ST ANDREW'S CATHEDRAL SCHOOL
Founded 1885

YEAR 12 SEMESTER ONE EXAMINATION 2010

EXTENSION 2 MATHEMATICS

This Examination Paper does not necessarily reflect the content or format of the final Higher School Certificate Examination paper for this subject.

INSTRUCTIONS

TIME ALLOWED: 2 HOURS PLUS 5 MINUTES READING TIME

- Attempt **all** questions.
- **All** necessary working must be shown in **all** questions.
- Marks may **not** be awarded for careless **or** badly arranged work.
- **All** questions are of equal value.
- Start **each** question in a new booklet.
- Approved calculators and templates **may** be used.

Question 1 (27 Marks) START NEW BOOKLET.

(a) Let $z = -\sqrt{3} + i$

(i) Write z in modulus-argument form 2

(ii) Hence find z^8 in the form $x + iy$ where x and y are real. 2

(iii) Find the least positive integer value of n such that z^n is real. 1

(b) Solve $x^2 - 3ix + 4 = 0$ 2

(c) If $z = r(\cos\theta + i\sin\theta)$, show that $\dfrac{z}{z^2 + r^2}$ is purely real, and give its value. 3

(d) Describe, in geometric terms, the locus (in the Argand Diagram) represented by : 4

$$2|z| = z + \bar{z} + 4$$

(e) Sketch on the Argand Diagram $|z - 4 - 3i| = 1$
Hence or otherwise determine the least value of mod z 4

(f) $OABC$ is a square. O represents the origin, A represents $\sqrt{3} + i$, B represents z and lies in the first quadrant, C represents w and D is the point where the diagonals meet.

(i) Represent this information on the Argand Diagram. 1

(ii) Find the complex numbers represented by C and D in the form $x + iy$ 2

(iii) Find $\arg\left(\dfrac{w}{z}\right)$ 2

g) z is a complex number such that $|z - 3| + |z + 3| = 10$. Describe the locus of z and find the cartesian equation of this locus. 4

Question 2 (21 Marks) START NEW BOOKLET.

(a) The equation $x^3 - 3x + 3 = 0$ has roots which are α, β and γ . Find the equation in x where the roots are α^2, β^2 and γ^2. **3**

(b) If the polynomial $P(x)$ has a zero of multiplicity n at $x = a$, show that its derivative $P'(x)$ will have a zero of multiplicity $n-1$ at $x = a$. **3**

(c) Solve $x^5 + 2x^4 - 2x^3 - 8x^2 - 7x - 2 = 0$ if it has a root of multiplicity of 4. **3**

(d) Consider the polynomial $P(x) = x^4 - 4x^3 + 11x^2 - 14x + 10$. **6**

(i) If $P(x)$ has roots $a + bi, a - 2bi$ (where a,b are real) find the values of a and b.

(ii) Hence, find the zeros of $P(x)$ over the complex field and express $P(x)$ as the product of two quadratic factors.

(e) Express $\cos 2\theta$ and $\cos 3\theta$ in terms of $\cos\theta$, and show that the equation $\cos 3\theta = \cos 2\theta$ can be expressed as $4x^3 - 2x^2 - 3x + 1 = 0$ where $x = \cos\theta$.

By solving this equation for x, find the exact value of $\cos\frac{2\pi}{5}$ **6**

Question 3 (22 Marks) START NEW BOOKLET.

(a) The ellipse E has equation $\frac{x^2}{4} + \frac{y^2}{3} = 1$. P is a point on E.

(i) Calculate the eccentricity and write down the coordinates of the foci S and S'. **2**

(ii) Write down the equation of each directrix. **1**

(iii) Sketch E showing all important features. **2**

(b) Derive the equation of the tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at the point $P(a\cos\theta, b\sin\theta)$ **3**

(c) The point $P(a\cos\theta, b\sin\theta)$ lies on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. **6**

Q is the foot of the perpendicular from P to the x-axis. The normal at P cuts the x-axis at N.

Show that the length of NQ is: $\left|\frac{b^2\cos\theta}{a}\right|$

Question 3 (continued)

(d) $T(x_1, y_1)$ is a point on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ with centre O. A line MN drawn through O parallel to the tangent to the ellipse at T meets the ellipse at M and N. **8**

(i) Show that the coordinates of M and N are respectively $\left(\frac{-ay_1}{b}, \frac{bx_1}{a}\right)$ and $\left(\frac{ay_1}{b}, \frac{-bx_1}{a}\right)$

(ii) Hence prove that the area of the triangle TMN is independent of the position of T.

Question 4 (10 Marks) START NEW BOOKLET.

(a) Prove by mathematical induction that: $(a+b)^n \geq a^n + b^n$ for $n \geq 1$ where $a > 0$ and $b > 0$. **5**

(b) If $\frac{1}{a}, \frac{1}{b}$ and $\frac{1}{c}$ form an arithmetic progression, then the numbers a,b,c are said be in harmonic progression and b is said to be the harmonic mean of a and c.

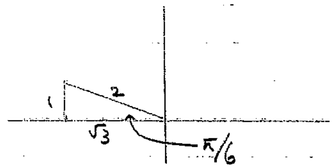
(i) Show that 2,3 and 6 are in harmonic progression. **1**

(ii) Show that the harmonic mean of a and c is equal to $\frac{2ac}{a+c}$ **2**

(iii) If $a > 0, c > 0$ show that $\frac{2ac}{a+c} \leq \sqrt{ac}$ **3**

Solutions to Semester 1. Ex. 2. Exam.

(a)



(i) $z = 2\,\text{cis}\,\frac{5\pi}{6}$

(ii) $z^8 = 2^8\,\text{cis}\,\frac{40\pi}{6}$

$= 2^8\,\text{cis}\,\frac{2\pi}{3}$

$= -128 + 128\sqrt{3}\,i$

(iii) $z^6 = 2^6\,\text{cis}\,5\pi$

$= -2^6$

$\therefore n = 6$

(b) $x^2 - 3ix + 4 = 0$

$x = \dfrac{3i \pm \sqrt{-9-16}}{2}$

$= \dfrac{3i \pm 5i}{2}$

$\therefore x = 4i \quad \text{or} \quad x = -i$

(c) $z^2 = r^2(\cos 2\theta + i \sin 2\theta)$

$z^2 + r^2 = r^2(\cos 2\theta + 1 + i \sin 2\theta)$

$= r^2(2\cos^2\theta + 2i\sin\theta\cos\theta)$

$= 2r^2(\cos\theta + i\sin\theta)\cos\theta$

$\therefore \dfrac{z}{z^2+r^2} = \dfrac{r(\cos\theta + i\sin\theta)}{2r^2(\cos\theta + i\sin\theta)\cos\theta}$

$= \dfrac{1}{2r\cos\theta}$

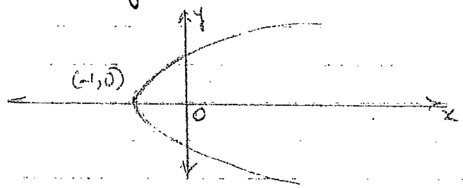
$\therefore$ real.

(d) $2\sqrt{x^2+y^2} = x + iy + x - iy + 4$

$\sqrt{x^2+y^2} = x + 2$

$x^2 + y^2 = x^2 + 4x + 4$

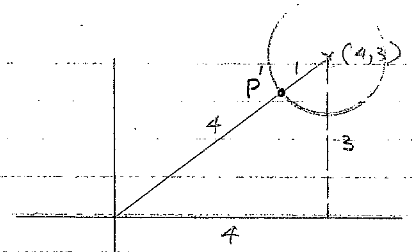
$y^2 = 4(x+1)$



Parabola, vertex at $(-1,0)$

domain : $x \geq -1$.

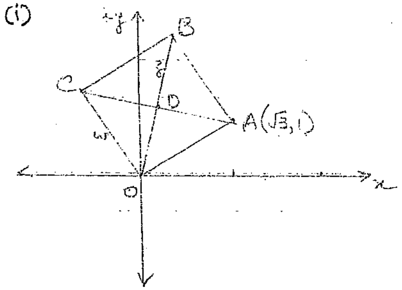
(e)



Least value will occur at P'.

$\therefore |z| = 4$.

(f) (i)



(ii) $w = i(\sqrt{3} + i)$

$= -1 + i\sqrt{3}$

$z = \sqrt{3} + i + -1 + i\sqrt{3}$

$= (\sqrt{3} - 1) + (\sqrt{3} + 1)i$

$\therefore C \rightarrow -1 + i\sqrt{3}$

$D \rightarrow \left(\dfrac{\sqrt{3}-1}{2}\right) + \left(\dfrac{\sqrt{3}+1}{2}\right)i$

(iii) $\arg \dfrac{w}{z} = \arg w - \arg z$

$= \dfrac{\pi}{4}$

(Since $\angle AOC = \pi/2$ and OB is a diagonal bisecting $\angle AOC$)

g) $|z-3| + |z+3| = 10$ means

the sum of the distances from $(3,0)$ and $(-3,0)$ is 10 units.

This is an ellipse where $(3,0)$ & $(-3,0)$ are the foci and $2a = 10 \rightarrow a = 5$, the length of the semi major axis. Now $ae = 3$

$\therefore e = \frac{3}{5}$

$\therefore \dfrac{b^2}{a^2} = 1 - \dfrac{9}{25}$

$b^2 = 25\left(\dfrac{25-9}{25}\right)$

$\therefore b^2 = 16$

$\therefore$ The equation is

$\dfrac{x^2}{25} + \dfrac{y^2}{16} = 1$

OR Algebraically.

$\sqrt{(x-3)^2+y^2} + \sqrt{(x+3)^2+y^2} = 10$

$\sqrt{(x-3)^2+y^2} = 10 - \sqrt{(x+3)^2+y^2}$

$(x-3)^2 + y^2 = 100 - 20\sqrt{(x+3)^2+y^2} + (x+3)^2 + y^2$

$20\sqrt{(x+3)^2+y^2} = 100 + (x+3)^2 - (x-3)^2$

$20\sqrt{(x+3)^2+y^2} = 100 + 12x$

$5\sqrt{(x+3)^2+y^2} = 25 + 3x$

$25\left((x+3)^2+y^2\right) = (25+3x)^2$

$25\left((x+3)^2+y^2\right) = 625 + 150x + 9x^2$

$25x^2 + 150x + 225 + 25y^2 = 625 + 150x + 9x^2$

$16x^2 + 25y^2 = 400$

$\therefore \dfrac{x^2}{25} + \dfrac{y^2}{16} = 1$

a)

If α is a root of $x^3-3x+3=0$ then α^2 is a root of:

$$x\sqrt{x}-3\sqrt{x}+3=0$$
$$\sqrt{x}(x-3)=-3$$
$$x(x-3)^2=9$$
$$x(x^2-6x+9)=9$$

$x^3-6x^2+9x-9=0$ is the required equation.

b) $P(x)=(x-a)^n Q(x)$

$$P'(x)=(x-a)^n Q'(x)+n(x-a)^{n-1}Q(x)$$
$$=(x-a)^{n-1}\{(x-a)Q'(x)+nQ(x)\}$$

Now $M(x)=(x-a)Q'(x)+nQ(x)$

$$\therefore P'(x)=(x-a)^{n-1}M(x)$$

$\therefore P'(x)$ will have a zero of multiplicity $n-1$ at $x=a$

c) $P(x)=x^5+2x^4-2x^3-8x^2-7x-2$

$$P'(x)=5x^4+8x^3-6x^2-16x-7$$
$$P''(x)=20x^3+24x^2-12x-16$$
$$P'''(x)=60x^2+48x-12$$
$$=12(5x-1)(x+1)$$

$\therefore x=\frac{1}{5}$ or $x=-1$

$P(-1)=0$

$\therefore P(x)=(x+1)^4(x+k)$

by inspection $k=-2$

i.e. $P(x)=(x+1)^4(x-2)$

$\therefore x=-1,-1,-1,-1,2$

d) $P(x)=x^4-4x^3+11x^2-14x+10$

Let $\alpha=a+bi \rightarrow \beta=a-bi$

Let $\gamma=a+2bi \rightarrow \delta=a-2bi$

$\Sigma\alpha=4a \quad \therefore 4a=4 \rightarrow a=1$

$\alpha\beta\gamma\delta=(1+bi)(1-bi)(1+2bi)(1-2bi)=10$

$$(1+b^2)(1+4b^2)=10$$
$$1+5b^2+4b^4=10$$
$$4b^4+5b^2-9=0$$
$$(4b^2+9)(b^2-1)=0$$
$$\therefore b=\pm 1$$

(ii) $\alpha=1+i \quad \beta=1-i$

$\gamma=1+2i \quad \delta=1-2i$

$\therefore (x^2-2x+2)(x^2-2x+5)$ are the two quadratic factors.

(e) $\cos 2\theta=2\cos^2\theta-1$

$$\cos 3\theta=\cos(2\theta+\theta)$$
$$=\cos 2\theta\cos\theta-\sin 2\theta\sin\theta$$
$$=(2\cos^2\theta-1)\cos\theta-2\sin^2\theta\cos\theta$$
$$=2\cos^3\theta-\cos\theta-2\cos\theta(1-\cos^2\theta)$$
$$=2\cos^3\theta-\cos\theta-2\cos\theta+2\cos^3\theta$$
$$=4\cos^3\theta-3\cos\theta$$

$\therefore$ If $\cos 3\theta=\cos 2\theta$ then

$$4\cos^3\theta-3\cos\theta=2\cos^2\theta-1$$
$$4\cos^3\theta-2\cos^2\theta-3\cos\theta+1=0$$

when $x=\cos\theta$

$$4x^3-2x^2-3x+1=0$$

)

$$P(x)=4x^3-2x^2-3x+1$$
$$P(1)=4-2-3+1$$
$$=0$$

$$\begin{array}{r}
4x^2+2x-1 \\
x-1\,\overline{)\,4x^3-2x^2-3x+1} \\
\underline{4x^3-4x^2} \\
2x^2-3x+1 \\
\underline{2x^2-2x} \\
-x+1 \\
\underline{-x+1} \\
\end{array}$$

$\therefore P(x)=(x-1)(4x^2+2x-1)$

Consider $4x^2+2x-1=0$

$$x=\frac{-2\pm\sqrt{4+4\times4}}{8}$$
$$=\frac{-2\pm\sqrt{20}}{8}$$
$$=\frac{-1\pm\sqrt{5}}{4}$$

$\therefore \cos\theta=+1$ or $\cos\theta=\frac{-1+\sqrt{5}}{4}$

$\cos\theta=\frac{-1-\sqrt{5}}{4}$

when $\theta=\frac{2\pi}{5}$ $\quad \cos 2\theta=\cos 4\pi/5$

$=-\cos\pi/5$

$\cos 3\theta=\cos\frac{6\pi}{5}$

$=-\cos\pi/5$

$\therefore \theta=\frac{2\pi}{5}$ is a solution

Now $\cos\frac{2\pi}{5}=\frac{-1+\sqrt{5}}{4}$

since $\frac{-1+\sqrt{5}}{4}>0$.

3 a)

(i) $\frac{b^2}{a^2}=1-e^2$

$\frac{3}{4}=1-e^2 \rightarrow e=\frac{1}{2}$.

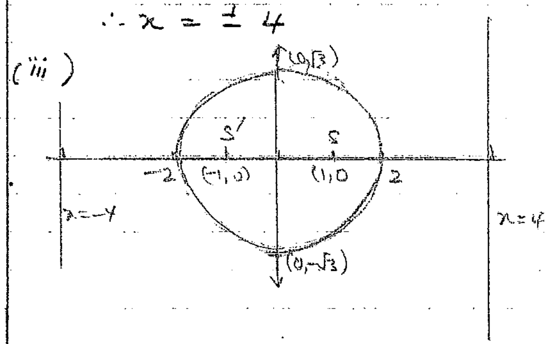
(i) $S(ae,0)$, $S'(-ae,0)$

$\therefore S(1,0)$, $S'(-1,0)$

(ii) $x=\pm\frac{a}{e}$

$\therefore x=\pm 4$

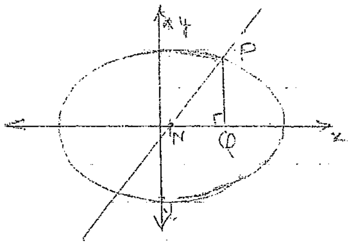
(iii)



b)

$$\frac{d}{dx}\left\{\frac{x^2}{a^2}+\frac{y^2}{b^2}\right\}=\frac{d}{dx}(1)$$
$$\frac{2x}{a^2}+\frac{2y}{b^2}\frac{dy}{dx}=0$$
$$\frac{dy}{dx}=\frac{-b^2}{a^2}\frac{x}{y}$$
$$=\frac{-b^2 a\cos\theta}{a^2 b\sin\theta}=\frac{-b\cos\theta}{a\sin\theta}$$
$$y-b\sin\theta=\frac{-b\cos\theta}{a\sin\theta}(x-a\cos\theta)$$
$$ay\sin\theta-ab\sin^2\theta=-b\cos\theta x+ab\cos^2\theta$$
$$b\cos\theta x+a\sin\theta y=ab$$
$$\frac{x\cos\theta}{a}+\frac{y\sin\theta}{b}=1$$

(c)



$m_T = -\frac{b\cos\theta}{a\sin\theta}$ from b)

$m_N = \frac{a\sin\theta}{b\cos\theta}$

eq. of normal:

$y - b\sin\theta = \frac{a\sin\theta}{b\cos\theta}(x - a\cos\theta)$

when $y = 0$

$-b^2\cancel{\sin\theta}\cos\theta = a\cancel{\sin\theta}(x - a\cos\theta)$

$-\frac{b^2\cos\theta}{a} = x - a\cos\theta$

$x = a\cos\theta - \frac{b^2\cos\theta}{a}$

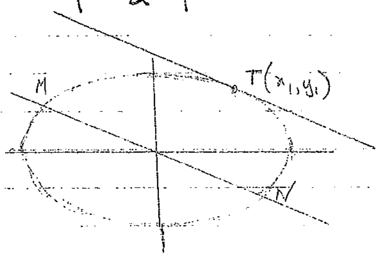
$= \frac{\cos\theta}{a}(a^2 - b^2)$

$\therefore NQ = \left| a\cos\theta - \frac{\cos\theta(a^2-b^2)}{a} \right|$

$= \left| \frac{a^2\cos\theta - a^2\cos\theta + b^2\cos\theta}{a} \right|$

$= \left| \frac{b^2\cos\theta}{a} \right|$

3(d)



Equation of the tangent is

$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$

i.e. $m_T = -\frac{b^2}{a^2}\frac{x_1}{y_1}$

Equation of MN

$y = -\frac{b^2}{a^2}\frac{x_1}{y_1}\cdot x$

Solving with $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$\frac{x^2}{a^2} + \frac{1}{b^2}\times\frac{b^4x_1^2}{a^4y_1^2}x^2 = 1$

$\frac{x^2}{a^2} + \frac{b^2x_1^2}{a^4y_1^2}x^2 = 1$

$x^2\left\{\frac{a^2y_1^2 + b^2x_1^2}{a^4y_1^2}\right\} = 1$

Now (x_1, y_1) lies on $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$\therefore \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} = 1 \rightarrow b^2x_1^2 + a^2y_1^2 = a^2b^2$

$\therefore x^2\left\{\frac{a^2b^2}{a^4y_1^2}\right\} = 1$

$\therefore x^2 = \frac{a^2y_1^2}{b^2}$

$x = \pm\frac{ay_1}{b}$

$y = -\frac{b^2}{a^2}\frac{x_1}{y_1} \times \pm\frac{ay_1}{b}$

$= \mp\frac{bx_1}{a}$

$\therefore$ Co-ordinates are $\left(-\frac{ay_1}{b}, \frac{bx_1}{a}\right), \left(\frac{ay_1}{b}, -\frac{bx_1}{a}\right)$

(d)

$MN = \sqrt{\left(\frac{2ay_1}{b}\right)^2 + \left(\frac{2bx_1}{a}\right)^2}$

$= \sqrt{\frac{4a^2y_1^2}{b^2} + \frac{4b^2x_1^2}{a^2}}$

$= \frac{2}{ab}\sqrt{a^4y_1^2 + b^4x_1^2}$

eq. of MN is

$y = -\frac{b^2}{a^2} \times \frac{x_1}{y_1} \times x$.

$a^2y_1y = -b^2x_1x$

$b^2x_1x + a^2y_1y = 0$

p = perpendicular distance from (x_1, y_1) to MN

i.e. $p = \frac{|b^2x_1^2 + a^2y_1^2|}{\sqrt{b^4x_1^2 + a^4y_1^2}}$

$= \frac{a^2b^2}{\sqrt{b^4x_1^2 + a^4y_1^2}}$

$\therefore$ Area $= \frac{1}{2} \times \frac{2}{ab}\sqrt{a^4y_1^2 + b^4x_1^2} \times \frac{a^2b^2}{\sqrt{a^4y_1^2 + b^4x_1^2}}$

$= ab$.

4(a) Prove $(a+b)^n \geq a^n + b^n$ (*) for $n \geq 1$.

For $n = 1$ LHS $= a + b$

RHS $= a + b$

$\therefore$ LHS = RHS

$\therefore$ True for $n = 1$

Assume * is true for $n = k$

i.e. $(a+b)^k \geq a^k + b^k$ (A)

Now it is necessary to prove that * is true for $n = k+1$

i.e. $(a+b)^{k+1} \geq a^{k+1} + b^{k+1}$ (P)

Multiply both sides of (A) by $(a+b)$

$(a+b)^{k+1} \geq (a+b)(a^k + b^k)$

$\geq a^{k+1} + ba^k + ab^k + b^{k+1}$

$\geq a^{k+1} + b^{k+1} + ba^k + ab^k$

Now $ba^k + ab^k > 0$.

$\therefore a^{k+1} + b^{k+1} + ba^k + ab^k > a^{k+1} + b^{k+1}$

$\therefore (a+b)^{k+1} \geq a^{k+1} + b^{k+1}$.

Now $(a+b)^k \geq a^k + b^k$ is true for $n = k$ and $(a+b)^{k+1} \geq a^{k+1} + b^{k+1}$ i.e. true for $n = k+1$, so since $(a+b)^n \geq a^n + b^n$ is true for $n = 1$ then it must be true for $n = 2, 3, \ldots$ i.e. all $n \geq 1$.

(b)

i) 2, 3 and 6 are in harmonic progression if $\frac{1}{2}, \frac{1}{3}$ and $\frac{1}{6}$ are in arithmetic progression ie

if $\frac{1}{3} - \frac{1}{2} = \frac{1}{6} - \frac{1}{3}$

Now $\frac{1}{3} - \frac{1}{2} = -\frac{1}{6}$

and $\frac{1}{6} - \frac{1}{3} = -\frac{1}{6}$

$\frac{1}{2}, \frac{1}{3}$ & $\frac{1}{6}$ are in arithmetic progression with $d = -\frac{1}{6}$.

ii) Since $\frac{1}{a}, \frac{1}{b}$ & $\frac{1}{c}$ are in arithmetic progression then

$$\frac{1}{b} - \frac{1}{a} = \frac{1}{c} - \frac{1}{b}$$

$$\frac{1}{b} + \frac{1}{b} = \frac{1}{a} + \frac{1}{c}$$

$$\frac{2}{b} = \frac{a+c}{ac}$$

$$\frac{b}{2} = \frac{ac}{a+c}$$

$$\therefore \quad b = \frac{2ac}{a+c}$$

iii) $(a-c)^2 \geq 0$

$$a^2 + c^2 \geq 2ac$$

$$a^2 + 2ac + c^2 \geq 4ac$$

$$\therefore (a+c)^2 \geq 4ac$$

$$ac(a+c)^2 \geq 4a^2c^2$$

$$ac \geq \frac{4a^2c^2}{(a+c)^2}$$

$$\therefore \quad \sqrt{ac} \geq \sqrt{\frac{4a^2c^2}{(a+c)^2}}$$

$$\therefore \sqrt{ac} \geq \frac{2ac}{a+c}$$

i.e. $\frac{2ac}{a+c} \leq \sqrt{ac}$

i.e. the harmonic mean is less than the geometric mean.

If a, b, c are in geometric progression then

$$\frac{b}{a} = \frac{c}{b}$$

$$b^2 = ac$$

$$b = \pm\sqrt{ac}$$

Taking $\underline{b = \sqrt{ac}}$ $\quad a, b, c$ all > 0.