Student Number:___________________

St. Catherine's School
Waverley

August 2008

TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

Extension II Mathematics

Time allowed: *3 Hours + 5 mins Reading Time*

INSTRUCTIONS

- Write your STUDENT NUMBER on each page
- All questions are of equal value
- Marks for each part of a question are indicated
- All questions should be attempted on the separate paper provided
- All necessary working should be shown
- Start each question on a NEW page
- Approved scientific calculators and drawing templates may be used
- Standard integrals are printed at the end of the paper

QUESTION 1 **(15 marks)** **Marks**

a)

(i) Show that the equation of the tangent at a point P $(4\cos\theta, 3\sin\theta)$ to the ellipse $\frac{x^2}{16}+\frac{y^2}{9}=1$ is $3x\cos\theta+4y\sin\theta=12$ **3**

(ii) Find the eccentricity, the coordinates of the foci and the equations of the directrices of the hyperbola $\frac{x^2}{16}-\frac{y^2}{9}=1$ **2**

(iii) If the tangent to the ellipse at P (in part (i)) goes through a focus of the hyperbola (in part (ii)), show that P must lie on a the corresponding directrix of the hyperbola. **2**

(iv) Show that the gradient of the tangent at P is either 1 or -1. **2**

b) Consider the Hyperbola $x^2-y^2=16$

(i) Show that the eccentricity of the Hyperbola is $\sqrt{2}$ **1**

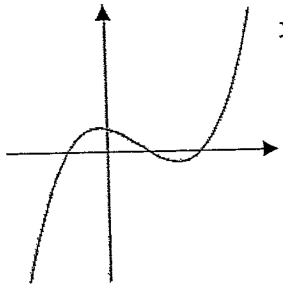
(ii) State the equation of the asymptotes **1**

(iii) This hyperbola is rotated anticlockwise through 45^0 to assume the equation $xy=c^2$, explain why $c^2=8$ **2**

(iv) Find the coordinates of the foci to $xy=c^2$ **2**

QUESTION 2 (15 marks) Start a *new* page. Marks

a) The graph shown is $y = f(x)$, where $f(x) = (x-1)(x+1)(x-2)$



Sketch the graph of each of the following graphs on separate Number Planes:

(approx $\frac{1}{3}$ page each)

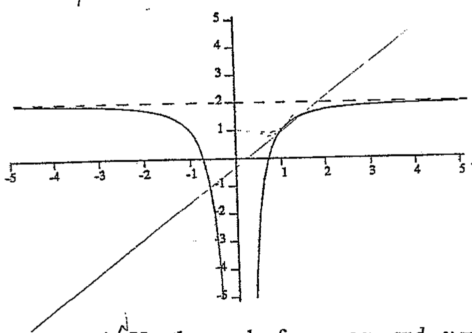
(i) $y = |f(x)|$ 1

(ii) $y = f(|x|)$ 2

(iii) $y^2 = f(x)$ 2

(iv) $y = f(x-1)$ 1

b) The graph shown is of $y = f(x)$, where $f(x) = 2 - \frac{1}{x^2}$



(i) Sketch the graph of $y = (f(x))^2$ 2

(ii) Graph $y = x$ on the same Number Plane as $y = f(x)$ and state the values of x for which $2 - \frac{1}{x^2} > x$ 3

c) Use the graph of $u = \cos x$ *and* $y = e^u$ to sketch the graph $y = e^{\cos x}$ clearly labelling key points. 4

QUESTION 3 (15 marks) Start a *new* page. Marks

a) Integrate $\int \frac{2x+3}{x^2+2x+5}dx$ 4

b) Integrate $\int \frac{e^x}{\sqrt{1-e^{2x}}}dx$ 3

c) (i) Use the substitution $x = a - t$ where a is a constant, to prove that 2

$$\int_0^a f(x)dx = \int_0^a f(a-x)dx$$

(ii) Hence or otherwise show that $\int_0^1 x(1-x)^{99}\,dx = \frac{1}{10100}$ 2

d) (i) Show that $(1-\sqrt{x})^{n-1}\sqrt{x} = (1-\sqrt{x})^{n-1} - (1-\sqrt{x})^n$ 1

(ii) If $I_n = \int_0^1 (1-\sqrt{x})^n dx$, for $n \geq 0$, show that $I_n = \frac{n}{n+2} I_{n-1}$, 3

QUESTION 4 (15 marks) Start a *new* page. **Marks**

a) Factorise $x^4 + x^2 + 1$ over the set of real numbers **1**

b) The equation $x^3 - 5x^2 + 5 = 0$ has roots $\alpha,\ \beta\ and\ \gamma$.

(i) Find the cubic equation with integer coefficients whose roots are $\alpha - 1, \beta - 1$ and $\gamma - 1$ **2**

(ii) Find the cubic equation with integer coefficients whose roots are **2**

α^2, β^2 and γ^2

(iii) Find the value of $\alpha^3 + \beta^3 + \gamma^3$ **1**

c) (i) Prove that the equation $ax^2 + bx + c = 0$ has a double root if $b^2 - 4ac = 0$ **2**

(ii) Prove that the equation $ax^3 + bx^2 + cx + d = 0$ has a triple root at $x = h$ **3**

if $\frac{b}{a} = \frac{c}{b} = \frac{d}{c} = -h$

d) The real number x is a solution of $x^2 - x - 1 = 0$.

Consider the series $1 + x + x^2 + x^3 + \dots\dots\dots + x^{2n-1}$

(i) Write down S, the sum to n terms of this series **1**

(ii) Use the binomial theorem to show that $S = \sum_{r=1}^{n} {}^nC_r\ x^{r+1}$ **3**

QUESTION 5 (15 marks) Start a *new* page. **Marks**

a) For a sequence of numbers $a_1 = 2$; $a_2 = 3$ and $a_n = 3\ a_{n-1} - 2\ a_{n-2}$, **4**

for all integers $n \geq 3$, prove by mathematical induction that

$a_n = 2^{n-1} + 1$ for all $n \geq 1$

b) The point P (x_1, y_1) lies on the curve $\sqrt{x} + \sqrt{y} = \sqrt{a}$

(i) Show that the equation of the tangent of the tangent at P is **3**

$\sqrt{y_1}\ x + \sqrt{x_1}\ y = \sqrt{a}\sqrt{x_1 y_1}$

(ii) The tangent meets the coordinate axes at S and T, show that $OS + OT = a$ **2**

c) (i) Show that $x > \tan^{-1} x$ *for* $x > 0$ **3**

(ii) By evaluating $\int_0^1 x\ dx$ *and* $\int_0^1 \tan^{-1} x\ dx$, show that $2 > \pi - \ln 4$ **3**

QUESTION 6 (15 marks) Start a *new* page. **Marks**

a) The region between the curve $y = e^x$, the y-axis and the line $y = e$ is rotated about the line $y = e$. Use the method of slicing to find the volume of the solid generated. 4

b) (i) The ellipse $\frac{x^2}{9}+\frac{y^2}{4}=1$ is rotated about the line $x = 4$. Use the method of cylindrical shells to show that the volume V of the solid generated is given by 3

$$V = \frac{8\pi}{3}\int_{-3}^{3}(4-x)\sqrt{9-x^2}\,dx$$

(ii) Hence find the Volume 3

c) If $f(x) = \cos^{-1}(\sin x)$

(i) Show $f'(x) = \pm 1$ 2

(ii) Hence or otherwise sketch the graph of $y = f(x)$ for $-2\pi \le x \le 2\pi$ 3

QUESTION 7 (15 marks) Start a *new* page. **Marks**

a) The If $\frac{a}{c}=\frac{a-b}{b-c}$, then b is called the Harmonic Mean of a and c.

(i) Show that $b=\frac{2ac}{a+c}$ 1

(ii) Prove that the reciprocals of a,b and c are in Arithmetic progression. 2

b) (i) Show that the condition for the line y = mx+c to be a tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is $c^2=a^2m^2+b^2$ 2

(ii) Hence show that the pair of tangents from the point (3,4) to the ellipse $\frac{x^2}{16}+\frac{y^2}{9}=1$ are at right angles to each other. 3

c) Use De Moivre's theorem to show that $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$ 1

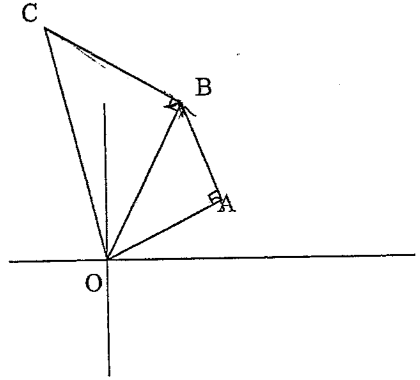
(i) Deduce that $8x^3-6x+1=0$ has solutions x= $\sin\theta$, where $\sin 3\theta=\frac{1}{2}$ 2

(ii) Find the roots of $8x^3-6x+1=0$ in terms of $\sin\theta$ 2

(iii) Hence evaluate $\sin\frac{\pi}{18}\sin\frac{5\pi}{18}\sin\frac{13\pi}{18}$ 1

QUESTION 8 **(15 marks) Start a *new* page.** **Marks**

a)



OAB is an isosceles right angled triangle, right angled at A.

Also OBC is an isosceles right angled triangle, right angled at B.

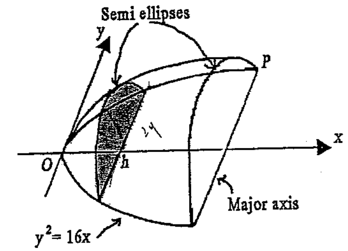
If the point A and C represent complex numbers α and β respectively, show that

(i) $OC = 2 \times OA$ **2**

(ii) $4\alpha^2 + \beta^2 = 0$ **3**

Question 8 continued on next page

b)



The base of a solid P is the region in the xy plane enclosed by the parabola $y^2 = 16x$ and the line $x = 6$, and each cross-section perpendicular to the x axis is a semi-ellipse with the minor axis one half of the major axis and the major axis is on the parabola.

i) Show that the area of the semi-ellipse at $x = h$ is $4\pi h$ **2**

(You may assume the area of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ to be πab).

ii) Find the volume of the solid P. **2**

c) Let $p = \cos\frac{2\pi}{7} + i\sin\frac{2\pi}{7}$. The complex number $\alpha = p + p^2 + p^4$ is a root of the quadratic equation $x^2 + ax + b = 0$, where a and b are real.

Noting that p is a complex root of $z^7 = 1$

(i) Prove that $1 + p + p^2 + \ldots + p^6 = 0$ **1**

(ii) The second root of the quadratic equation $x^2 + ax + b = 0$ is β. **3**

Show that $\beta = p^3 + p^5 + p^6$.

(iii) Hence find the values of the coefficients a and b **2**

END OF PAPER

Solutions	Marks	Comments

Question 1 a) (i) $\frac{x^2}{16}+\frac{y^2}{9}=1$ $\quad P(4\cos\theta, 3\sin\theta)$

$9x^2+16y^2=144$ ---- ①.

$18x+32y\frac{dy}{dx}=0$

$\therefore \frac{dy}{dx}=\frac{-18x}{32y}=\frac{-9x}{16y}$

at P. $\frac{dy}{dx}=-\frac{36\cos\theta}{48\sin\theta}=\frac{-3\cos\theta}{4\sin\theta}$

Now. $y-3\sin\theta=\frac{-3\cos\theta}{4\sin\theta}(x-4\cos\theta)$.

$4y\sin\theta-12\sin^2\theta=-3x\cos\theta+12\cos^2\theta$

$\therefore 3x\cos\theta+4y\sin\theta=12(\sin^2\theta+\cos^2\theta)$

$\therefore 3x\cos\theta+4y\sin\theta=12$

(ii) $\frac{x^2}{16}-\frac{y^2}{9}=1$ $\quad$ foci $(\pm ae,0)$, directrices $x=\pm\frac{a}{e}$

$b^2=a^2(e^2-1)$

$9=16(e^2-1)$

$9=16e^2-16$

$\therefore e^2=\frac{25}{16}$.

$e=\frac{5}{4}$ $(e>0)$ $\quad\therefore$ foci $(\pm5,0)$, directrices $x=\pm\frac{16}{5}$

(iii) tangent $3x\cos\theta+4y\sin\theta=12$

If tangent passes through a focus of hyperbola

$\pm15\cos\theta=12$

$\therefore \cos\theta=\pm\frac{4}{5}$

$\therefore$ the x coordinate of P is $\pm\frac{16}{5}$

$\therefore$ P lies on corresponding directrix of hyperbola

(iv) gradient of tangent at P is $\frac{-3\cos\theta}{4\sin\theta}$

now $\cos\theta=\pm\frac{4}{5}$

$\therefore \sin\theta=\pm\frac{3}{5}$ $\quad\therefore$ gradient is $\frac{\mp\frac{12}{5}}{\pm\frac{12}{5}}$

ie $m=\pm1$

Solutions	Marks	Comments

Question 1 b) $x^2-y^2=16$ $\quad \frac{x^2}{16}-\frac{y^2}{16}=1$

(i) $a=b=4$

now $b^2=a^2(e^2-1)$

$16=16(e^2-1)$

$1=e^2-1$

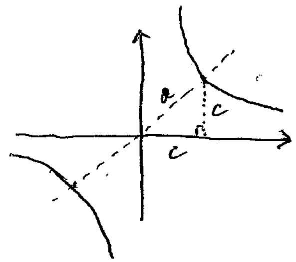
$2=e^2$

$\therefore e=\sqrt2$ $\quad$ 1

(ii) $y=\pm\frac{b}{x}$ $\quad\therefore$ asymptotes are $y=\pm x$ $\quad$ 1

(iii) $xy=c^2$ $\quad$ Coords of vertices of original hyperbola are $(a,0)$ $(-a,0)$

on rotation these become (c,c) and $(-c,-c)$



$\therefore a^2=c^2+c^2$

$\therefore 2c^2=a^2$

$\sqrt2c=a$

$c=\frac{a}{\sqrt2}$ $\quad$ or use $xy=\frac{1}{2}a^2$

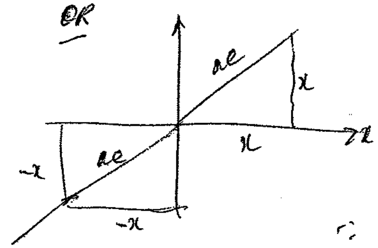
but $a=4$

$\therefore c^2=\left(\frac{4}{\sqrt2}\right)^2=8$

(iv) foci $(\pm c\sqrt2, \pm c\sqrt2)$

$\therefore$ foci $(4,4)$ $(-4,-4)$

OR



$2x^2=a^2e^2$

$2x^2=16\times2$

$x^2=16$

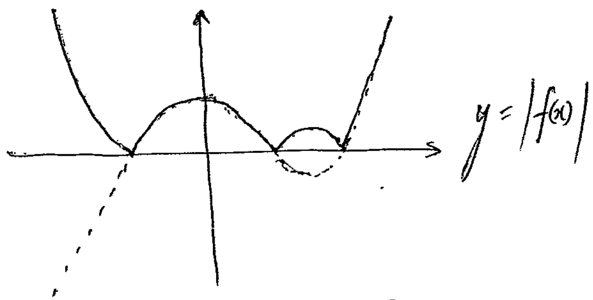
$x=\pm4$

$\therefore$ foci $(4,4)$ $(-4,-4)$

Solutions	Marks	Comments

Question 2

(a) (i)



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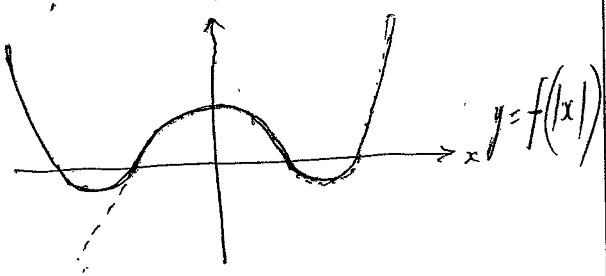
(ii)



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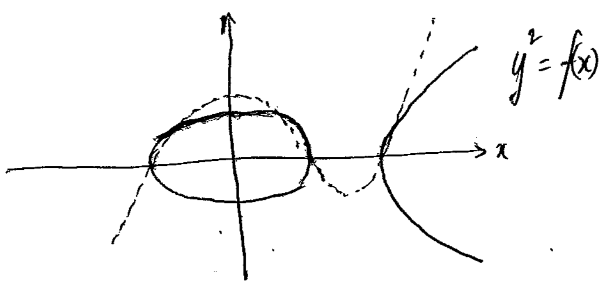
(iii)



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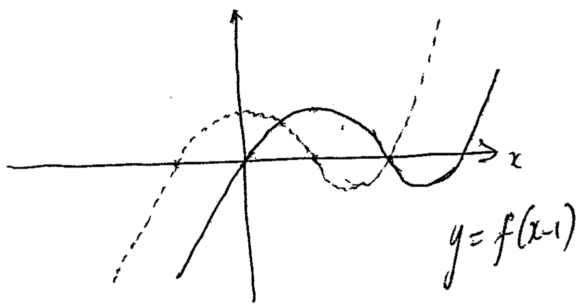
(iv)



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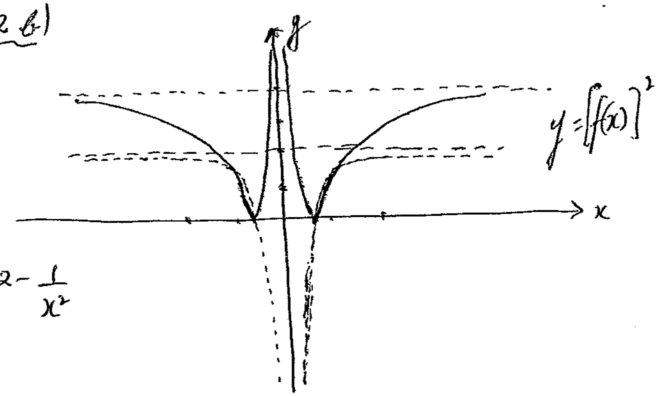
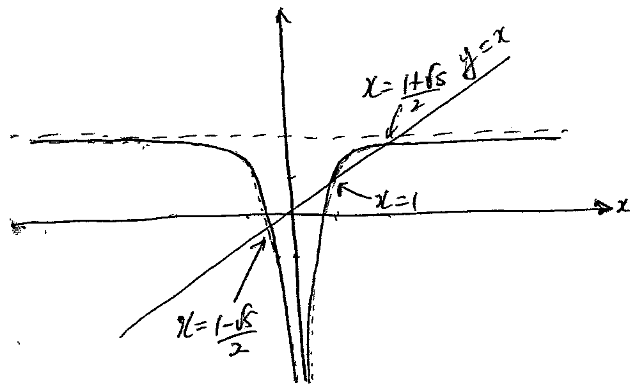
Solutions	Marks	Comments

Question 2 (b)



$f(x) = 2 - \frac{1}{x^2}$



$2x^2 - 1 > x^3$

$x^3 - 2x^2 + 1 < 0$

$(x-1)(x^2 - x - 1) < 0$

$(x-1)\left(x - \frac{1+\sqrt{5}}{2}\right)\left(x - \frac{1-\sqrt{5}}{2}\right) < 0$

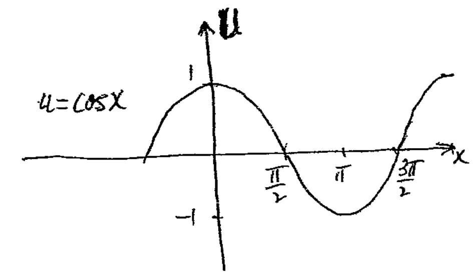
$\therefore$ curves intersect at $x = 1, \frac{1+\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2}$

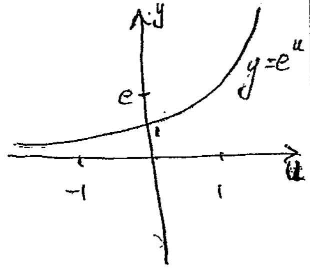
from graph $2 - \frac{1}{x^2} > x$ for

$x < \frac{1-\sqrt{5}}{2}$ and $1 < x < \frac{1+\sqrt{5}}{2}$

Solutions	Marks	Comments

Question 2 c)

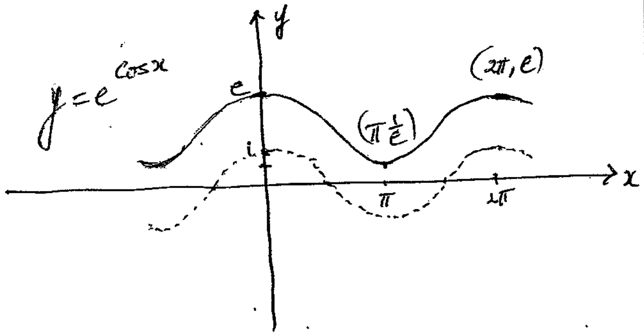




when $u = 0$ $y = 1$, $-1 \le u \le 1$, $\frac{1}{e} \le y \le e$

$y = e^u$ is monotonic ie. $e^{\cos x}$ increases as $\cos x$ increases

$e^{\cos x}$ decreases as $\cos x$ decreases.



Marks: 4

Solutions	Marks	Comments

Question 3: a)

$$\int \frac{2x+3}{x^2+2x+5}\,dx = \int \frac{2x+2}{x^2+2x+5}\,dx + \int \frac{1}{x^2+2x+5}\,dx$$

$$= \ln(x^2+2x+5) + \int \frac{1}{(x+1)^2+4}\,dx$$

$$= \ln(x^2+2x+5) + \frac{1}{2}\tan^{-1}\left(\frac{x+1}{2}\right) + c$$

Marks: 4

b)

$$\int \frac{e^x}{\sqrt{1-e^{2x}}}\,dx \qquad u = e^x,\quad du = e^x\,dx$$

$$= \int \frac{du}{\sqrt{1-u^2}} \cancel{dx} \qquad u = \sin x \quad x = \sin^{-1} u,\quad du = \cos x\,dx$$

$$= \int \frac{\cos x\,dx}{\cos x}$$

$$= \int 1\,dx$$

$$= x$$

$$= \sin^{-1} u$$

$$= \sin^{-1}(e^x) + c$$

Marks: 3

c) (i) $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$ (show)

$$\int_0^a f(x)\,dx \qquad \text{let } x = a-t,\ t = a-x,\ dt = -dx,\ x=0\ t=a,\ x=a\ t=0$$

$$= \int_0^a f(t)\,dt$$

$$= -\int_a^0 f(t)\,dt$$

$$= \int_0^a f(a-x)\,dx$$

Marks: 2

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Marking Scheme for Task:

Solutions	Marks	Comments
<u>Question 3 c (ii)</u> $\int_0^1 x(1-x)^{99}\,dx = \int_0^1 (1-x)(x)^{99}\,dx$		
$= \int_0^1 (x^{99} - x^{100})\,dx$		
$= \left[\frac{x^{100}}{100} - \frac{x^{101}}{101}\right]_0^1$		
$= \left(\frac{1}{100} - \frac{1}{101}\right) - (0)$	2	
$= \frac{1}{10100}$		
d) (i) $(1-\sqrt{x})^{n-1}\sqrt{x} = (1-\sqrt{x})^{n-1} - (1-\sqrt{x})^n$ (prove)		
$RHS = (1-\sqrt{x})^{n-1}\left[1-(1-\sqrt{x})\right]$		
$= (1-\sqrt{x})^{n-1}(\sqrt{x})$	1	
$= LHS.$		
(ii) $I_n = \int_0^1 (1-\sqrt{x})^n\,dx \quad n \geq 0$		
let $u = (1-\sqrt{x})^n \quad v = x$		
$u' = -n(1-\sqrt{x})^{n-1} \cdot \frac{1}{2\sqrt{x}} \quad v' = 1$		
$= \left[x(1-\sqrt{x})^n\right]_0^1 + \int_0^1 n(1-\sqrt{x})^{n-1}\frac{\sqrt{x}}{2}\,dx$		
$\therefore I_n = 0 + \frac{n}{2}\left[\int_0^1 (1-\sqrt{x})^{n-1}dx - \int_0^1 (1-\sqrt{x})^n dx\right]$ from part (i)		
$= \frac{n}{2}I_{n-1} - \frac{n}{2}I_n$		
$\therefore 2I_n = nI_{n-1} - nI_n$		
$(n+2)I_n = nI_{n-1}$		
$\therefore I_n = \frac{n}{n+2}I_{n-1}$		

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Marking Scheme for Task:

Solutions	Marks	Comments
<u>Question 4:</u>		
a) $x^4+x^2+1 = (x^2+1)^2 - x^2$		
$= (x^2+1-x)(x^2+1+x)$	1	
$= (x^2-x+1)(x^2+x+1)$		
b) $x^3-5x^2+5=0 \quad \alpha, \beta, \gamma$		
(i) let $y = x-1 \quad \therefore x = y+1$		
$(y+1)^3 - 5(y+1)^2 + 5 = 0$		
$y^3+3y^2+3y+1-5y^2-10y-5+5=0$		
$y^3-2y^2-7y+1=0$		
$\therefore$ required cubic is $x^3-2x^2-7x+1=0$		
(ii) let $y = x^2 \quad \therefore x = \sqrt{y}$		
$(\sqrt{y})^3 - 5(\sqrt{y})^2 + 5 = 0$		
$y^{3/2} - 5y + 5 = 0$		
$y^{3/2} = 5y - 5$		
Square both sides $y^3 = 25y^2 - 50y + 25$		
$\therefore$ required cubic is $x^3-25x^2+50x-25=0$		
(iii) $\alpha^3-5\alpha^2+5=0$ — (1) since α is a root		
$\beta^3-5\beta^2+5=0$ — (2)		
$\gamma^3-5\gamma^2+5=0$ — (3)		
(1) + (2) + (3)		
$\alpha^3+\beta^3+\gamma^3 - 5(\alpha^2+\beta^2+\gamma^2) + 15 = 0$		
$\therefore \alpha^3+\beta^3+\gamma^3 = 5\left[(\alpha+\beta+\gamma)^2 - 2(\alpha\beta+\alpha\gamma+\beta\gamma)\right] - 15$		
$= 5\left[25 - 0\right] - 15$		
$= 110$		

Solutions	Marks	Comments
Question 4 c) (i) $ax^2+bx+c=0$ —— ① If double root then root is a zero of $f'(x)$ $f'(x) = 2ax+b = 0$ $\therefore x = -\frac{b}{2a}$ Sub in ① $a\left(\frac{b^2}{4a^2}\right) - \frac{b^2}{2a} + c = 0$ $\frac{b^2}{4a} - \frac{b^2}{2a} + c = 0$ $(\times 4a)$ $b^2 - 2b^2 + 4ac = 0$ $\therefore b^2 - 4ac = 0$ (ii)		

Question 5

$a_1 = 1$; $a_2 = 3$

$a_n = 3a_{n-1} - 2a_{n-2}$ $n \geq 3$

Let $P(n)$: $a_n = 2^{n-1} + 1$

Consider $P(1)$: $a_1 = 1$ (given)

$2^{1-1} + 1 = 1$

$\therefore P(1)$ is true

Consider $P(2)$: $a_2 = 3$ (given)

$2^{2-1} + 1 = 3$

$\therefore P(2)$ is true. (1M)

Let $P(k)$, be true for all $k \leq n$ —— (A)

Consider $P(n+1)$

$$
\begin{aligned}
a_{n+1} &= 3a_n - 2a_{n-1} \quad \text{(given)} \\
&= 3(2^{n-1}+1) - 2(2^{n-2}+1) \quad \text{by (A)} \\
&= 3(2^{n-1}) + 3 - 2^{n-1} - 2 \\
&= 2(2^{n-1}) + 1 \\
&= 2^n + 1.
\end{aligned}
$$

(2)

Thus $P(n+1)$ is true if $P(k)$ is true for all $k \leq n$.

Thus by the principle of mathematical induction, $P(n)$ is true $\forall n \geq 1$.

b) $\sqrt{x} + \sqrt{y} = \sqrt{a}$.

differentiate w.r.to x:

$$\frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{y}}\, y' = 0$$

$$\therefore\ y' = -\frac{2\sqrt{y}}{2\sqrt{x}} \quad (1M)$$

$$y'_{\text{at } P(x_1, y_1)} = -\frac{\sqrt{y_1}}{\sqrt{x_1}}$$

Equation of the tangent at P:

$$y - y_1 = -\frac{\sqrt{y_1}}{\sqrt{x_1}}(x - x_1) \quad (1M)$$

$$y\sqrt{x_1} - \sqrt{x_1}\, y_1 = -x\sqrt{y_1} + x_1\sqrt{y_1}$$

$$\sqrt{y_1}\, x + \sqrt{x_1}\, y = x_1\sqrt{y_1} + \sqrt{x_1}\, y_1$$

$$= \sqrt{x_1 y_1}\,(\sqrt{x_1} + \sqrt{y_1})$$

$$= \sqrt{x_1 y_1}\,(\sqrt{a}) \quad (1M)$$

S: put $y = 0$

$$x = \frac{\sqrt{x_1 y_1}\,\sqrt{a}}{\sqrt{y_1}}$$

$$= \sqrt{a}\sqrt{x_1} \quad (1M)$$

T: put $x = 0$;

$$y = \sqrt{a}\sqrt{y_1}$$

$$\therefore OS + OT = \sqrt{a}(\sqrt{x_1} + \sqrt{y_1}) \quad (1M)$$

$$= \sqrt{a}\sqrt{a} = a$$

c) i) Consider $f(x) = x - \tan^{-1}x$.

$$f(0) = 0 - \tan^{-1}0 = 0 \quad (1)$$

$$f'(x) = 1 - \frac{1}{1+x^2}$$

$$= \frac{1+x^2-1}{1+x^2}$$

$$= \frac{x^2}{1+x^2}$$

$f'(x) > 0$ for all x. (1)

$\therefore f(x)$ is an increasing function. (1)

also $f(0) = 0$

$\therefore f(x) > 0$ for all $x > 0$ (1)

i.e. $x - \tan^{-1}x > 0$

$x > \tan^{-1}x$ for $x > 0$

---

ii) $$\int_0^1 x\,dx = \left(\frac{x^2}{2}\right)_0^1 = \frac{1}{2} \quad \left(\tfrac{1}{2}\right)$$

$$\int_0^1 \tan^{-1}x\,dx = \left(\tan^{-1}x \cdot x\right)_0^1 - \int_0^1 \frac{x}{1+x^2}\,dx$$

$$= (\tan^{-1}1 - 0) - \frac{1}{2}\left(\ln(1+x^2)\right)_0^1$$

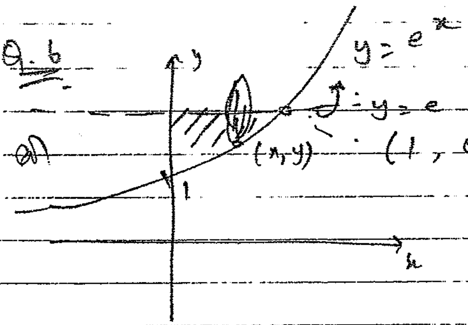
$$= \frac{\pi}{4} - \frac{1}{2}\left(\ln(2) - \ln(1)\right)$$

$$= \frac{\pi}{4} - \frac{1}{2}\ln 2 \quad (2M)$$

$$\therefore \frac{1}{2} > \frac{\pi}{4} - \frac{1}{2}\ln 2 \quad (1/2)$$

$$1 > \frac{\pi}{2} - 2\ln 2 \quad \text{or} \quad 2 > \pi - \ln 4$$

Q. 6



Take a slice at (x,y) perpendicular to $y = e$.

Δv, the volume of that slice is

$$\Delta v = \pi (e-y)^2 \Delta x.$$

$$\therefore V = \pi \int_0^1 (e - e^x)^2 dx.$$

$$= \pi \int_0^1 (e^2 + e^{2x} - 2e.e^x)\, dx$$

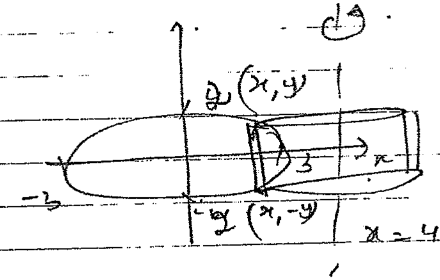
$$= \pi \left[e^2 x + \frac{e^{2x}}{2} - 2e e^x \right]_0^1$$

$$= \pi \left[\left(e^2 + \frac{e^2}{2} - 2e^2\right) - \left(0 + \frac{1}{2} - 2e\right) \right]$$

$$= \pi \left[-\frac{e^2}{2} + 2e - \frac{1}{2} \right]$$

$$= \frac{\pi}{2} \left[4e - e^2 - 1 \right] \quad \text{units}^3.$$

b)



A shell at a distance x from the y axis.

Δv, its volume

$$\Delta v = 2\pi (4-x)\, 2y\, \Delta x.$$

$$= 4\pi (4-x)\, y\, \Delta x$$

$$\frac{x^2}{9} + \frac{y^2}{4} = 1$$

$$4x^2 + 9y^2 = 36$$

$$9y^2 = 36 - 4x^2$$
$$= 4(9-x^2)$$

$$y^2 = \frac{4}{9}(9-x^2)$$

$$y = \pm \frac{2}{3}\sqrt{9-x^2}. \qquad (1M)$$

$$\therefore \Delta v = 4\pi (4-x) \cdot \frac{2}{3}\sqrt{9-x^2}\, \Delta x$$

$$= \frac{8\pi}{3} (4-x) \sqrt{9-x^2}\, \Delta x$$

$$\therefore V = \frac{8\pi}{3} \int^3 (4-x) \sqrt{9-x^2}\, dx.$$

$$V = \frac{8\pi}{3}\left[\int_{-3}^{3} 4\sqrt{9-x^2}\,dx - \int_{-3}^{3} x\sqrt{9-x^2}\,dx\right]$$

$x\sqrt{9-x^2}$ is an odd fn.

$$\therefore \int_{-3}^{3} x\sqrt{9-x^2} = 0 \qquad (1M)$$

$\sqrt{9-x^2}$ is an even fn.

$$\therefore \int_{-3}^{3} \sqrt{9-x^2}\,dx = 2\int_{0}^{3} \sqrt{9-x^2}\,dx. \qquad (1M)$$

Let $x = 3\sin\theta$

$dx = 3\cos\theta\,d\theta$

$x=0;\ \theta=0$

$x=3;\ \theta=\frac{\pi}{2}$

$$= 2\int_0^{\pi/2} \sqrt{9-9\sin^2\theta}\;3\cos\theta\,d\theta$$

$$= 2\int_0^{\pi/2} 3\cos\theta \cdot 3\cos\theta\,d\theta$$

$$= 18\int_0^{\pi/2} \cos^2\theta\,d\theta \qquad (2M)$$

$$= 9\int_0^{\pi/2} \frac{1+\cos 2\theta}{\cancel{2}}\,d\theta$$

$$= 9\left[\theta + \frac{\sin 2\theta}{2}\right]_0^{\pi/2} = 9\left(\frac{\pi}{2}\right) = \frac{9\pi}{2}$$

$$\therefore V = \frac{8\pi}{3} \times 4 \times \frac{9\pi}{2}$$

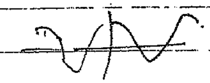
$$= 48\pi^2 \text{ units}^3$$

c).

$$f(x) = \cos^{-1}(\sin x)$$

$$f'(x) = \frac{-1}{\sqrt{1-\sin^2 x}}\,(+\cos x)$$

$$= \frac{-\cos x}{|\cos x|}$$



$$= \frac{-\cos x}{\cos x} = -1 \text{ when } \cos x > 0$$

$-\frac{\pi}{2} < x < \frac{\pi}{2}$ OR

$-2\pi < x < -\frac{3\pi}{2}$

$\frac{3\pi}{2} < x < 2\pi$

$$= \frac{-\cos x}{-\cos x} = 1 \text{ when } \cos x < 0$$

$\frac{\pi}{2} < x < \frac{3\pi}{2}$

$-\frac{3\pi}{2} < x < -\frac{\pi}{2}$

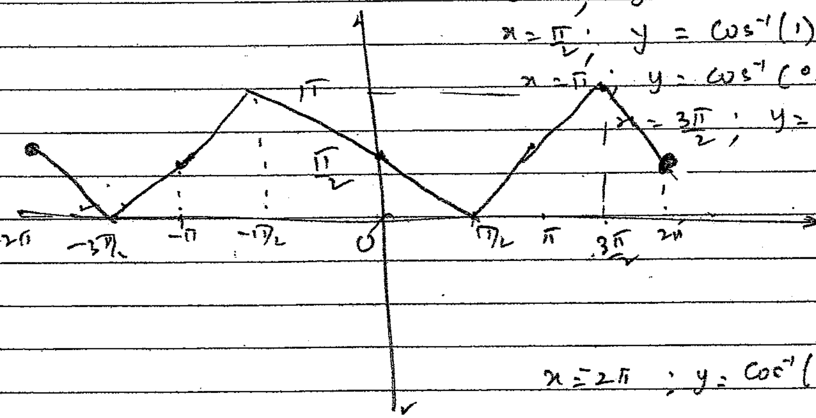
Thus grad. of $y = f(x)$ is ± 1.

$x=0;\ y = \cos^{-1}(\sin 0) = \frac{\pi}{2}$

$x=\frac{\pi}{2};\ y = \cos^{-1}(1) = 0$

$x=\pi;\ y = \cos^{-1}(0) = \pi/2$

$x=\frac{3\pi}{2};\ y = \cos^{-1}(-1) = \pi$



$x = 2\pi;\ y = \cos^{-1}(0) = \pi/2$

Q-7

a) $\dfrac{a}{c} = \dfrac{a-b}{b-c}$

$\therefore ab - ac = ac - bc$ — (1)

$ab + bc = 2ac$

$b(a+c) = 2ac$

$b = \dfrac{2ac}{a+c}$

Consider $\dfrac{1}{a}, \dfrac{1}{b}, \dfrac{1}{c}$

$\dfrac{1}{b} - \dfrac{1}{a} = \dfrac{a-b}{ab}$

$\dfrac{1}{c} - \dfrac{1}{b} = \dfrac{b-c}{bc}$

Since $\dfrac{ac-bc}{abc} = \dfrac{ab-ac}{abc}$ from (1)

$\dfrac{a-b}{ab} = \dfrac{b-c}{bc}$

$\therefore \dfrac{1}{b} - \dfrac{1}{a} = \dfrac{1}{c} - \dfrac{1}{b}$

Hence $\dfrac{1}{a}, \dfrac{1}{b}, \dfrac{1}{c}$ are in Arithmetic seq.

b) $y = mx + c$ meets $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

at $\dfrac{x^2}{a^2} + \dfrac{(mx+c)^2}{b^2} = 1$

$b^2x^2 + a^2(mx+c)^2 = a^2b^2.$

$x^2(b^2 + a^2m^2) + 2a^2mc\,x + a^2c^2 - a^2b^2 = 0$ (A)

$y = mx + c$ is a tangent to the ellipse if Δ of A is 0.

$4a^4m^2c^2 - 4(b^2 + a^2m^2)(a^2c^2 - a^2b^2) = 0$

$(\div 4a^2)$

$a^2m^2c^2 - (b^2 + a^2m^2)(c^2 - b^2) = 0.$

$a^2\cancel{m^2}c^2 - (b^2c^2 + a^2\cancel{m^2}c^2 - b^4 - a^2m^2b^2) = 0$

$b^4 - b^2c^2 + a^2m^2b^2 = 0$

$\div b^2$ $\quad b^2 - c^2 + a^2m^2 = 0$

$c^2 = b^2 + a^2m^2$

(ii) if $y = mx + c$ is a tangent to $\dfrac{x^2}{16} + \dfrac{y^2}{9} = 1$ $\quad a^2 = 16$, $b^2 = 9$

$4 = 3m + c$ — (1)

also $c^2 = 9 + 16m^2$ — (2)

Combine (1) and (2)

$$(4-3m)^2 = 9+16m^2$$

$$16+9m^2-24m = 9+16m^2$$

or

$$7m^2+24m-7=0$$

note if m_1 and m_2 are the two gradients of the two tangents from $(3,4)$ $\quad m_1 m_2 = \frac{-7}{7} = -1$.

c). $(\cos\theta + i\sin\theta)^3 = \cos 3\theta + i\sin 3\theta$

$\cos3\theta + i\sin3\theta$

$$= \cos^3\theta + 3\cos^2\theta(i\sin\theta) + 3\cos\theta(i\sin\theta)^2 + (i\sin\theta)^3.$$

Equating Imaginary parts

$$\sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta$$
$$= 3(1-\sin^2\theta)\sin\theta - \sin^3\theta$$
$$= 3\sin\theta - 4\sin^3\theta$$

i) when $\sin 3\theta = \frac{1}{2}$.

$$3\sin\theta - 4\sin^3\theta = \frac{1}{2}$$

or $\quad 6\sin\theta - 8\sin^3\theta = 1$

$$8\sin^3\theta - 6\sin\theta + 1 = 0.$$

Thus $x = \sin\theta$ is a solution to
$$8x^3 - 6x + 1 = 0$$

ii) The three roots to $8x^3-6x+1=0$ are the 3 values of $\sin\theta$, where θ is a solution to
$$\sin 3\theta = \frac{1}{2}.$$

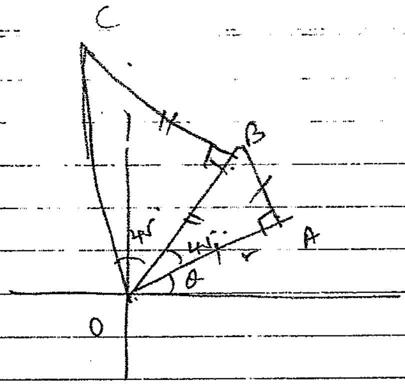
$$3\theta = \frac{\pi}{6}, \left(2\pi+\frac{\pi}{6}\right), \left(\pi-\frac{\pi}{6}\right)$$

$$\theta = \frac{\pi}{18}, \frac{5\pi}{18}, \frac{13\pi}{18}$$

$\sin\frac{\pi}{18}$, $\sin\frac{5\pi}{18}$, $\sin\frac{13\pi}{18}$ are roots of
$$8x^3 - 6x + 1 = 0$$

iii) $\therefore$ The product of the roots $= \frac{1}{8}$.

Q.8



Let $\alpha = r\,\text{cis}\,\theta$, where $r = OA$

In $\Delta\, OAB$, rt $\angle$d. at A :

$r^2 + r^2 = OB^2$

$\therefore OB = \sqrt{2}\, r$

In $\Delta\, OBC$, rt $\angle$d. at B

$(2r^2) + (2r^2) = OC^2$

$\therefore OC = 2r$

$= 2 \times OA$

(ii)

$\therefore \beta = 2r\,\text{cis}\left(\theta + \frac{\pi}{2}\right)$

$4\alpha^2 + \beta^2$

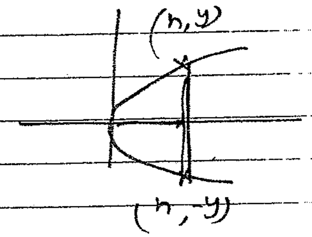
$= 4(r^2 \text{cis}\, 2\theta) + 4r^2 \text{cis}\, 2\left(\theta + \pi/2\right)$

$= 4r^2\left(\cos 2\theta + i\sin 2\theta + \cos(2\theta + \pi) + i\sin(2\theta + \pi)\right.$

$= 4r^2\left(\cos 2\theta + i\sin 2\theta - \cos 2\theta - i\sin 2\theta\right)$

$= 0$

b) (i) Consider a slice at a distance h from the origin along the x axis.



$y^2 = 16h$

$y = \pm 4\sqrt{h}$.

$2y = 8\sqrt{h}$.

$4\sqrt{h}$ is the length of the semi major axis & $2\sqrt{h}$ is the length of the semi minor axis (1M)

Not necessary

[Equ of the ellipse - $\frac{x^2}{16h} + \frac{y^2}{4h} = 1$]

Area of the semi ellipse

$= \frac{1}{2}\left(\pi \times 4\sqrt{h} \times 2\sqrt{h}\right)$

$= 4\pi h$. (1M)

(ii) $V = \int_0^6 4\pi\, x\, dx$

$= 4\pi\left(\frac{x^2}{2}\right)_0^6$

$= 72\pi$

c) p is a complex root of $z^7=1$

$\therefore p^2, p^3, \ldots p^6$ are the other complex roots

$1, p, p^2, \ldots p^6$ are the roots of $z^7=1$

$\therefore \underbrace{1+p+p^2+\ldots+p^6}_{\text{Sum of roots}} = 0$

(ii) α is a root of $x^2+ax+b=0$

β, the other root is $\bar{\alpha}$

~~$\beta = 1+p$~~ $\beta = \overline{p+p^2+p^4}$

$= \bar{p} + \overline{p^2} + \overline{p^4}$

$\bar{p} = \overline{\text{cis}\frac{2\pi}{7}} = \text{cis}(2\pi - 2\pi/7) = \text{cis}\frac{12\pi}{7}$

$= p^6$

Similarly $\overline{p^2} = \text{cis}(2\pi - 4\pi/7) = \text{cis}\,10\pi/7 = p^5$

$\overline{p^4} = \text{cis}(2\pi - 8\pi/7) = \text{cis}\,6\pi/7 = p^3$

$\therefore \beta = p^3+p^5+p^6.$

$-a =$ sum of roots

$= p+p^2+p^4+p^3+p^5+p^6$

$= -1$ from (1)

$\therefore \underline{a=1}$

$b =$ product of roots

$(p+p^2+p^4)(p^3+p^5+p^6)$

$= p^4+p^6+p^7=1+p^5+(p^7=1)+(p^8=p)$

$+(p^7=1)+(p^9=p^2)+(p^{10}=p^3)$

$= p^4+p^6+1+p^5+1+p+1+p^2+p^3$

$b = \underline{0+2}.$