St George Girls High School

Trial Higher School Certificate Examination

2012

Mathematics Extension 2

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using blue or black pen.
- Write your student number on each booklet.
- Board-approved calculators may be used.
- A table of standard integrals is provided at the back of this paper.
- The mark allocated for each question is listed at the side of the question.
- Marks may be deducted for careless or poorly presented work.

Total Marks – 100

Section I – Pages 2 – 4
10 marks

- Attempt Questions 1 – 10.
- Allow about 15 minutes for this section.
- Answer on the sheet provided.

Section II – Pages 5 – 13
90 marks

- Attempt Questions 11 – 16
- Allow about 2 hours 45 minutes for this section.
- Begin each question in a new booklet.
- Show all necessary working in Questions 11 – 16.
- Templates for Q12(a) to be detached and placed in Q3 answer booklet.

Students are advised that this is a Trial Examination only and does not necessarily reflect the content or format of the Higher School Certificate Examination.

STUDENT NUMBER/NAME: ....

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \ -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE: $\ln x = \log_e x, \quad x > 0$

Section I – (10 marks)

Marks

Answer this section on the answer sheet provided at the back of this paper.
Select the alternative A, B, C or D that best answers the question. Fill in the response oval completely.

1. The maximum value of y reached by the ellipse with equation

$$\frac{3(x+3)^2}{5}+\frac{(y-4)^2}{6}=3$$

is:

A. $-4+3\sqrt{2}$
B. $4+\sqrt{5}$
C. $3\sqrt{2}$
D. $4+3\sqrt{2}$

2. The graph of $f(x)=\frac{1}{x^2+mx-n}$, where m and n are real constants, has no vertical asymptotes if

A. $m^2<4n$
B. $m^2>4n$
C. $m^2=-4n$
D. $m^2<-4n$

3. The number of real solutions to $x^4-x^3=\operatorname{cosec}^2(x)-\cot^2(x)$ is:

A. 0
B. 1
C. 2
D. 3

4. If $z=\frac{3+4i}{1+2i}$, the imaginary part of z is:

A. -2 B. $-\frac{2}{5}i$ C. $-\frac{2}{5}$ D. $-2i$

Section I (cont'd)

Marks

5. If $I=\int_0^{\ln 2}\frac{e^x}{e^x+e^{-x}}\,dx$ and $J=\int_0^{\ln 2}\frac{e^{-x}}{e^x+e^{-x}}\,dx$, then the exact value of $I-J$ is:

A. $\ln\left(\frac{5}{2}\right)$ B. $\ln 2$ C. $\ln(5)$ D. $\ln\left(\frac{5}{4}\right)$

6. If $z=\sqrt{3}+i$ then in modulus/argument form $z=2\operatorname{cis}\frac{\pi}{6}$. If $z^n+(\bar{z})^n$ is to be rational, then the integer $'n'$ can not be:

A. 2
B. 3
C. 5
D. 6

7. Given hyperbola $\mathcal{H}$ with equation $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ has eccentricity e then the ellipse E with equation $\frac{x^2}{a^2+b^2}+\frac{y^2}{b^2}=1$ has eccentricity.

A. $-e$ B. $\frac{1}{e}$ C. $\sqrt{e}$ D. e^2

8. What restrictions must be placed on p if α,β,γ are the three, non-zero real roots of the equation $x^3+px-1=0$?

A. $p>0$, p is real
B. $p<0$, p is real
C. $p\geq 0$, p is real
D. $p\leq 0$, p is real

Section I (cont'd)

Marks

9. Given that $\frac{dy}{dx} = y^2 + 1$, and that $y = 1$ at $x = 0$, then

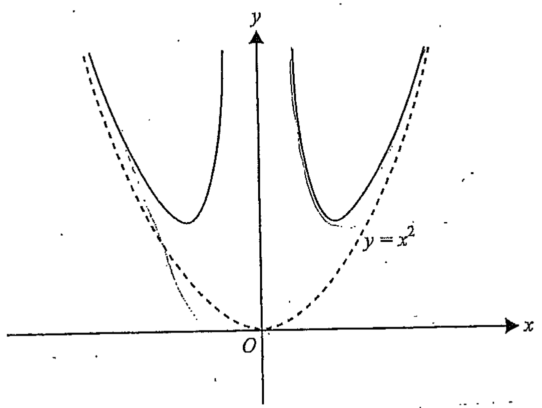
A. $y = \tan\left(x - \frac{\pi}{4}\right)$

B. $y = \tan\left(x + \frac{\pi}{4}\right)$

C. $x = \log_e\left(\frac{y^2+1}{2}\right)$

D. $y = \frac{1}{3}y^3 + y - \frac{1}{3}$

10.



A possible equation for the graph of the curve shown above is

A. $y = \frac{x^3+a}{x}, \quad a > 0$

B. $y = \frac{x^3+a}{x}, \quad a < 0$

C. $y = \frac{2x^4+a}{x^2}, \quad a > 0$

D. $y = \frac{x^4+a}{x^2}, \quad a < 0$

Section II – Show all working

Question 11 – Start A New Booklet – (15 marks)

Marks

a) Find $\int \frac{dx}{\sqrt{3 - 4x - 4x^2}}$ **2**

b) Evaluate $\int_0^{\frac{\pi}{6}} \frac{d\theta}{9 - 8\cos^2\theta}$ using the substitution $t = \tan\theta$ **3**

c) Find $\int \frac{dx}{(x+1)(x^2+4)}$ **3**

d) Evaluate $\int_0^1 \tan^{-1}x \, dx$ **2**

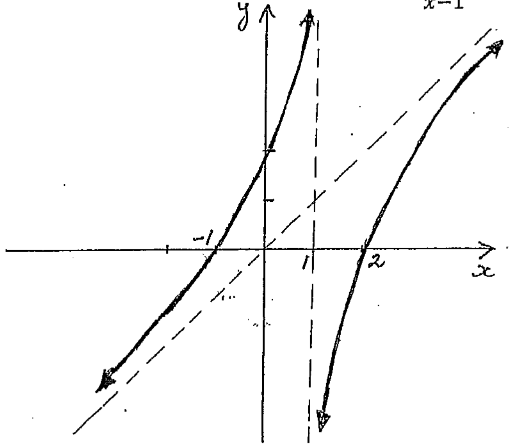
e) If $I_n = \int_0^{\frac{\pi}{2}} \sin^n x \, . dx$ show that $I_n = \frac{n-1}{n} \, . \, I_{n-2}$ **3**

Hence evaluate $\int_0^{\frac{\pi}{2}} \sin^5 x \, . dx$ **2**

Question 12 – Start A New Booklet – (15 marks)

Marks

a) The sketch of $y = f(x)$ is shown below where $f(x) = \frac{x^2-x-2}{x-1}$



(i) Show that $y = x$ is an asymptote. 2

(ii) Sketch each of the following on the template provided.

(α) $y = |f(x)|$ 2

(β) $y = f(1-x)$ 2

(γ) $y^2 = f(x)$ 2

b) Consider the curve C: $x^2 + xy + y^2 = 9$

(i) Find $\frac{dy}{dx}$ 1

(ii) Find all stationary points and points where $\frac{dy}{dx}$ is not defined. 4

(iii) Sketch C clearly showing the above features and intercepts on the x, y axes. 2

Question 13 – Start A New Booklet – (15 marks)

Marks

a) If $z = (1+i)^{-1}$.

(i) Express $\bar{z}$ in modulus-argument form. 2

(ii) If $(\bar{z})^9 = a + ib$ where a and b are real numbers, find the values of a and b. 2

b) Sketch each of the following on separate Argand diagrams.

(i) $|z - 2 + 3i| = |z + 2 - 3i|$ 2

(ii) $\arg(z + 3 - i) = \frac{3\pi}{4}$ 2

c) (i) On an Argand diagram sketch $|z - \sqrt{2} - \sqrt{2}\,i| = 1$ 2

(ii) Find the minimum values of $|z|$ and $\arg z$ 3

d)

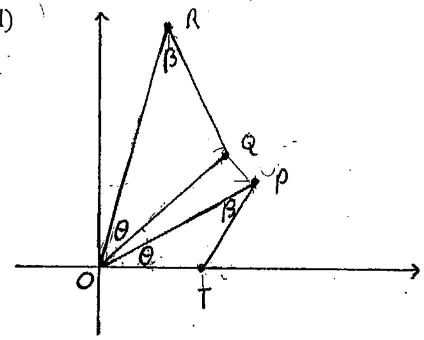


Fig I

The points T, P and Q in the complex plane correspond to the complex numbers 1, $\sqrt{3} + i$ and $2 + 2i$ respectively. 2

Triangles OTP and OQR are similar with corresponding angles as shown in Fig I. Find the complex number represented by R (in modulus argument form).

Question 14 – Start A New Booklet – (15 marks) **Marks**

a) The polynomial equation $x^3 - 6x^2 + 3x - 2 = 0$ has roots α, β, γ. 2

Evaluate $\alpha^3 + \beta^3 + \gamma^3$

b) Prove that if a polynomial $P(x)$ has a zero of multiplicity 'm' then the derived polynomial $P'(x)$ has that same zero with multiplicity '$m - 1$' 1

c) Given that $-2 - i$ is a zero of $P(x) = x^4 + 6x^3 + 14x^2 + 14x + 5$, find all zeros of $P(x)$ 3

d) (i) Prove that $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$ by use of de Moivre's theorem. 2

(ii) Find the general solution of $\cos 3\theta = \frac{1}{2}$ 1

(iii) Solve for x : $8x^3 - 6x - 1 = 0$ 3

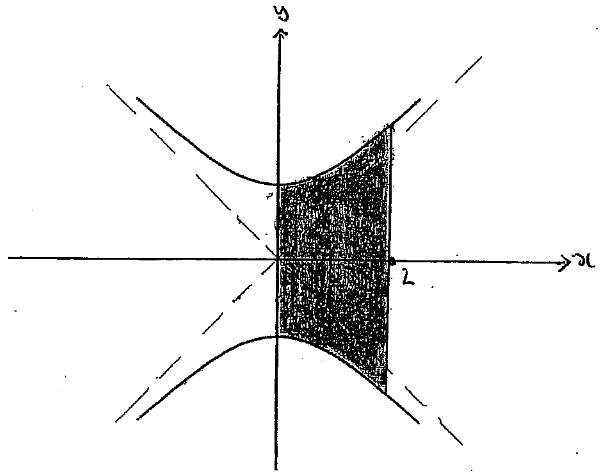
(iv) Find a polynomial of least degree which has zeros

$\sec^2\frac{\pi}{9}, \sec^2\frac{5\pi}{9}, \sec^2\frac{7\pi}{9}$ 2

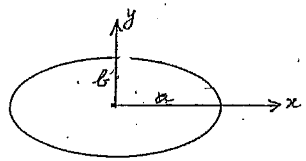
(v) Hence evaluate $\sec^2\frac{\pi}{9} + \sec^2\frac{5\pi}{9} + \sec^2\frac{7\pi}{9}$ 1

Question 15 – Start A New Booklet – (15 marks) **Marks**

a) Using the method of cylindrical shells, find the volume generated by revolving the area bounded by the lines $\begin{cases} x = 2 \\ x = 0 \end{cases}$ and the two branches of the hyperbola $\frac{y^2}{9} - \frac{x^2}{4} = 1$ about the y-axis (as shown in the diagram) 3



b) (i)

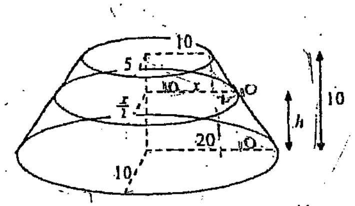


The ellipse shown has equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Prove that the area enclosed by this ellipse is πab 3

Question 15 (cont'd) **Marks**

b) (ii)



A solid of height 10 m stands on horizontal ground.

- The base of the solid is an ellipse with semi-axes of 20 m and 10 m.
- The top of the solid is an ellipse with semi-axes of 10 m and 5 m.

Horizontal cross-sections taken parallel to the base and at height h metres above the base are ellipses with semi-axes x metres and $\frac{x}{2}$ metres.

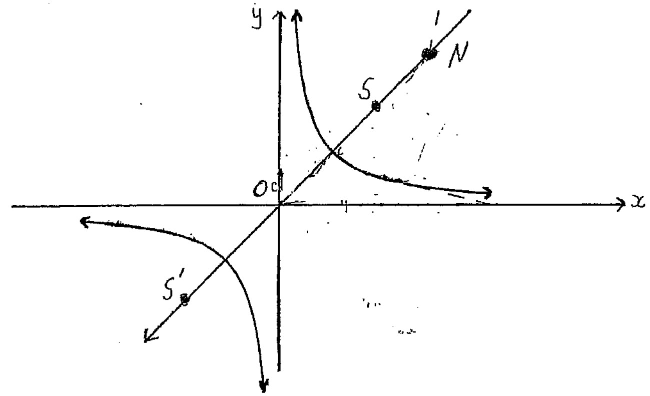
The centres of these elliptical cross-sections and the base lie on a vertical straight line, and the extremities of their semi-axes lie on sloping straight lines as shown in the diagram.

(α) Prove that $x = 20 - h$ **2**

(β) Find the volume of the solid correct to the nearest cubic metre. **3**

Question 15 (cont'd) **Marks**

c) The diagram shows the hyperbola $xy = 4$



(i) What are the coordinates of the foci S and S' ? **1**

(ii) The point $P(2t, \frac{2}{t})$ lies on the curve, where $t \neq 0$. The normal at P intersects the straight line $y = x$ at N. O is the origin.

Given the equation of the normal at P is $y = t^2x + \frac{2}{t} - 8$

(α) Find the coordinates of N **1**

(β) Show that the triangle OPN is isosceles **2**

Question 16 – Start A New Booklet – (15 marks) **Marks**

4

a) A parachutist of mass M is initially located travelling downward in a straight line with a speed of v_0. [let $x = 0$ at $t = 0$]

If the resistance on the parachute is proportional to the speed and the gravitational force is g.

(i) Show that the speed, v, can be given as 3

$$v = \frac{g}{k} - \left(\frac{g}{k} - v_0\right) e^{-kt}$$

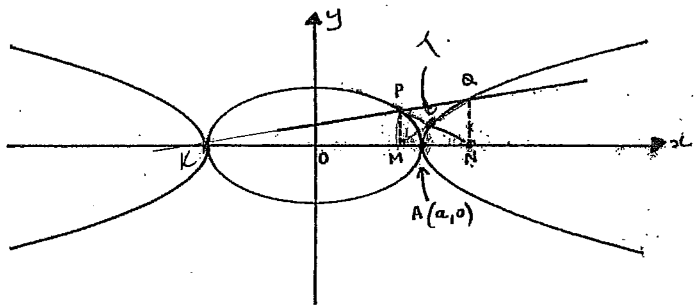
(k) is constant of proportionality.

(ii) Find the parachutist's "terminal" velocity. 1

Questions 16 b) continued on next page

Question 16 (cont'd)

b) $P(a\cos\theta,\ b\sin\theta)$ and $Q(a\sec\theta,\ b\tan\theta)$ lie on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, respectively as shown.



M and N are the feet of the perpendicular from P and Q respectively to the x-axis. $0 < \theta < \frac{\pi}{2}$, and QP meets the x-axis at K. A is the point $(a, 0)$.

(i) Given $\Delta KPM \parallel \Delta KQN$, show that $\frac{KM}{KN} = \cos\theta$ 1

(ii) Hence, show that K has coordinates $(-a, 0)$ 2

(iii) Show that the tangent to the ellipse at P has equation $\frac{x\cos\theta}{a} + \frac{y\sin\theta}{b} = 1$, and deduce it passes through N 3

(iv) Given that the tangent to the hyperbola at Q has equation $\frac{x\sec\theta}{a} - \frac{y\tan\theta}{b} = 1$, show that the tangent passes through M. 2

If T is the point of intersection of PN and QM, show that AT is perpendicular to the x-axis.

c) Using mathematical induction prove that 3

$$\sum_{r=1}^{n} r^3 < n^2(n+1)^2$$

End of Paper

Trial Hsc Ext 2 - 2012

SECTION I

1. $\frac{(x+3)^2}{15} + \frac{(y-4)^2}{18} = 1$ Ellipse centre $(-3,4)$

$a = \sqrt{15}$

$b = \sqrt{18}$

$= 3\sqrt{2}$ D

$\therefore$ maximum value of $y = 4 + 3\sqrt{2}$

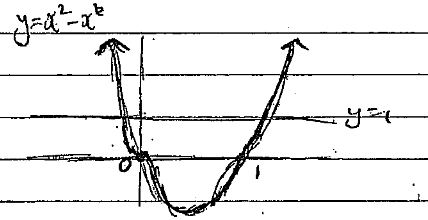
2. No vertical asymptots then $x^2 + mx - n \neq 0$

Then $\Delta < 0$, so $m^2 - 4.1.-n < 0$ D

$m^2 < -4n$

3. $x^3(x-1) = 1$ sketch $y = x^4 - x^3$ C

$x^4 - x^3 - 1 = 0$ two pts of intersection



or $y = x^4 - x^3 - 1$

$y' = 4x^3 - 3x^2$

$= x^2(4x-3)$

$y'' = 12x^2 - 6x$

$= 6x(2x-1)$

etc.

4. $z = \frac{3+4i}{1+2i} \times \left(\frac{1-2i}{1-2i}\right)$ B

$= \frac{3-6i+4i+8}{1+4}$

$= \frac{11}{5} - \frac{2}{5}i$

imaginary part

5. $I - J = \int_0^{\ln 2} \frac{e^x - e^{-x}}{e^x + e^{-x}}\,dx$ $\left[\frac{f'(x)}{f(x)}\right]$

$= \left[\ln\left(e^x + e^{-x}\right)\right]_0^{\ln 2}$ D

$= \ln\left(e^{\ln 2} + e^{\ln(2^{-1})}\right) - \ln(1+1)$

$= \ln\left[2 + \frac{1}{2}\right] - \ln(2)$

$= \ln\left(\frac{5}{4}\right)$

6. $z^n + (\bar{z})^n = 2\left(\cos\left(\frac{n\pi}{6}\right) + i\sin\left(\frac{n\pi}{6}\right)\right) + 2\left(\cos\left(\frac{n\pi}{6}\right) - i\sin\left(\frac{n\pi}{6}\right)\right)$

$= 4\cos\left(\frac{n\pi}{6}\right)$ C

$n=2$, $4\cos\frac{\pi}{3} = \frac{1}{2} \times 4 = 2$

$n=3$, $4\cos\frac{\pi}{2} = 0 \times 4 = 0$

$n=5$, $4\cos\frac{5\pi}{6} = -\frac{\sqrt{3}}{2} \times 4 = -2\sqrt{3}$

$n=6$, $4\cos\pi = -1 \times 4 = -4$

7. $b^2 = a^2(e^2-1)$

$\therefore e^2 = 1 + \frac{b^2}{a^2}$

$= \frac{a^2+b^2}{a^2}$ $\Big\}$ $e^2a^2 = a^2 + b^2$

then for E

$b^2 = (a^2+b^2)(1-E^2)$ $\therefore E = \frac{1}{e}$ B

$\frac{b^2}{a^2+b^2} = 1 - E^2$

$E^2 = 1 - \frac{b^2}{a^2+b^2}$

$E^2 = 1 - \frac{b^2}{e^2a^2}$

$= 1 - \frac{1}{e^2}(e^2-1)$

$= 1 - 1 + \frac{1}{e^2}$

$= \frac{1}{e^2}$

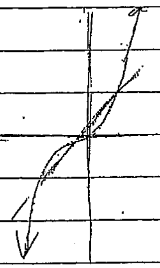
8. Require curve $x^3+px-1=0$ to cut x-axis at three distinct places.

B

Consider

$x^3 = 1 - px$ [p gradient of line]

$p < 0$



9. $\frac{dx}{dy} = \frac{1}{y^2+1}$

B

$x = \tan^{-1} y + C$

at $x = 0$, $y = 1$

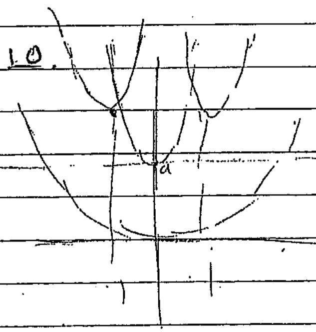
$0 = \frac{\pi}{4} + C$

$C = -\frac{\pi}{4}$

$\therefore x + \frac{\pi}{4} = \tan^{-1} y$

$y = \tan\left(x + \frac{\pi}{4}\right)$

10.



C.

QUESTION 11:

(a) $\displaystyle\int \frac{dx}{\sqrt{3-4x-4x^2}} = \int \frac{dx}{\sqrt{-1(4x^2+4x-3)}}$

$\displaystyle = \int \frac{dx}{\sqrt{-1[(2x+1)^2-4]}}$

$\displaystyle = \int \frac{dx}{\sqrt{4-(2x+1)^2}}$

let $2x+1 = 2\sin\theta$

$2dx = 2\cos\theta\, d\theta$

$dx = \cos\theta\, d\theta$

$\displaystyle = \int \frac{\cos\theta\, d\theta}{2\cos\theta}$

$= \frac{\theta}{2} + C$

$= \frac{1}{2}\sin^{-1}\left(\frac{2x+1}{2}\right) + C$

$\displaystyle * = \int \frac{dx}{\sqrt{2^2 - u^2}}$ $\quad u = 2x+1$, $du = 2dx$

$\displaystyle = \frac{1}{2}\int \frac{du}{\sqrt{2^2-u^2}}$

$= \frac{1}{2}\sin^{-1}\left(\frac{u}{2}\right) + C$

$= \frac{1}{2}\sin^{-1}\left(\frac{2x+1}{2}\right) + C$

(b) $\displaystyle\int_0^{\frac{\pi}{6}} \frac{d\theta}{9-8\cos^2\theta}$

$t = \tan\theta$ ——— (1)

$dt = \sec^2\theta\, d\theta$

ie $d\theta = \frac{dt}{\sec^2\theta}$

$= \frac{dt}{1+t^2}$

from (1): $\sec^2\theta = 1+t^2$

$\therefore \cos^2\theta = \frac{1}{1+t^2}$

$\displaystyle = \int_0^{\frac{1}{\sqrt{3}}} \frac{\frac{dt}{1+t^2}}{9-8\left(\frac{1}{1+t^2}\right)}$

$\displaystyle = \int_0^{\frac{1}{\sqrt{3}}} \frac{dt}{9+9t^2-8}$

$\displaystyle * = \int_0^{\frac{1}{\sqrt{3}}} \frac{dt}{1+(3t)^2}$ $\quad 3t = \tan\alpha$, $3dt = \sec^2\alpha\, d\alpha$

$\displaystyle = \int_0^{\frac{\pi}{3}} \frac{1}{3} \cdot \frac{\sec^2\alpha}{\sec^2\alpha}\, d\alpha$

$= \frac{1}{3}\cdot\left[\alpha\right]_0^{\frac{\pi}{3}}$

$= \frac{\pi}{9}$

$\displaystyle * \quad \frac{1}{9}\int_0^{\frac{1}{\sqrt{3}}} \frac{dt}{\left(\frac{1}{3}\right)^2+t^2}$

$= \frac{1}{9}\left[\frac{1}{\frac{1}{3}}\tan^{-1}\left(\frac{t}{\frac{1}{3}}\right)\right]_0^{\frac{1}{\sqrt{3}}}$

$= \frac{1}{3}\left[\tan^{-1}\left(\frac{3}{\sqrt{3}}\right) - \tan^{-1} 0\right]$

$= \frac{1}{3} \times \frac{\pi}{3} - 0$

(c) $\int \frac{dx}{(x+1)(x^2+4)}$

let $\frac{1}{(x+1)(x^2+4)} = \frac{a}{x+1} + \frac{bx+c}{x^2+4}$

ie $1 = a(x^2+4) + (bx+c)(x+1)$

$x=-1 \Rightarrow 1 = 5a$

$\therefore a = \frac{1}{5}$

co-eff of $x^2 \Rightarrow 0 = a + b$

$\therefore b = -\frac{1}{5}$

constant $\Rightarrow 1 = 4a + c$

$= \frac{4}{5} + c$

$\therefore c = \frac{1}{5}$

$$= \int \left(\frac{\frac{1}{5}}{x+1} + \frac{\frac{1}{5} - \frac{1}{5}x}{x^2+4} \right) dx$$

$$= \frac{1}{5} \ln|x+1| - \frac{1}{5} \int \frac{x-1}{x^2+4} \, dx$$

$$= \frac{1}{5} \ln|x+1| - \frac{1}{5} \int \left(\frac{1}{2} \cdot \frac{2x}{x^2+4} - \frac{1}{x^2+4} \right) dx$$

$$= \frac{1}{5} \ln|x+1| - \frac{1}{10} \cdot \ln|x^2+4| + \frac{1}{5} \cdot \frac{1}{2} \tan^{-1}\left(\frac{x}{2}\right) + c$$

$$= \frac{1}{5} \ln|x+1| - \frac{1}{10} \ln(x^2+4) + \frac{1}{10} \tan^{-1}\left(\frac{x}{2}\right) + c$$

(d) $\int_0^1 \underbrace{1}_{du} \underbrace{\tan^{-1} x}_{v} \, dx = x \tan^{-1} x \Big]_0^1 - \int_0^1 x \cdot \frac{1}{1+x^2} \, dx$

$$= (1 \tan^{-1} 1 - 0) - \frac{1}{2} \Big[\ln(1+x^2) \Big]_0^1$$

$$= \frac{\pi}{4} - \left(\frac{1}{2} \ln 2 - \frac{1}{2} \ln 1 \right)$$

$$= \frac{\pi}{4} - \frac{1}{2} \ln 2$$

(d) $I_n = \int_0^{\frac{\pi}{2}} \sin^n x \, dx$

$$= \int_0^{\frac{\pi}{2}} \underbrace{\sin x}_{du} \cdot \underbrace{\sin^{n-1} x}_{v} \, dx$$

$$= \Big[-\cos x \cdot \sin^{n-1} x \Big]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} -\cos x \cdot (n-1) \sin^{n-2} x \cdot \cos x$$

$$= 0 + \int_0^{\frac{\pi}{2}} (n-1) \cdot \cos^2 x \cdot \sin^{n-2} x \, dx$$

$$= (n-1) \int_0^{\frac{\pi}{2}} (1 - \sin^2 x) \sin^{n-2} x \, dx$$

$$= (n-1) \cdot I_{n-2} - (n-1) I_n$$

ie $I_n + (n-1) I_n = (n-1) I_{n-2}$

$I_n (1 + n - 1) = (n-1) I_{n-2}$

$\therefore I_n = \left(\frac{n-1}{n} \right) \cdot I_{n-2}$

Then $\int_0^{\frac{\pi}{2}} \sin^5 x \, dx = I_5$

$$= \frac{4}{5} \times I_3$$

$$= \frac{4}{5} \times \frac{2}{3} \times I_1$$

$$= \frac{8}{15} \int_0^{\frac{\pi}{2}} \sin x \, dx$$

$$= \frac{8}{15} \cdot \Big[-\cos x \Big]_0^{\frac{\pi}{2}}$$

$$= \frac{8}{15} \Big[0 - -1 \Big]$$

$$= \frac{8}{15}$$

QUESTION 12:

(a) (i) $y = f(x) = \dfrac{(x-2)(x+1)}{(x-1)}$

$= \dfrac{x^2 - x}{x-1} - \dfrac{2}{x-1}$

$= \dfrac{x(x-1)}{x-1} - \dfrac{2}{x-1} \qquad x \neq 1$

$= x - \dfrac{2}{x-1}$

as $x \to \pm\infty$, $\dfrac{2}{x-1} \to 0$

$\therefore$ $y = x$ is an asymptote

* See templates (ii)

(b) (i) $2x + 1.y + x.\dfrac{dy}{dx} + 2y\dfrac{dy}{dx} = 0$

then $\dfrac{dy}{dx}(x+2y) = -(2x+y)$

$\dfrac{dy}{dx} = -\dfrac{(2x+y)}{(x+2y)}$

(ii) Stationary when $2x + y = 0$

i.e. $y = -2x$

Then in C: $x^2 + x.-2x + (-2x)^2 = 9$

$x^2 - 2x^2 + 4x^2 = 9$

$3(x^2 - 3) = 0$

$x = -\sqrt{3}$ and $x = \sqrt{3}$

$y = 2\sqrt{3}$ $\qquad$ $y = -2\sqrt{3}$

Stationary $(-\sqrt{3}, 2\sqrt{3})$ $\qquad$ $(\sqrt{3}, -2\sqrt{3})$

Not defined when $x + 2y = 0$

$x = -2y$

Sub in C: $4y^2 - 2y^2 + y^2 = 9$

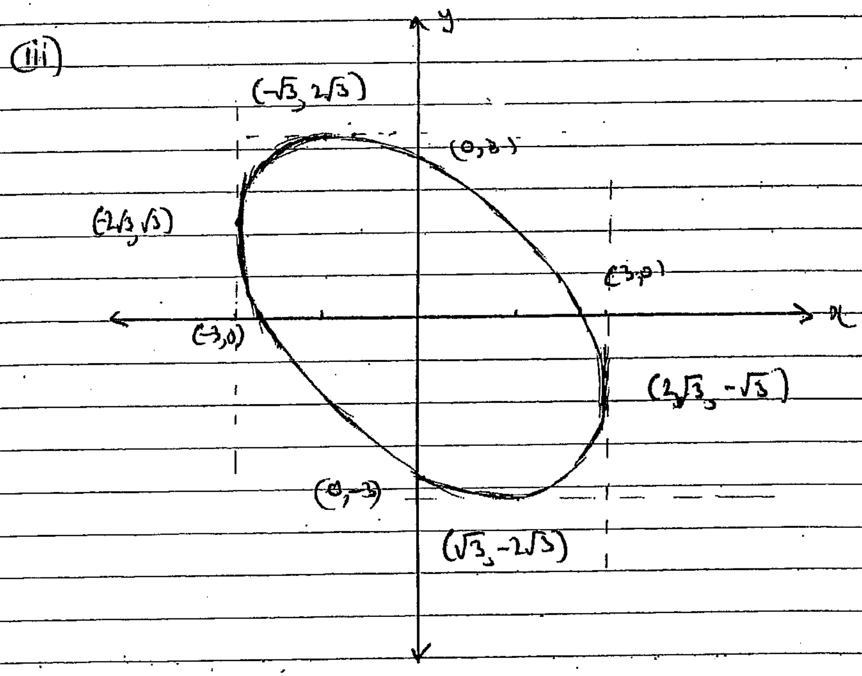
$3(y^2 - 3) = 0$

When $y = -\sqrt{3}$ and $y = \sqrt{3}$

$x = 2\sqrt{3}$ $\qquad$ $x = -2\sqrt{3}$

Not defined at $(2\sqrt{3}, -\sqrt{3})$ and $(-2\sqrt{3}, \sqrt{3})$

(iii)



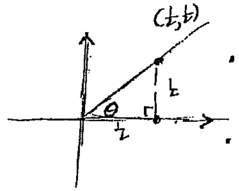
Intercepts: When $x = 0$; $y^2 = 9$ $\qquad (0,-3)$; $(0,3)$

$y = 0$; $x^2 = 9$ $\qquad (-3,0)$; $(3,0)$

QUESTION 13:

(a) $z = \frac{1}{1+i} \cdot \frac{1-i}{1-i}$

$= \frac{1}{2} - \frac{1}{2}i$



$\tan\theta = 1$

$\theta = \frac{\pi}{4}$

$\sqrt{(\frac{1}{2})^2+(\frac{1}{2})^2} = \frac{1}{\sqrt{2}}$

(i) $\bar{z} = \frac{1}{2} + \frac{1}{2}i$

$= \frac{1}{\sqrt{2}} \operatorname{cis} \frac{\pi}{4}$ $\quad$ [ie $\frac{1}{\sqrt{2}}(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4})$]

(ii) $(\bar{z})^9 = (\frac{1}{\sqrt{2}})^9 \operatorname{cis} \frac{9\pi}{4}$

$= \frac{1}{16\sqrt{2}}(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i)$

$= \frac{1}{32} + \frac{1}{32}i$

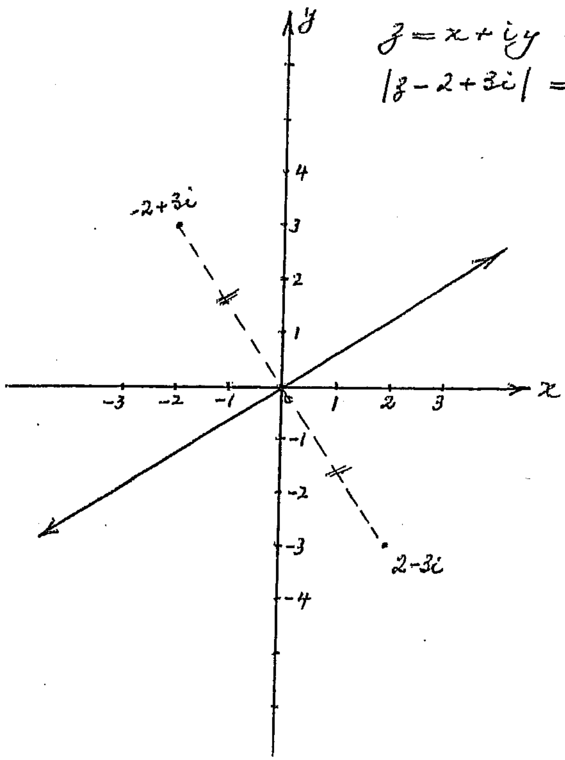
$\therefore a = b = \frac{1}{32}$

(b) (i) $|z - 2 + 3i| = |z + 2 - 3i|$

$\Rightarrow |z - (2-3i)| = |z - (-2+3i)|$ $\quad$ ie all points which are equidistant from $2-3i$ and $-2+3i$

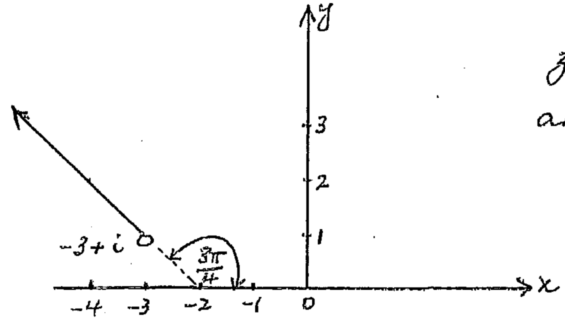
$z = x + iy$ where

$|z - 2 + 3i| = |z + 2 - 3i|$



(ii) $\arg(z + 3 - i) = \frac{3\pi}{4}$

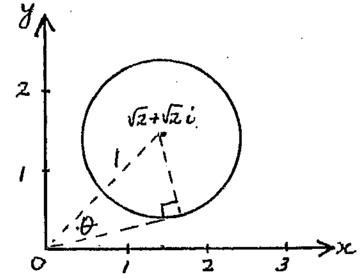
$\Rightarrow \arg[z - (-3+i)] = \frac{3\pi}{4}$



$z = x + iy$ where

$\arg(z + 3 - i) = \frac{3\pi}{4}$

(c) (i)



$z = x + iy$ where

$|z - \sqrt{2} - \sqrt{2}i| = 1$

$\Rightarrow |z - (\sqrt{2} + \sqrt{2}i)| = 1$

Circle centred at $\sqrt{2} + \sqrt{2}i$ with radius 1.

(ii) See dotted lines in (i) above

$|\sqrt{2} + \sqrt{2}i| = 2$

Hence minimum value of $|z|$ is $2 - 1 = 1$

Then the minimum value of $\arg z$ is

$\arg(\sqrt{2} + \sqrt{2}i) - \theta$ where $\sin\theta = \frac{1}{2}$

$\theta = \frac{\pi}{6}$

$\Rightarrow \frac{\pi}{4} - \frac{\pi}{6}$

$= \frac{\pi}{12}$

(d) $T \equiv 1$

$P \equiv \sqrt{3} + i$

$Q \equiv 2 + 2i$

By similar triangles

$$\frac{|OR|}{|OP|} = \frac{|OQ|}{|OT|}$$

ie $|OR| = \frac{|OP| . |OQ|}{|OT|}$

$$= \frac{2 . 2\sqrt{2}}{1}$$

$$= 4\sqrt{2}$$

and $\arg \vec{OR} = \arg OQ + \theta$

$$= \frac{\pi}{4} + \arg OP$$

$$= \frac{\pi}{4} + \frac{\pi}{6}$$

$$= \frac{5\pi}{12}$$

$\therefore R \equiv 4\sqrt{2} \text{ cis } \frac{5\pi}{12}$

QUESTION 14:

(a) Since α, β, γ satisfy equation

$$\alpha^3 - 6\alpha^2 + 3\alpha - 2 = 0 \quad \text{--- (i)}$$

$$\beta^3 - 6\beta^2 + 3\beta - 2 = 0 \quad \text{--- (ii)}$$

$$\gamma^3 - 6\gamma^2 + 3\gamma - 2 = 0 \quad \text{--- (iii)}$$

Sum (i), (ii), (iii) $\Rightarrow \alpha^3 + \beta^3 + \gamma^3 - 6(\alpha^2+\beta^2+\gamma^2) + 3(\alpha+\beta+\gamma) - 6 = 0$

Now $\alpha^2+\beta^2+\gamma^2 = (\alpha+\beta+\gamma)^2 - 2(\alpha\beta+\alpha\gamma+\beta\gamma)$

$$= 6^2 - 2 \times 3$$

$$= 30$$

So $\alpha^3+\beta^3+\gamma^3 - 6 \times 30 + 3 \times 6 - 6 = 0$

$$\alpha^3+\beta^3+\gamma^3 = 168$$

(b) Let α be zero of multiplicity m

then $P(x) = (x-\alpha)^m Q(x)$ [α not a zero of $Q(x)$]

Differentiate $P'(x) = m(x-\alpha)^{m-1} Q(x) + Q'(x)(x-\alpha)^m$

$$= (x-\alpha)^{m-1}[m\,Q(x) + Q'(x)(x-\alpha)]$$

$\therefore$ α is a zero of multiplicity $(m-1)$ of $P'(x)$

(c) Since coefficient integers then if z is a zero so is $\bar{z}$.

$\therefore \; P(x) = [x-(-2-i)][x-(-2+i)](ax^2+bx+c)$

a, b, c real $\quad = [(x+2)+i][(x+2)-i](ax^2+bx+c)$

$= [(x+2)^2 - i^2][ax^2+bx+c]$

$= (x^2+4x+5)(ax^2+bx+c)$

Since $P(x)$ is monic, $a=1$

$= (x^2+4x+5)(x^2+bx+c)$

- constant 5 gives $c=1$

$= (x^2+4x+5)(x^2+bx+1)$

- by observation $b=2$

$P(x) = (x^2+4x+5)(x^2+2x+1)$

$= (x^2+4x+5)(x+1)^2$

Zeros $-2-i, -2+i, -1, -1$

(d) (i) Let $z = \cos\theta + i\sin\theta$

then $z^3 = (\cos\theta + i\sin\theta)^3$

- by de Moivre's theorem $z^3 = \cos 3\theta + i.\sin 3\theta$ (I)

- on expansion $z^3 = \cos^3\theta + 3\cos^2\theta.i\sin\theta + 3\cos\theta(i\sin\theta)^2 + (i\sin\theta)^3$

$= \cos^3\theta - 3\cos\theta.\sin^2\theta + i[3\cos^2\theta\sin\theta - \sin^3\theta]$ II

Equating real parts from (I) and II

$\cos 3\theta = \cos^3\theta - 3\cos\theta.\sin^2\theta$

$\cos 3\theta = \cos^3\theta - 3\cos\theta[1-\cos^2\theta]$

$= 4\cos^3\theta - 3\cos\theta$

(ii) Let $\cos 3\theta = \frac{1}{2}$, related acute angle $\frac{\pi}{3}$ in 1st & 4th quad.

$\therefore 3\theta = \frac{\pi}{3} + 2n\pi, \; -\frac{\pi}{3} + 2n\pi$

gives $\theta = \frac{2n\pi}{3} \pm \frac{\pi}{9}$ $\quad * \left(\frac{\pi}{9}(6n \pm 1)\right)$

(iii) $8x^3 - 6x - 1 = 0$

Equivalent to $2(4x^3-3x) = 1$

$4x^3 - 3x = \frac{1}{2}$

let $\cos\theta = x$ then $4\cos^3\theta - 3\cos\theta = \frac{1}{2}$

equivalent to $\cos 3\theta = \frac{1}{2}$

So solutions from (ii)

$n=0$; $\theta = \pm\frac{\pi}{9}$ $\quad \cos\frac{\pi}{9} \; \left[= \cos\left(-\frac{\pi}{9}\right)\right]$

$n=1$; $\theta = \frac{5\pi}{9}$ and $\frac{7\pi}{9}$

Cubic has three solutions $\cos\frac{\pi}{9}, \cos\frac{5\pi}{9}, \cos\frac{7\pi}{9}$

$x_1 = \cos\frac{\pi}{9}$

$x_2 = \cos\frac{5\pi}{9}$

$x_3 = \cos\frac{7\pi}{9}$

(iv) If $\alpha = \cos\frac{\pi}{9}$ then $\frac{1}{\alpha^2} = \sec^2\frac{\pi}{9}$

ie $\alpha = a_1$ then $\frac{1}{\alpha^2} = \frac{1}{a_1^2} = X$

Requir polynomial with X as a zero, $\alpha = \pm\frac{1}{\sqrt{X}}$

Since α is a solution of $8a^3 - 6a - 1 = 0$

then $8\left(\frac{\pm 1}{\sqrt{x}}\right)^3 - 6\left(\frac{\pm 1}{\sqrt{x}}\right) - 1 = 0$

$[X \neq 0]$ $\quad \frac{8}{X\sqrt{X}} - \frac{6}{\sqrt{X}} - 1 = 0 \qquad \left[-\frac{8}{X\sqrt{X}} + \frac{6}{\sqrt{X}} - 1 = 0\right]$

$8 - 6X - X\sqrt{X} = 0$

$8 - 6X = X\sqrt{X} \qquad [-8 + 6X = X\sqrt{X}]$

So $\quad 64 - 96X + 36X^2 = X^3 \qquad [64 - 96X + 36X^2 = X^3]$

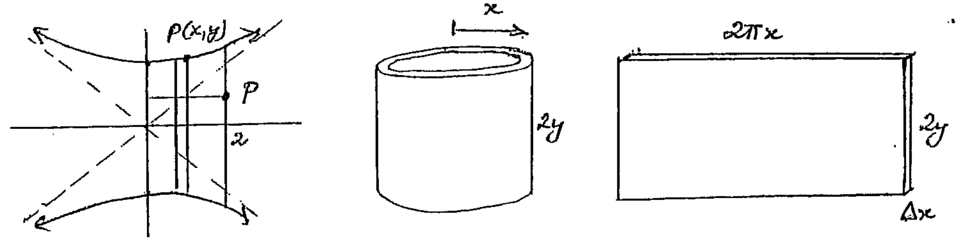
Required polynomial
$X^3 - 36X^2 + 96X - 64 = 0$

(v) Sum of roots of polynomial $-\frac{b}{a}$

$\therefore \sec^2\frac{\pi}{9} + \sec^2\frac{5\pi}{9} + \sec^2\frac{7\pi}{9} = 36.$

QUESTION 15:

(a)



Volume of shell is $\Delta V = 2\pi x . 2y \Delta x$

$= 4\pi x y \Delta x \quad \text{——— (1)}$

where $\frac{y^2}{9} - \frac{x^2}{4} = 1$

ie $\frac{y^2}{9} = 1 + \frac{x^2}{4}$

$\therefore y^2 = \frac{9}{4}(4 + x^2)$

$\therefore y = \frac{3}{2}\sqrt{4 + x^2}$

Then (1) $\Rightarrow \Delta V = 4\pi x . \frac{3}{2} . \sqrt{4 + x^2}\, \Delta x$

$= 6\pi x \sqrt{4 + x^2}\, \Delta x$

Then the volume of the solid is

$V = \lim_{\Delta x \to 0} \sum_{x=0}^{2} 6\pi x \sqrt{4 + x^2}\, \Delta x$

$= 6\pi \int_0^2 x\sqrt{4 + x^2}\, dx \qquad$ let $u = 4 + x^2$, $du = 2x\,dx$

$= 3\pi \int_0^2 2x\sqrt{4 + x^2}\, dx$

$= 3\pi \int_4^8 \sqrt{u}\, du$

$= 3\pi . \frac{2}{3}\left[\sqrt{u^3}\right]_4^8$

$= 2\pi\left[16\sqrt{2} - 8\right]$

$= 16\pi\left(2\sqrt{2} - 1\right)$ units^3

(b) (i) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$\Rightarrow \frac{y^2}{b^2} = 1 - \frac{x^2}{a^2}$

$y^2 = \frac{b^2}{a^2}(a^2 - x^2)$

ie $y = \frac{b}{a}\sqrt{a^2 - x^2}$

Then the area enclosed

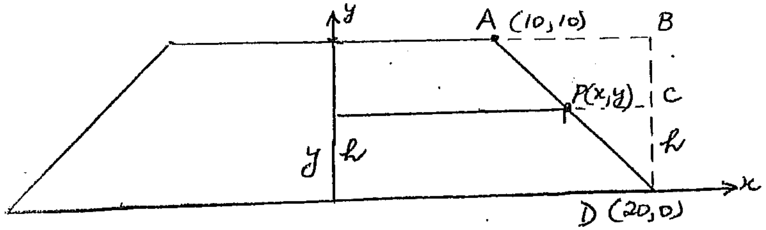
$= 4\int_0^a \frac{b}{a}.\sqrt{a^2-x^2}\,dx$

$= \frac{4b}{a}\cdot \underbrace{\int_0^a \sqrt{a^2-x^2}\,dx}_{\text{quadrant of a circle radius 'a'.}}$

$= \frac{4b}{a}\cdot\frac{1}{4}\cdot\pi a^2$

$= \pi ab$

(ii) (a) Front view



$\Delta ABD \parallel\!\!| \Delta PCD$ (equiangular)

$\therefore \frac{AB}{PC} = \frac{BD}{CD}$

$\therefore \frac{10}{20-x} = \frac{10}{h}$

$\therefore 20 - x = h$

$\therefore x = 20 - h$

The area of the ellipse at height h is

$A = \pi ab$ from (i)

$= \pi x.\frac{x}{2}$

$= \frac{\pi x^2}{2}$

$\therefore$ Volume of slice is

$\Delta V = \frac{\pi x^2}{2}\Delta h$

$= \frac{\pi}{2}(20-h)^2\Delta h$

$\therefore$ Volume of solid is

$V = \lim_{\Delta h \to 0}\sum_{h=0}^{10}\frac{\pi}{2}(20-h)^2\Delta h$

$= \frac{\pi}{2}\int_0^{10}(20-h)^2\,dh$

$= \frac{\pi}{2}\cdot\left[\frac{(20-h)^3}{-3}\right]_0^{10}$

$= \frac{\pi}{2}\left[\frac{10^3}{-3} - \frac{20^3}{-3}\right]$

$= \frac{\pi}{2}\left[\frac{20^3}{3} - \frac{10^3}{3}\right]$

$= \frac{3500\pi}{3}$ units^3

(c) (i) $xy = 4$

$= c^2$ where $c = 2$

$= \frac{1}{2}a^2$

$\therefore a^2 = 8$

$a = 2\sqrt{2} \quad (a > 0)$

$\therefore$ Foci are at (a, a) and $(-a, -a)$

i.e. $(2\sqrt{2}, 2\sqrt{2})$ and $(-2\sqrt{2}, -2\sqrt{2})$

(ii) (α) normal at P: $y = t^2x + \frac{2}{t} - 8$

cuts $y = x$ when

$x = t^2x + \frac{2 - 8t}{t}$

$x(t^2 - 1) = \frac{8t - 2}{t}$

$\therefore x = \frac{8t-2}{t(t^2-1)}$

$\therefore N \equiv \left(\frac{8t-2}{t(t^2-1)}, \frac{8t-2}{t(t^2-1)}\right)$

(β) Gradient of OP: $m_1 = \frac{\frac{2}{t}}{2t}$

$= \frac{1}{t^2}$

Gradient of PN : $m_2 = t^2$
(normal at P)

Let $P\hat{O}N = \alpha$ (angle between $y = x$ and OP)

Then $\tan\alpha = \left|\frac{1 - \frac{1}{t^2}}{1 + \frac{1}{t^2}}\right|$

$= \left|\frac{t^2-1}{t^2+1}\right|$

Let $P\hat{N}O = \theta$ (angle between $y = x$ and PN)

Then $\tan\theta = \left|\frac{1-t^2}{1+t^2}\right|$

$= \tan\alpha$

Hence $\theta = \alpha$

Then ΔPON is isosceles.

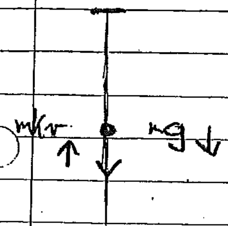
QUESTION 16:

(a) (i) Equation of motion : $F = mg - mkv$

[k positive constant of proportionality]

$\therefore m\ddot{x} = m(g - kv)$

$\ddot{x} = g - kv$



then $\frac{dv}{dt} = g - kv$

so $\frac{dt}{dv} = \frac{1}{g - kv}$

integrate with respect to v $\quad t = -\frac{1}{k}\int \frac{-k}{g-kv}\,dv$

$t = -\frac{1}{k}\ln(g - kv) + c$

when $t = 0$, $v = v_0$ $\therefore$ $0 = -\frac{1}{k}\ln(g - kv_0) + c$

$c = \frac{1}{k}\ln(g - kv_0)$

So $t = -\frac{1}{k}\left[\ln(g-kv) - \ln(g - kv_0)\right]$

$= -\frac{1}{k}\ln\left[\frac{g-kv}{g-kv_0}\right]$

gives $-kt = \ln\left[\frac{g-kv}{g-kv_0}\right]$

$e^{-kt} = \frac{\frac{g}{k} - v}{\ldots}$

$$e^{-kt}\left(\frac{g}{k} - v_0\right) = \frac{g}{k} - v$$

So $$v = \frac{g}{k} - \left(\frac{g}{k} - v_0\right) e^{-kt}$$

as required.

(ii) When $\ddot{x} = 0$, $g - kv = 0$

$v = \frac{g}{k}$ terminal velocity

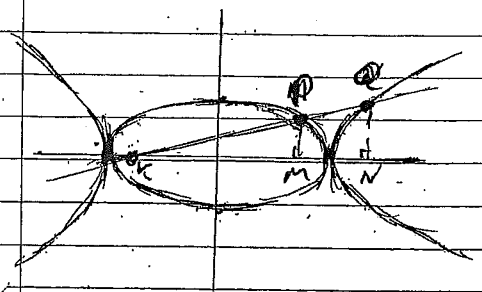
or As $t \to \infty$, $e^{-kt} \to 0$

$\therefore v = \frac{g}{k} - \left(\frac{g}{k} - v_0\right) \times 0$

ie terminal velocity $v = \frac{g}{k}$

(b) (i) $M(a\cos\theta, 0)$ and $N(a\sec\theta, 0)$

Since $\triangle KPM \,|||\, \triangle KQN$ (equiangular)



Corresponding sides in proportion

$\therefore \frac{KM}{KN} = \frac{MP}{NQ}$

$= \frac{b\sin\theta}{b\tan\theta}$

$= \cos\theta$

(ii) Let distance $OK = d$

then $KM = d + a\cos\theta$

$KN = d + a\sec\theta$

So $$\frac{KM}{KN} = \frac{d + a\cos\theta}{d + a\sec\theta} = \cos\theta$$

$$d + a\cos\theta = d\cos\theta + a$$

$$d(1 - \cos\theta) = a(1 - \cos\theta)$$

$$d = a$$

This gives $K(-a, 0)$

(iii) $$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

then $$\frac{2x}{a^2} + \frac{2y}{b^2}\frac{dy}{dx} = 0$$

So $$\frac{dy}{dx} = \frac{-b^2 x}{a^2 y}$$

at $P(a\cos\theta, b\sin\theta)$, gradient of tangent

$$m = \frac{-b^2(a\cos\theta)}{a^2(b\sin\theta)}$$

$$m = \frac{-b\cos\theta}{a\sin\theta}$$

Equation of tangent

$$y - b\sin\theta = \frac{-b\cos\theta}{a\sin\theta}\left(x - a\cos\theta\right)$$

$$y\sin\theta - b\sin^2\theta = -\frac{b}{a}\cos\theta . x + b\cos^2\theta$$

$$\frac{b\cos\theta}{a}x + y.\sin\theta = b\left(\sin^2\theta + \cos^2\theta\right)$$

$$\frac{x.\cos\theta}{a} + \frac{y\sin\theta}{b} = 1$$

Substitute $N(a\sec\theta, 0) \Rightarrow \frac{a.\sec\theta.\cos\theta}{a} + 0$

$$= 1$$

Tangent passes through N.

(iv) Substitute $M(a\cos\theta, 0)$

Then $\frac{a\cos\theta.\sec\theta}{a} - \frac{\cancel{0}\tan\theta}{b} = 1$

$$1 - 0 = 1$$

M lies on tangent.

Solving $\frac{x\cos\theta}{a} + \frac{y\sin\theta}{b} = 1$ ... (I)

$\frac{x\sec\theta}{a} - \frac{y\tan\theta}{b} = 1$ ... II

(I) $\times\tan\theta$ $\quad \frac{x}{a}\cos\theta\tan\theta + \frac{y}{b}\sin\theta.\tan\theta = \tan\theta$ $\quad +$

(II) $\sin\theta$ $\quad \frac{x}{a}\sin\theta.\sec\theta - \frac{y}{b}\sin\theta.\tan\theta = \sin\theta$

$$\frac{x}{a}\left[\sin\theta + \tan\theta\right] = \tan\theta + \sin\theta$$

$$\therefore \frac{x}{a} = 1$$

$$x = a$$

Gives A and T on line $x = a$, vertical
$\therefore \perp$ to x-axis.

(c) Sum $\sum_{r=1}^{n} r^3 = 1 + 8 + 27 + \ldots$

Let $n = 1$; LHS $1^3 = 1$ $\quad$ RHS $1^2(1+1)^2 = 4$

LHS < RHS

true for $n = 1$

Let Proposition be true for $n = k$,
k a positive integer,

then $1 + 8 + 27 + \ldots + k^3 < k^2(k+1)^2$

For next $n = k+1$,

$$1 + 8 + 27 + \ldots + k^3 + (k+1)^3$$

[from above

$$< k^2(k+1)^2 + (k+1)^3$$

$$= (k+1)^2\left[k^2 + k + 1\right]$$

$$< (k+1)^2\left[k^2 + k + 1 + 3k + 3\right]$$

Since $(k+1)^2(3k+3)$ has $(k+1)^2 > 0$
and $3(k+1) > 0$

$$= (k+1)^2(k^2+4k+4)$$

$$= (k+1)^2(k+2)^2$$

$$= (k+1)^2[(k+1)+1]^2 \quad \text{as required.}$$

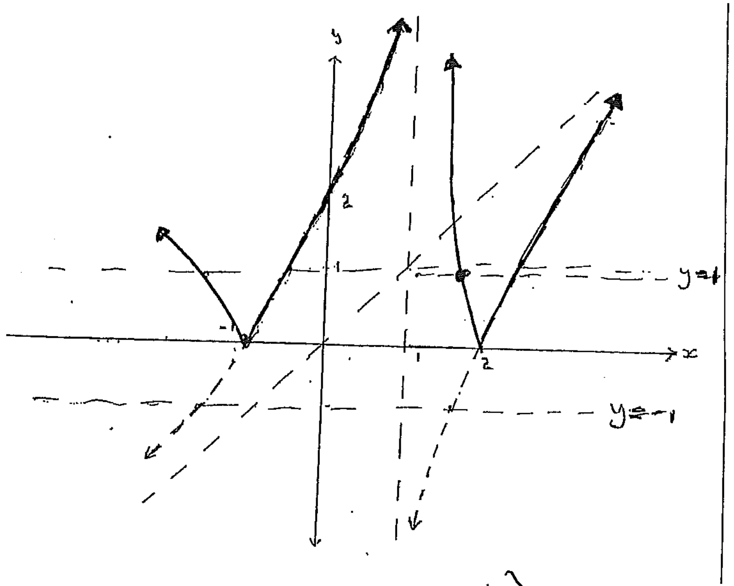
So, if true for $n=k$ we have shown it is true for next $n=k+1$.

Since true for $n=1$ then it is true for $n=2$ and by induction, true for all n.

Template Question 12

$y = |f(x)|$

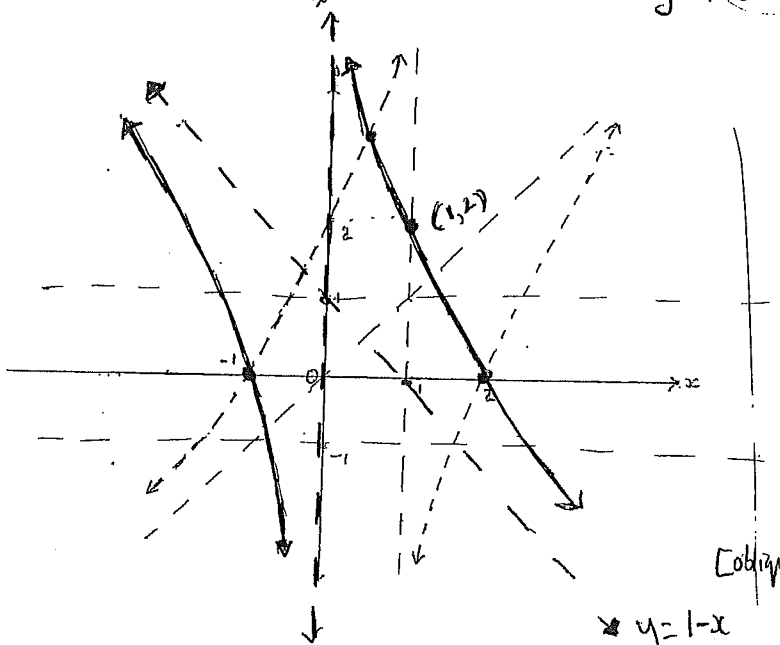
(α)



Template Question 12

$y = f(1-x)$

(β)

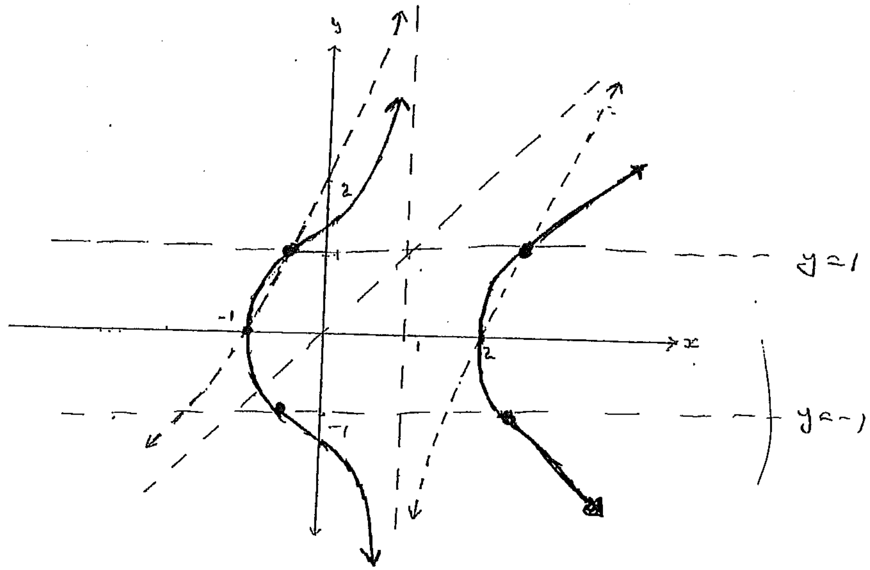


x - /

Template Question 12

$y^2 = f(x)$

(γ)

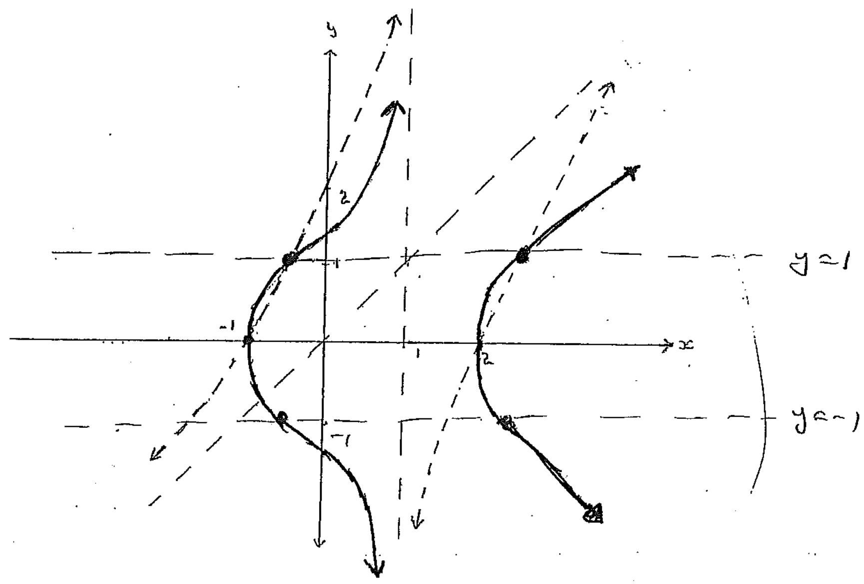


$f(x+1)$, $x = -1$

Template Question 12

$y^2 = f(x)$

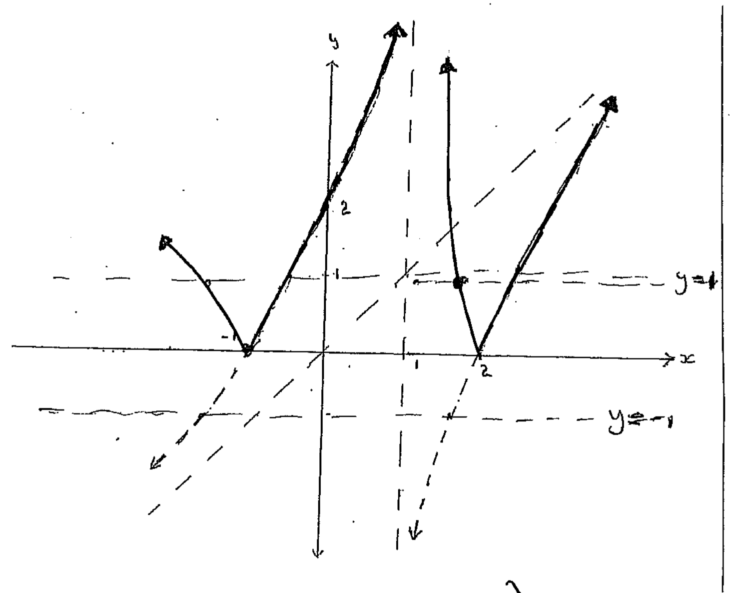
(γ)



Template Question 12

$y = |f(x)|$

(α)



Template Question 12

$y = f(1-x)$

(β)