ST ANDREW'S CATHEDRAL SCHOOL

Founded 1885

MATHEMATICS

EXTENSION 1

H.S.C. COURSE

2010 ASSESSMENT TASK 2

(Weighting: 20%)

Time Allowed: 60 minutes

Instructions

* Attempt all questions.
* All necessary working must be shown in all questions.
* Approved calculators and templates may be use

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE: $\ln x = \log_e x, \quad x > 0$

Marks

1. Find $\frac{d}{dx}\sin^{-1}5x$ — 1

2. Find $\int \cos^2 6x\,dx$ — 2

3. Find $\int \frac{1}{9+4x^2}\,dx$ — 2

4. Find the exact value of $\sin\left[\cos^{-1}\left(-\frac{2}{3}\right)\right]$ — 2

5. (a) State the domain and range of $y = 2\cos^{-1}(x-1)$. — 2

   (b) Hence sketch the curve $y = 2\cos^{-1}(x-1)$. — 1

6. Write, in terms of π, the general solution of $2\sin 2\theta + 1 = 0$. — 2

7. Find the term independent of x in the expansion of $x^3\left(3x^2 - \frac{2}{x}\right)^9$.
   (Give your answer as an integer.) — 3

8. Use the substitution $u = \cos^2 x$ to evaluate $\int_{\frac{\pi}{4}}^{\frac{2\pi}{3}} \frac{\cos x \sin x}{1+\cos^2 x}\,dx$ — 3
   (Give your answer in its simplest exact form.)

9. Newton's Law of Cooling states that the rate of cooling of a body is proportional to the excess of the temperature of the body ($T^\circ C$) over the surrounding temperature ($S^\circ C$).
   That is $\frac{dT}{dt} = -k(T-S)$ where k is a constant and $k > 0$.

   (a) If A is a constant, show that $T = S + Ae^{-kt}$ satisfies Newton's Law of Cooling. — 1

   (b) A cup of coffee with a temperature of $100^\circ C$ was too hot to drink. Two minutes later, the temperature had dropped to $94^\circ C$. If the surrounding temperature was $24^\circ C$, find the values of A and k. — 2

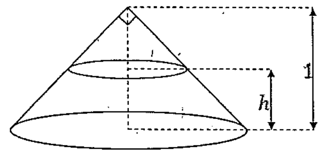
   (c) The coffee was drinkable when the temperature has dropped to $80^\circ C$. How long, to the nearest minute, did it take for the coffee to be drinkable ? — 2

10. A particle is moving in a horizontal straight line. At time t seconds, the displacement of the particle from a fixed point O is x metres, its velocity is $v\,ms^{-1}$ and its acceleration is $a\,ms^{-2}$. Given $a = 2x(4-x^2)$ and initially the particle is 2cm to the right of O and is travelling at a speed of $3\,ms^{-1}$:

    (a) find an expression for v in terms of x. — 3

    (b) hence find a possible set of values for x. — 2

11.



A closed right hollow cone has a height 1 metre and semi-vertical angle of 45°. The cone stands with its base on a horizontal surface. Water is poured into the cone through a hole in its apex at the constant rate of $0.1\,m^3$ per minute.

   (a) Show that when the depth of water in the cone is h metres (where $0 < h < 1$), the volume of water V m^3 in the cone is given by $V = \frac{\pi}{3}(h^3 - 3h^2 + 3h)$. — 2

   (b) Find the rate at which the depth of water is increasing when the depth is 0.6 m.
   (Give your answer in its simplest exact form.) — 3

12. (a) Use the Binomial Theorem to write the expansion of $(1-x)^{2n}$. — 1

    (b) Hence show that $\sum_{k=1}^{2n} \frac{(-1)^{k+1}}{k+1}\,{}^{2n}C_k = \frac{2n}{2n+1}$. — 3

13. (a) Show that $\frac{d}{dx}\frac{x}{\sqrt{1-x^2}} = (1-x^2)^{-\frac{3}{2}}$ — 1

    (b) Hence find $\frac{d}{dx}\tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)$ — 2

    (c) Hence deduce that $\tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right) = \sin^{-1}x$ for $0 \le x < 1$. — 2

1.

$$\frac{d}{dx}\sin^{-1}5x$$
$$= \frac{\frac{d}{dx}5x}{\sqrt{1-(5x)^2}}$$
$$= \frac{5}{\sqrt{1-25x^2}}$$

$\boxed{1}$

2.

$$\int \cos^2 6x\,dx$$
$$= \int \tfrac{1}{2}(1+\cos 12x)\,dx$$
$$= \tfrac{1}{2}\left[x + \frac{\sin 12x}{12}\right] + c$$
$$= \frac{x}{2} + \frac{\sin 12x}{24} + c$$

$\boxed{2}$

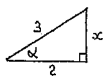
3.

$$\int \frac{1}{9+4x^2}\,dx$$
$$= \frac{1}{4}\int \frac{1}{\frac{9}{4}+x^2}\,dx$$
$$= \frac{1}{4}\int \frac{1}{\left(\frac{3}{2}\right)^2+x^2}\,dx$$
$$= \frac{1}{4}\times\frac{1}{\frac{3}{2}}\tan^{-1}\left(\frac{x}{\frac{3}{2}}\right)+c$$
$$= \frac{1}{6}\tan^{-1}\frac{2x}{3} + c$$

$\boxed{2}$

4.

<u>let</u> $\alpha = \cos^{-1}\frac{2}{3}$



$x^2 = 3^2 - 2^2$

$x = \sqrt{5}$

$$\sin\left[\cos^{-1}\left(-\tfrac{2}{3}\right)\right]$$
$$= \sin\left[\pi - \cos^{-1}\tfrac{2}{3}\right]$$
$$= \sin(\pi-\alpha)$$
$$= \sin\alpha$$
$$= \frac{\sqrt{5}}{3}$$

$\boxed{2}$

5. a/

$$y = 2\cos^{-1}(x-1)$$

<u>Domain</u>: $-1 \le x-1 \le 1$

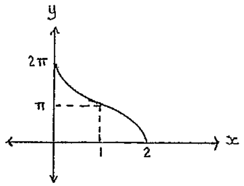
$0 \le x \le 2$

<u>Range</u>: $0 \le \cos^{-1}(x-1) \le \pi$

$0 \le 2\cos^{-1}(x-1) \le 2\pi$

$\boxed{2}$

5. b/



$\boxed{1}$

6.

$$2\sin 2\theta + 1 = 0$$
$$\sin 2\theta = -\tfrac{1}{2}$$

$$2\theta = n\pi + (-1)^n\sin^{-1}\left(-\tfrac{1}{2}\right)$$
$$= n\pi + (-1)^n \times -\sin^{-1}\tfrac{1}{2}$$
$$= n\pi + (-1)^{n+1}\frac{\pi}{6}$$
$$\therefore\ \theta = \frac{n\pi}{2} + (-1)^{n+1}\frac{\pi}{12}$$

(where n is an integer)

$\boxed{2}$

7.

<u>for</u> $x^3\left(3x^2 - \frac{2}{x}\right)^9$

$$T_{r+1}$$
$$= x^3 \times {}^9C_r \times (3x^2)^{9-r} \times \left(-\frac{2}{x}\right)^r$$
$$= x^3 \times {}^9C_r \times 3^{9-r} \times x^{18-2r} \times (-2)^r \times x^{-r}$$
$$= {}^9C_r \times 3^{9-r} \times (-2)^r \times x^{21-3r}$$

for term independent of x

$21 - 3r = 0$

$r = 7$

$$T_8 = {}^9C_7 \times 3^2 \times (-2)^7$$
$$= -41472$$

$\boxed{3}$

8.

$$u = \cos^2 x$$
$$\frac{du}{dx} = -2\cos x.\sin x$$

<u>when $x = \frac{2\pi}{3}$</u>

$u = \cos^2\left(\frac{2\pi}{3}\right)$

$= \left(-\frac{1}{2}\right)^2$

$= \frac{1}{4}$

<u>when $x = \frac{\pi}{4}$</u>

$u = \cos^2\frac{\pi}{4}$

$= \left(\frac{1}{\sqrt{2}}\right)^2$

$= \frac{1}{2}$

$$\int_{\frac{\pi}{4}}^{\frac{2\pi}{3}} \frac{\cos x\sin x}{1+\cos^2 x}\,dx$$
$$= -\frac{1}{2}\int_{\frac{\pi}{4}}^{\frac{2\pi}{3}} \frac{-2\cos x.\sin x}{1+\cos^2 x}\,dx$$
$$= -\frac{1}{2}\int_{\frac{1}{2}}^{\frac{1}{4}} \frac{du}{1+u}$$
$$= \frac{1}{2}\int_{\frac{1}{4}}^{\frac{1}{2}} \frac{du}{1+u}$$
$$= \frac{1}{2}\Big[\log_e(1+u)\Big]_{\frac{1}{4}}^{\frac{1}{2}}$$
$$= \frac{1}{2}\left[\log_e\tfrac{3}{2} - \log_e\tfrac{5}{4}\right]$$
$$= \frac{1}{2}\log_e\left(\tfrac{3}{2}\div\tfrac{5}{4}\right)$$
$$= \frac{1}{2}\log_e\frac{6}{5}$$

$\boxed{3}$

9. a/

$$T = S + Ae^{-kt} \quad \text{——①}$$
$$\frac{dT}{dt} = -k \times Ae^{-kt} \quad \text{——②}$$

from ①

$$T - S = Ae^{-kt} \quad \text{——③}$$

sub ③ into ②

$$\frac{dT}{dt} = -k(T-S)$$

$\boxed{1}$

9. b/

$$T = 24 + Ae^{-kt}$$

<u>when $t=0$, $T=100$</u>

$100 = 24 + Ae^0$

$76 = A$

$\Rightarrow\ T = 24 + 76e^{-kt}$

<u>when $t=2$, $T=94$</u>

$94 = 24 + 76e^{-2k}$

$\frac{70}{76} = e^{-2k}$

$\log_e\left(\frac{35}{38}\right) = -2k$

$k = -\frac{1}{2}\log_e\left(\frac{35}{38}\right)$

$$\Rightarrow\ T = 24 + 76e^{\frac{1}{2}\log_e\left(\frac{35}{38}\right)}$$

$\boxed{2}$

9. c/

<u>when $T = 80$</u>

$$80 = 24 + 76e^{\frac{t}{2}\log_e\left(\frac{35}{38}\right)}$$
$$\frac{56}{76} = e^{\frac{t}{2}\log_e\left(\frac{35}{38}\right)}$$
$$\log_e\left(\frac{14}{19}\right) = \frac{t}{2}\log_e\left(\frac{35}{38}\right)$$
$$t = 2\log_e\left(\frac{14}{19}\right) \div \log_e\left(\frac{35}{38}\right)$$

$t = 7$ minutes

(to nearest minute)

$\boxed{2}$

10. a/

$$a = 2x(4-x^2)$$
$$\frac{d}{dx}\left(\tfrac{1}{2}v^2\right) = 8x - 2x^3$$
$$\tfrac{1}{2}v^2 = \int(8x-2x^3)\,dx$$
$$\tfrac{1}{2}v^2 = 4x^2 - \frac{x^4}{2} + c$$

<u>when $t=0$, $x=2$, $|v|=3$</u>

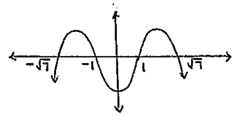
$$\frac{9}{2} = 16 - 8 + c$$
$$c = -\frac{7}{2}$$
$$\therefore\ \tfrac{1}{2}v^2 = 4x^2 - \frac{x^4}{2} - \frac{7}{2}$$
$$v^2 = 8x^2 - x^4 - 7$$
$$v = \pm\sqrt{8x^2 - x^4 - 7}$$

$\boxed{3}$

10. b/

since $v^2 \ge 0$

$$8x^2 - x^4 - 7 \ge 0$$
$$-(x^4 - 8x^2 + 7) \ge 0$$
$$-(x^2-7)(x^2-1) \ge 0$$
$$-(x-\sqrt{7})(x+\sqrt{7})(x-1)(x+1) \ge 0$$

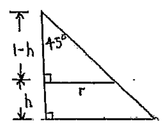


since initially particle is 2cm to the right of 0

$$1 \le x \le \sqrt{7}$$

$\boxed{2}$

11. a/



$$\tan 45^\circ = \frac{r}{1-h}$$
$$1 = \frac{r}{1-h}$$
$$r = 1-h$$

(since $V = \frac{1}{3}\pi r^2 h$ for cone)

<u>volume of water</u>

$$V = \tfrac{1}{3}\pi \times 1^2 \times 1 - \tfrac{1}{3}\pi \times (1-h)^2 \times (1-h)$$
$$= \tfrac{1}{3}\pi\left[1^3 - (1-h)^3\right]$$
$$= \tfrac{1}{3}\pi\left[1 - (1 - 3h + 3h^2 - h^3)\right]$$
$$= \tfrac{1}{3}\pi\left(h^3 - 3h^2 + 3h\right)$$

$\boxed{2}$

11. b/

$$V = \tfrac{1}{3}\pi\left(h^3 - 3h^2 + 3h\right)$$

<u>since</u> $\frac{dV}{dt} = \frac{dV}{dh} \times \frac{dh}{dt}$

$$\frac{dV}{dt} = \tfrac{1}{3}\pi(3h^2 - 6h + 3) \times \frac{dh}{dt}$$
$$= \pi(h^2 - 2h + 1) \times \frac{dh}{dt}$$

given $\frac{dV}{dt} = 0.1$, hence <u>when $h = 0.6$</u>

$$0.1 = \pi(0.36 - 1.2 + 1) \times \frac{dh}{dt}$$
$$\frac{dh}{dt} = \frac{0.1}{0.16\pi}$$
$$= \frac{0.625}{\pi}$$
$$= \frac{5}{8\pi}$$

$\therefore$ depth of water is increasing at $\frac{5}{8\pi}$ m/min.

$\boxed{3}$

13. a/

$$\frac{d}{dx}\frac{x}{\sqrt{1-x^2}}$$
$$= \frac{d}{dx}\,x(1-x^2)^{-\frac{1}{2}}$$
$$= x \times \frac{d}{dx}(1-x^2)^{-\frac{1}{2}} + (1-x^2)^{-\frac{1}{2}} \times \frac{d}{dx}x$$
$$= x \times -\tfrac{1}{2}(1-x^2)^{-\frac{3}{2}} \times -2x + (1-x^2)^{-\frac{1}{2}} \times 1$$
$$= (1-x^2)^{-\frac{3}{2}}\left[x^2 + (1-x^2)\right]$$
$$= (1-x^2)^{-\frac{3}{2}}$$

$\boxed{1}$

13. b/

$$\frac{d}{dx}\tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)$$
$$= \frac{\frac{d}{dx}\frac{x}{\sqrt{1-x^2}}}{1 + \left(\frac{x}{\sqrt{1-x^2}}\right)^2}$$
$$= (1-x^2)^{-\frac{3}{2}} \div \left(1 + \frac{x^2}{1-x^2}\right)$$
$$= (1-x^2)^{-\frac{3}{2}} \div \frac{1-x^2+x^2}{1-x^2}$$
$$= (1-x^2)^{-\frac{3}{2}} \times \frac{1-x^2}{1}$$
$$= (1-x^2)^{-\frac{1}{2}}$$
$$= \frac{1}{\sqrt{1-x^2}}$$

$\boxed{2}$

13. c/

from part b/

$$\int\frac{1}{\sqrt{1-x^2}}\,dx = \tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right) + c_1$$

<u>but</u>

$$\int\frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1}x + c_2$$

<u>hence</u>

$$\tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right) + c_1 = \sin^{-1}x + c_2$$
$$\tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right) = \sin^{-1}x + c_3$$

<u>when $x=0$</u>

$$\tan^{-1}0 = \sin^{-1}0 + c_3$$
$$0 = 0 + c_3$$
$$\therefore\ \tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right) = \sin^{-1}x$$

for $0 \le x < 1$

$\boxed{2}$

m.v.

12. a)

$$(1-x)^{2n} = {}^{2n}C_0 + {}^{2n}C_1 \times (-x) + {}^{2n}C_2 \times (-x)^2 + \ldots\ldots + {}^{2n}C_k \times (-x)^k + \ldots\ldots + {}^{2n}C_{2n} \times (-x)^{2n}$$

$$= {}^{2n}C_0 + (-1) \times {}^{2n}C_1 x + (-1)^2 \times {}^{2n}C_2 x^2 + \ldots\ldots + (-1)^k \times {}^{2n}C_k x^k + \ldots\ldots + (-1)^{2n} \times {}^{2n}C_{2n} x^{2n}$$

1

12. b)

$$\int_0^1 (1-x)^{2n}\,dx = \int_0^1 \left[{}^{2n}C_0 + (-1) x\, {}^{2n}C_1 x + (-1)^2 x\, {}^{2n}C_2 x^2 + \ldots\ldots + (-1)^k x\, {}^{2n}C_k x^k + \ldots\ldots + (-1)^{2n} x\, {}^{2n}C_{2n} x^{2n} \right] dx$$

$$\left[\frac{(1-x)^{2n+1}}{(2n+1) \times -1} \right]_0^1 = \left[{}^{2n}C_0 x + (-1) x\, {}^{2n}C_1 \frac{x^2}{2} + (-1)^2 x\, {}^{2n}C_2 \frac{x^3}{3} + \ldots\ldots + (-1)^k x\, {}^{2n}C_k \frac{x^{k+1}}{k+1} + \ldots\ldots + (-1)^{2n} x\, {}^{2n}C_{2n} \frac{x^{2n+1}}{2n+1} \right]_0^1$$

$$\frac{0-1}{-(2n+1)} = \left[{}^{2n}C_0 + (-1) \times {}^{2n}C_1 \times \frac{1}{2} + (-1)^2 \times {}^{2n}C_2 \times \frac{1}{3} + \ldots\ldots + (-1)^k \times {}^{2n}C_k \times \frac{1}{k+1} + \ldots\ldots + (-1)^{2n} \times {}^{2n}C_{2n} \times \frac{1}{2n+1} \right] - 0$$

$$\frac{1}{2n+1} - {}^{2n}C_0 = \frac{-1}{2} \times {}^{2n}C_1 + \frac{(-1)^2}{3} \times {}^{2n}C_2 + \ldots\ldots + \frac{(-1)^k}{k+1} \times {}^{2n}C_k + \ldots\ldots + \frac{(-1)^{2n}}{2n+1} \times {}^{2n}C_{2n}$$

$$\frac{1}{2n+1} - 1 = \sum_{k=1}^{2n} \frac{(-1)^k}{k+1}\, {}^{2n}C_k$$

$$\frac{1-(2n+1)}{2n+1} = \sum_{k=1}^{2n} \frac{(-1)^k}{k+1}\, {}^{2n}C_k$$

$$\frac{-2n}{2n+1} = \sum_{k=1}^{2n} \frac{(-1)^k}{k+1}\, {}^{2n}C_k$$

$$\therefore \sum_{k=1}^{2n} \frac{(-1)^{k+1}}{k+1}\, {}^{2n}C_k = \frac{2n}{2n+1}$$

3