^2 + 4k + 6$.

$= k^2 + 2k + 1 + 2k + 5$.

$= (k+1)^2 + 2(k+1) + 3$.

$\therefore P(k+1)$ is true

$P(1)$, $P(2)$ are true

$P(k+1)$ is true if $P(n)$ is true for $n \le k$

$\therefore$ By the principle of MI $P(n)$ is true $\forall n > 0$

Q.k

$0 \le \sqrt{x+1} < 1$

$0 \le x+1 < 1$

(D). $-1 \le x < 0$

b) $z = \cos\theta + i\sin\theta$

$\frac{1}{z} = z^{-1} = (\cos\theta + i\sin\theta)^{-1}$

$= \cos(-\theta) + i\sin(-\theta)$

$= \cos\theta - i\sin\theta$.

(ii). $z^n + z^{-n}$.

$= \cos n\theta + i\sin n\theta + \cos n\theta - i\sin n\theta$

$= 2\cos n\theta$.

(iii). ~~$\cos\theta + i\sin$~~

$(z+\frac{1}{z})^5 = z^5 + 5z^4 \cdot \frac{1}{z} + 10z^3 \cdot \frac{1}{z^2} + 10z^2 \cdot \frac{1}{z} + 5z \cdot \frac{1}{z^4} + (\frac{1}{z})^5$

$= (z^5 + \frac{1}{z^5}) + 5(z^3 + \frac{1}{z^3}) + 10(z + \frac{1}{z})$

$(2\cos\theta)^5 = 2\cos 5\theta + 10\cos 3\theta + 20\cos\theta$

$\cos^5\theta = \frac{1}{32}(2\cos 5\theta + 10\cos 3\theta + 20\cos\theta)$

$= \frac{1}{16}(\cos 5\theta + 5\cos 3\theta + 10\cos\theta)$

(iii) $xy = 16$

$xy' + y = 0$

$y' = -\frac{y}{x}$

$y'_{at\, P} = -\frac{4/p}{} = -\frac{1}{p^2}$

tgr. $y - \frac{4}{p} = -\frac{1}{p^2}(x - 4p)$

$p^2 y - 4p = -x + 4p$.

$\underline{x + p^2 y = 8p}$ — (1) 2.

(ii). tgr at Q: $x + q^2 y = 8q$ — (2)

(1) − (2)

~~$p^2(y$~~. $y(p^2 - q^2) = 8(p-q)$

$y = \frac{8}{p+q}$.

Sub. in (1)

$x = 8p - p^2\left(\frac{8}{p+q}\right)$

$= \frac{8p^2 + 8pq - 8p^2}{p+q}$

$= \frac{8pq}{p+q}$.

(iii). ~~Chord~~. gred. of $P(0,8)$

$\frac{8 - 4/p}{-4p} = \frac{8 - 4/q}{-4q}$

$(8q - \frac{4q}{p} = 8p - \frac{4p}{q})\ pq$

$8pq^2 - 4q^2 = 8p^2q - 4p^2$.

$4(p^2-q^2) = 8pq(p-q)$

$\therefore p+q = 2pq \qquad p \neq q.$

(iv) $x = \frac{8pq}{p+q} \qquad y = \frac{8}{p+q}.$

& $p+q = 2pq$

$\therefore x = \frac{8pq}{2pq} = 4.$ is the locus.

Q.5 $y = \sin^{-1}(\sin x)$

$y' = \frac{1}{\sqrt{1-\sin^2 x}} \cdot \cos x$

$= \frac{\cos x}{|\cos x|}$

$= \begin{cases} +1 & \cos x > 0 \\ -1 & \cos x < 0 \end{cases}$

(ii) $\frac{dy}{dx} = -1$ when $|\cos x| = -\cos x$

i.e when $\cos x < 0$

$\therefore \frac{\pi}{2} < x < \frac{3\pi}{2}.$

b)

$\tan\alpha + \tan\beta = a$

$\tan\alpha \tan\beta = 1 - a.$

$\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$

$= \frac{a}{1-(1-a)} = 1$

$\therefore \alpha+\beta = n\pi + \pi/4.$

c) $P(x) = x^3 - 6x^2 + 9x + k$

$P'(x) = 3x^2 - 12x + 9$

$P'(x) = 0 \qquad x^2 - 4x + 3 = 0$

$(x-3)(x-1) = 0$

$x = 3 \quad x = 1.$

(ii) $x = 3$ or $x = 1$ can be repeated roots for $P(x) = 0$

$P(3) = 0 \implies 27 - 54 + 27 + k = 0$

$k = 0$

$P(1) = 0 \qquad 1 - 6 + 9 + k = 0$

$k = -4$

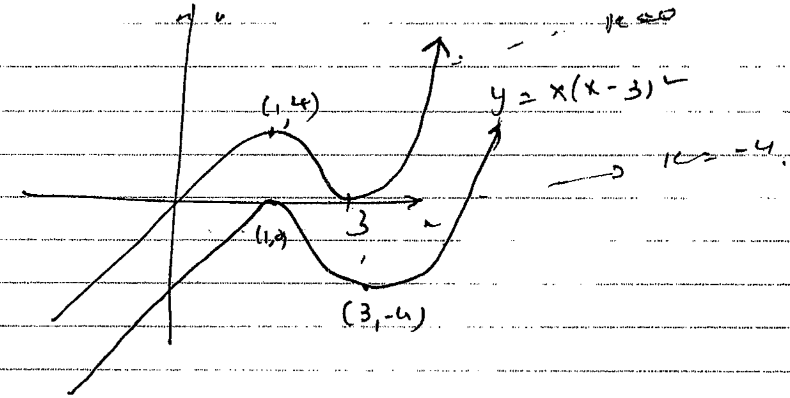
(iii) $y = x^3 - 6x^2 + 9x \qquad k = 0$

$= x(x^2 - 6x + 9)$

$= x(x-3)^2$

$y = x^3 - 6x^2 + 9x - 4$

$= x(x-3)^2 - 4$



(iv) One root $\implies$ one pt. of intersection with the x axis

happens when $k > 0$. or $k < -4$.