St Catherine's School
Waverley, Sydney

Student Number: ____________

Year 12
Assessment Task 1
27/2/2007

Mathematics Extension II

Student Number

Time allowed: 55 minutes

Reading time: NIL

Course weighting: 15%

General Instructions

- Attempt ALL questions
- Write your Student NUMBER at the top of this page and on the writing paper used

Sections	Marks
Total marks	

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1, \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

Extension II Assessment task 1-2007

Q.1 Find the Argument and modulus of $\dfrac{(1-i)^4}{(1+\sqrt{3}\ i)^2}$ (5m)

Q.2

(a) Sketch the locus of z:

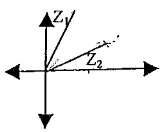
(i) $\arg(z-1-i)=\dfrac{\pi}{4}$ (2m)

(ii) $\arg(z-2i)=\arg(z+1)$ (2m)

(b) (i) Sketch the locus of z, such that $|z+i|=|z-1|$ (2m)

(ii) Describe the locus and find its Cartesian equation (2m)

Q.3. z_1 *and* z_2 are two complex numbers such that $|z_1|=|z_2|$, explain why $\dfrac{z_1+z_2}{z_1-z_2}$ is purely imaginary. (4m)



Q.4. (i) Solve for z, $z^6=1$ in the field of Complex Numbers. (2m)

(ii) Factorise z^6-1 in the field of Complex Numbers. (1m)

(iii) Factorise z^6-1 in the field of Real Numbers. (2m)

(iv) Explain why the roots of $z^4+z^2+1=0$ ~~is~~ are among the roots of $z^6-1=0$ (2m)

(v) State the roots of $z^4+z^2+1=0$ (1m)

Q.5. If $z=\cos\theta+i\sin\theta$,

(i) show that $z^n+\dfrac{1}{z^n}=2\cos n\theta$ (2m)

(ii) Hence show that $\cos^4\theta=\dfrac{1}{8}(\cos 4\theta+4\cos 2\theta+3)$ (3m)

(note: $(x+y)^4=x^4+4x^3y+6x^2y^2+4xy^3+y^4$)

Q.6. The equation $x^3+3x^2-2x+5=0$ has roots α, β *and* γ.

Find the equation whose roots are $\dfrac{1}{\alpha}$, $\dfrac{1}{\beta}$ *and* $\dfrac{1}{\gamma}$ (2m)

Q.7.

(i) . Given that α is a zero of multiplicity n for a polynomial P(x),
Show that α is a zero of multiplicity (n-1) for $P'(x)$ (2m)

(ii) Given that P(x): $2x^4-3x^3-3x^2+7x-3$ has a zero of multiplicity 3,
factorise P(x). (4m)

Q.8

(i) Show that $\dfrac{x^3-4x-10}{x^2-x-6}=x+1+\dfrac{3x-4}{x^2-x-6}$ (1m)

(ii) Hence express $\dfrac{x^3-4x-10}{x^2-x-6}$ as a sum of partial fractions. (3m)

End of Paper

Q.1

$|1-i| = \sqrt{2}$ $\quad |1+\sqrt{3}i| = 2$

$Arg(1-i) = -\frac{\pi}{4}$ $\quad Arg(1+\sqrt{3}i) = \frac{\pi}{3}$

$\left|\frac{(1-i)^4}{(1+\sqrt{3}i)^2}\right| = \frac{(\sqrt{2})^4}{2^2} = 1.$

$Arg\frac{(1-i)^4}{(1+\sqrt{3}i)^2} = 4\,Arg(1-i) - 2\,Arg(1+\sqrt{3}i) \pm 2n\pi$

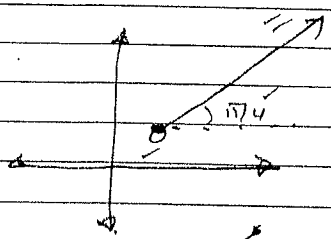
$= 4\left(\frac{-\pi}{4}\right) - 2\left(\frac{\pi}{3}\right) \pm 2n\pi$

$= -\pi - \frac{2\pi}{3} \quad (+2\pi)$

$= \frac{\pi}{3}.$

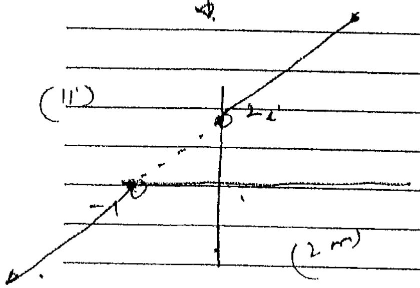
Q.2

(i)



$Arg(z-(1+i)) = \frac{\pi}{4}$

(ii)



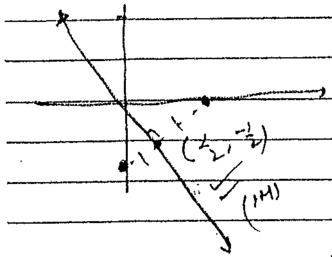
$Arg(z-2i) = Arg(z+1)$

(2 m)

(b) $|z+i| = |z-1|$

equidistant from $(0,-1)$ and $(1,0)$

Locus is the perpendicular bisector of the interval $(0,-1)$ and $(1,0)$



//1M – marking ⊥ and bisection.

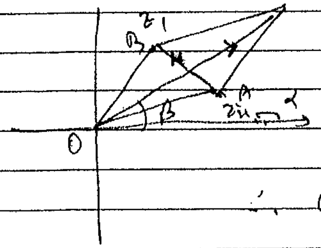
Cartesian equation is

$y+\frac{1}{2} = -1(x-\frac{1}{2})$

$2y+1 = -2x+1$ (1M)

$\underline{y = -x}$

Q.3



now

$\vec{OC}$ is z_1+z_2

$\vec{AB}$ is z_1-z_2

OABC is a rhombus

$\therefore OC \perp AB.$ — (1)

also $Arg\frac{z_1+z_2}{z_1-z_2}$

$= Arg(z_1+z_2) - Arg(z_1-z_2)$

$= \beta - \alpha$ (Ref: fig)

$= -(\alpha-\beta)$

$= -90°.$ from (1).

$\therefore \frac{z_1+z_2}{z_1-z_2}$ is purely imaginary

Q.4 $\quad z^6 = 1$

Let $z = r\,cis\,\theta$

$|z^6| = 1$

$\therefore |z|^6 = 1$

$r^6 = 1$

$r = 1$ (r is real) ✓

$\therefore z = cis\,\theta$

$cis\,6\theta = 1$

$\cos 6\theta = 1$; $\sin 6\theta = 0$

$6\theta = 0, \pm 2\pi, \pm 4\pi, 6\pi$

$\theta = 0, \pm \frac{\pi}{3}, \pm \frac{2\pi}{3}, \pi.$

$\therefore cis\,0$, $cis\,\pi/3$, $cis(-\pi/3)$, $cis(2\pi/3)$, $cis(-2\pi/3)$ and $cis\,\pi$ are the roots.

(ii)
$$z^6 - 1 = (z-1)(z - cis\,\pi/3)(z - cis(-\pi/3))(z - cis\,2\pi/3)(z - cis(-2\pi/3))(z+1)$$

$$z^6 - 1 = (z-1)(z+1)\left(z^2 - z(cis\,\pi/3 + cis(-\pi/3)) + cis\,\pi/3 \cdot cis(-\pi/3)\right)\left(z^2 - z(cis\,2\pi/3 + cis(-2\pi/3)) + cis\,2\pi/3 \cdot cis(-2\pi/3)\right)$$

$$= (z-1)(z+1)(z^2 - 2z\cos\pi/3 + 1)(z^2 - 2\cos 2\pi/3 + 1)$$

$$= (z-1)(z+1)(z^2 - z + 1)(z^2 + z + 1)$$

(iv) $z^6 - 1 = (z^2)^3 - 1$

$= (z^2 - 1)(z^4 + z^2 + 1)$

$\therefore$ The root of $z^4 + z^2 + 1 = 0$ are among the root of $z^6 - 1 = 0$

(v) $z^4 + z^2 + 1 = 0$

Roots are

$cis\,\frac{\pi}{3}$, $cis(-\pi/3)$, $cis\,2\pi/3$, $cis(-2\pi/3)$

Q.5 $\quad z = \cos\theta + i\sin\theta.$

$\frac{1}{z} = \cos\theta - i\sin\theta.$

$z^n = (\cos\theta + i\sin\theta)^n$ ✓

$= \cos n\theta + i\sin n\theta$ (De Moivre th.

$\left(\frac{1}{z}\right)^n = (\cos\theta + i\sin\theta)^{-n}.$ ✓

$= \cos(-n\theta) + i\sin(-n\theta)$ ✓

$= \cos n\theta - i\sin n\theta.$

$\therefore z^n + \frac{1}{z^n} = 2\cos n\theta$ ✓

(ii) Consider

$$\left(z+\frac{1}{z}\right)^4 = z^4 + 4.z^3.\frac{1}{z} + 6.z^2\frac{1}{z^2} + 4.z.\frac{1}{z^3} + \frac{1}{z^4}.$$

$$= \left(z^4 + \frac{1}{z^4}\right) + 4\left(z^2 + \frac{1}{z^2}\right) + 6$$

$$= 2\cos 4\theta + 4(2\cos 2\theta) + 6$$

also $z + \frac{1}{z} = 2\cos\theta$

$\therefore 16\cos^4\theta = 2\cos 4\theta + 8\cos 2\theta + 6$

$\therefore \cos^4\theta = \frac{1}{8}(\cos 4\theta + 4\cos 2\theta + 3)$

Q. 6

Let $P(x) : x^3 + 3x^2 - 2x + 5$.

α, β, γ are roots

$\therefore P(\alpha) = P(\beta) = P(\gamma) = 0$

Consider $P\left(\frac{1}{x}\right)$.

$P\left(\frac{1}{1/\alpha}\right) = P(\alpha) = 0$

Similarly $\frac{1}{\beta}, \frac{1}{\gamma}$ are roots of $P\left(\frac{1}{x}\right) = 0$

$\therefore \frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}$ are roots of

$P\left(\frac{1}{x}\right) = 0$

$$\frac{1}{x^3} + \frac{3}{x^2} - \frac{2}{x} + 5 = 0$$

$$1 + 3x - 2x^2 + 5x^3 = 0$$

Q. 7

$P(x) = (x-\alpha)^n \cdot Q(x)$

Consider

$P'(x) = (x-\alpha)^n Q'(x) + Q(x) \cdot n(x-\alpha)^{n-1}$

$= (x-\alpha)^{n-1}\left((x-\alpha) \cdot Q'(x) + nQ(x)\right)$

$(x-\alpha)Q'(x) + nQ(x)$ is a polynomial.

$\therefore \alpha$ is a root of multiplicity $(n-1)$ for $P'(x)$

(ii) Let α be the root of multiplicity 3 for $P(x)$

$\therefore \alpha$ is a root of multiplicity 2 for $P'(x)$ and 1 for $P''(x)$.

$P(x) = 2x^4 - 3x^3 - 3x^2 + 7x - 3$

$P'(x) = 8x^3 - 9x^2 - 6x + 7$

$P''(x) = 24x^2 - 18x - 6$

$P''(x) = 0 \implies 6(4x^2 - 3x - 1) = 0$

$6(4x+1)(x-1) = 0$

$x = -\frac{1}{4}$ or $x = 1$

now $P'(1) = 8 - 9 - 6 + 7$
$= 0$

& $P(1) = 2 - 3 - 3 + 7 - 3$
$= 0$

$\therefore$ 1 is a root of multiplicity 3 for $P(x) = 0$

$\therefore \quad P(x) = (x-1)^3 \cdot Q(x)$

$= (x-1)^3 (2x+3)$ (by observation)

Q. 8

$$\begin{array}{r} x + 1 \\ x^2 - x - 6 \;\big)\; \overline{x^3 - 4x - 10} \\ \underline{x^3 - x^2 - 6x} \\ x^2 + 2x - 10 \\ \underline{x^2 - x - 6} \\ 3x - 4 \end{array}$$

$\therefore \dfrac{x^3 - 4x - 10}{x^2 - x - 6} = x + 1 + \dfrac{3x - 4}{x^2 - x - 6}$.

Consider

$\dfrac{3x - 4}{(x-3)(x+2)} \equiv \dfrac{A}{x-3} + \dfrac{B}{x+2}$

$\therefore 3x - 4 \equiv A(x+2) + B(x-3)$

if $x = 3$. $\quad 5 = 5A \quad \therefore A = 1$

If $x = -2 \quad -10 = -5B \quad \therefore B = 2$

Thus $\dfrac{x^3 - 4x - 10}{x^2 - x - 6} = (x+1) + \dfrac{1}{x-3} + \dfrac{2}{x+2}$.