Student Number: ____________________

St. Catherine's School
Waverley

August 2009

TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

Extension II Mathematics

Time allowed: *3 Hours + 5 mins Reading Time*

INSTRUCTIONS

- Write your STUDENT NUMBER on each page
- All questions are of equal value
- Marks for each part of a question are indicated
- All questions should be attempted on the separate paper provided
- All necessary working should be shown
- Start each question on a NEW page
- Approved scientific calculators and drawing templates may be used
- Standard integrals are printed at the end of the paper

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1},\quad n \neq 1;\ x \neq 0,\ \text{if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x,\quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax},\quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax,\quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax,\quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax,\quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax,\quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a},\quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a},\quad a > 0,\quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right)\quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

QUESTION 1 (15 marks) **Marks**

a) Let $z=\dfrac{3+4i}{1+2i}$. Express z in the form $a+ib$, where a and b are real. **1**

b) Suppose $z=x+iy$ where x and y are real and z is a complex number

(i) Write $|z|^2$ and $\text{Re}(z^2)$ in terms of x and y **2**

(ii) Show that the locus of the points $z=x+iy$ in the complex plane such that **2**

$|z|^2+3\text{Re}(z^2)-4=0$ is given by $x^2-\dfrac{y^2}{2}=1$

(iii) Sketch the locus of z, stating and showing clearly the directrices and focus. **2**

c) Shade the region in the complex plane satisfying simultaneously the following inequalities. **4**

$$\begin{cases} -\dfrac{\pi}{6}<\arg z<\dfrac{\pi}{6} \\ 0< z-\bar{z} <4 \\ |z| \le |z+i| \end{cases}$$

d) (i) Express $\sqrt{3}-i$ in the form $r(\cos\theta+i\sin\theta)$ where $r>0$ and $-\pi<\theta\le\pi$ **2**

(ii) Show, using deMoivre's Theorem, that if n is a positive integer, then **2**

QUESTION 2 (15marks) **Start a *new* page.** **Marks**

a) Find $\displaystyle\int \frac{2x}{\sqrt{1+x^2}}\,dx$ **2**

b) Evaluate $\displaystyle\int_0^{\frac{\pi}{6}} x\cos 3x\ dx$ **3**

c) Evaluate $\displaystyle\int_2^5 \frac{1}{\sqrt{5+4x-x^2}}\,dx$ **3**

d) (i) Determine the values of A, B and C in the identity **2**

$$\frac{5x^2+3x+13}{(x+1)(x^2+4)}\equiv\frac{A}{x+1}+\frac{Bx+c}{x^2+4}$$

(ii) Hence find $\displaystyle\int \frac{5x^2+3x+13}{(x+1)(x^2+4)}\,dx$ **2**

e) Evaluate $\displaystyle\int_0^{\frac{\pi}{3}} \frac{dx}{1-\sin x}$ using the substitution $t=\tan\dfrac{x}{2}$ **3**

QUESTION 3 **(15 marks) Start a *new* page.** **Marks**

a) Given that $x = 2 + i$ is a zero of the polynomial **4**

$$P(x) = x^4 - 2x^3 - 7x^2 + 26x - 20$$

Solve the polynomial equation $P(x) = 0$

b) The cubic $x^3 + 5x^2 + 11 = 0$ has roots α, β and γ **3**

Find the cubic with roots α^2, β^2 and γ^2

c) Given that $P(x) = x^3 + 3px + q$ has a factor $(x - k)^2$

(i) Show that $p = -k^2$ **1**

(ii) Find q in terms of k **1**

(iii) Hence verify that $4p^3 + q^2 = 0$ **2**

d) Suppose 1, w, w^2 are the roots of the equation $x^3 - 1 = 0$ where w, w^2 are the complex roots.

(i) Show that $1 + w + w^2 = 0$ **1**

(ii) If the equation $x^3 - 1 = 0$ and $px^5 + qx + r = 0$ have a common root

then evaluate $(p + q + r)(pw^5 + qw + r)(pw^{10} + qw^2 + r)$ **3**

QUESTION 4 **(15 marks) Start a *new* page.** **Marks**

a) Let $f(x) = -x^2 + 6x - 8$

On separate diagrams ($\frac{1}{3}$ page minimum) and without using calculus, sketch the following graphs. Indicate clearly any asymptotes and intercepts with the axes.

(i) $y = f(x)$ **2**

(ii) $y = \dfrac{1}{f(x)}$ **2**

(iii) $y = |f(x)|$ **2**

(iv) $y = e^{f(x)}$ **2**

(v) $y^2 = f(x)$ **2**

Question was incorrect!!!

b) A sequence of terms u_n, $n = 1, 2, 3, \ldots\ldots$ is defined by the recurrence relation **5**

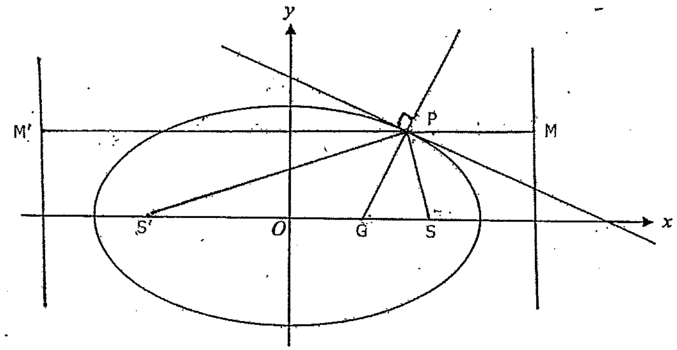
$u_n = 4u_{n-1} - 5u_{n-2}$ $+2u_{n-3}$, $n = 4, 5, 6, \ldots\ldots$ together with the initial conditions

$u_1 = 3,\ u_2 = 1,\ u_3 = 0.$

Show by mathematical induction that $\boxed{u_n = 2^{n-1} - 3n + 5}$ for all $n \geq 1$

QUESTION 5 (15 marks) Start a *new* page. **Marks**

a)



P is the point $(a\cos\theta,\ b\sin\theta)$ on the ellipse with equation $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$.

Without loss of generality assume $a>b$. S and S' are the two foci and M and M' are the feet of the perpendiculars from P onto the directrices corresponding to S and S' respectively

The normal to the ellipse at P meets the major axis of the ellipse at G.

(i) Using the fact that $SP=ePM$ where $0<e<1$, or otherwise, prove that $SP+S'P=2a$ **1**

(ii) Show that the equation of the normal at P is **2**

$$y-b\sin\theta=\frac{a\tan\theta}{b}(x-a\cos\theta)$$

(iii) Show that the coordinates of G are $\left(\dfrac{(a^2-b^2)\cos\theta}{a},0\right)$ **2**

(iv) Show that $\dfrac{GS}{GS'}=\dfrac{1-e\cos\theta}{1+e\cos\theta}=\dfrac{PS}{PS'}$ **4**

Question 5 continued next page.

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b) When the polynomial $P(x)$ is divided by $x-2$ the remainder is 4, and when Divided by $x-3$ the remainder is 7. Find the remainder when $P(x)$ is divided by x^2-5x+6 **3**

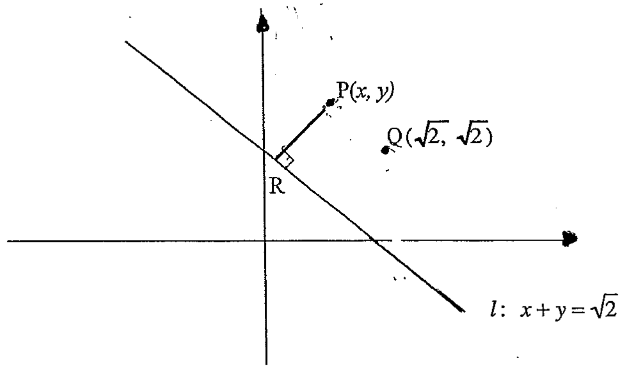
c) The sum of the first k positive integers can be written as **3**

$$1+2+3+.......+(k-1)+k=\frac{k(k+1)}{2}$$

Given that n is a non-zero positive integer ~~show that~~ the sum of the integers between 1 and $15n$ inclusive which are ~~not divisible by 3~~ is $75n^2$

QUESTION 6 (15 marks) Start a *new* page.

a)



The foot of the perpendicular from a variable point $P(x, y)$ to the straight line $x+y=\sqrt{2}$ is the point R, and Q is the point with coordinates $(\sqrt{2},\sqrt{2})$

$P(x, y)$ varies in such a way that $PQ^2 = 2PR^2$

(i) Show that that the locus of $P(x, y)$ is the rectangular hyperbola $xy=1$ **2**

(ii) Show that the equation of the tangent to the hyperbola $xy=1$ at the point $\left(t,\ \frac{1}{t}\right)$ $t \neq 0$ is $yt^2 + x - 2t = 0$ **2**

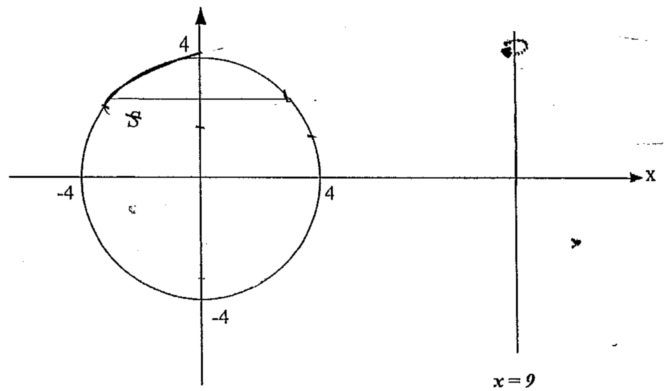
(iii) The tangent found in (ii) cuts the x-axis at A and the y-axis at B. Find the coordinates of A and B **2**

(iv) If C is the internal point on AB such that the ratio $AC : CB = a : b$ **2**

Show that the locus of C as t varies is the rectangular hyperbola

$$xy = \frac{4ab}{(a+b)^2}$$

b)



The area inside the circle $x^2+y^2=16$ is rotated about the line $x=9$ to form a torus (doughnut). When the circle is rotated, the line segment S at height y sweeps out an annulus.

The x coordinates of the end-points of S are x_1 and $-x_1$ where $x_1=\sqrt{16-y^2}$

(i) Show that the area of the annulus swept out by S is **4**

$$36\pi\sqrt{16-y^2}$$

(ii) Hence, find the volume of the torus. **3**

QUESTION 7 (15 marks) Start a *new* page. **Marks**

a) A solid figure has its base in the ellipse $\frac{x^2}{16}+\frac{y^2}{9}=1$

Cross sections perpendicular to the major axis produce right angled isosceles triangles with one of the equal sides in the ellipse.

(i) At a point where $x=k$ show that the area of the cross sectional triangle is; $\frac{9}{8}(16-k^2)$ **2**

(ii) Hence calculate the volume of the solid. **3**

b) Suppose k is a constant greater than 1. Let $f(x)=\frac{1}{1+(\tan x)^k}$ where $0\le x\le\frac{\pi}{2}$

It is given that $f\left(\frac{\pi}{2}\right)=0)$

(i) Show that $f(x)+f\left(\frac{\pi}{2}-x\right)=1$ for $0\le x\le\frac{\pi}{2}$ **2**

(ii) Show that $f'(x)=f'\left(\frac{\pi}{2}-x\right)$ **1**

(iii) Sketch $y=f(x)$ for $0\le x\le\frac{\pi}{2}$ **4**

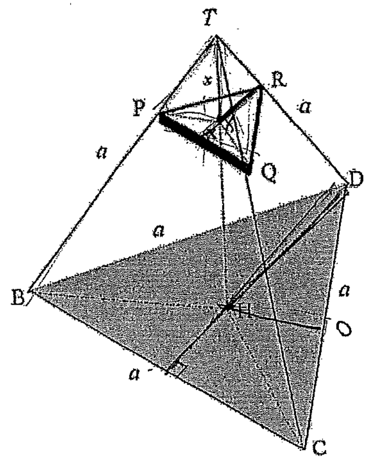
There is no need to find $f'(x)$ but assume $y=f(x)$ has a horizontal tangent at $x=0$.

[Hint: Your graph should exhibit the property of b) (i)]

(iv) Hence, or otherwise evaluate $\int_0^{\frac{\pi}{2}}\frac{dx}{1+(\tan x)^k}$ **3**

QUESTION 8 (15 marks) Start a *new* page. **Marks**

a)



A regular tetrahedron *TBCD* has 6 sides of length a units.
The point H marks the centre of the equilateral. The line TXH is perpendicular to the plane BDC. The plane PQR is parallel to the plane BDC.
TX is taken x units from T such that $0<x\le TH$

(i) Given that BH is $\frac{2}{3}$ of the altitude from B to DC, Show that $TH^2=\frac{2}{3}a^2$ **2**

(ii) Explain why triangle THD is similar to triangle TXR **1**

(iii) Show, with reasons, that the cross sectional area of the slice ΔPQR is given by $\frac{3\sqrt{3}}{8}x^2$ **3**

(iv) Hence by considering the typical slice ΔPQR of thickness δx units, show that the volume of the tetrahedron $TBCD$ is $\frac{a^3\sqrt{2}}{12}$ cubic units **3**

b) (i) Show that $\dfrac{x^{n-2}-x^n}{\sqrt{1-x^2}} = x^{n-2}\sqrt{1-x^2}$ **1**

(ii) Let $I_n = \displaystyle\int_0^1 \frac{x^n}{\sqrt{1-x^2}}dx$. **2**

By writing $\dfrac{x^n}{\sqrt{1-x^2}}$ in the form $x^{n-1}.\dfrac{x}{\sqrt{1-x^2}}$ show that for n, a

positive integer greater than one that;

$$I_n = \left(\frac{n-1}{n}\right)I_{n-2}$$

(iii) Hence or otherwise, evaluate $\displaystyle\int_0^1 \frac{x^5}{\sqrt{1-x^2}}dx$ **3**

End of Paper

Question 1:

a) $z = \frac{3+4i}{1+2i} \times \frac{1-2i}{1-2i}$

$= \frac{11-2i}{5}$

$= \frac{11}{5} - \frac{2}{5}i$

b) (i) $|z|^2 = \left(\sqrt{x^2+y^2}\right)^2$

$= x^2+y^2$

$z^2 = (x+iy)^2$

$= (x^2-y^2)+2ixy$

$\therefore Re(z^2) = x^2-y^2$

(ii) $|z|^2 + 3Re(z^2) - 4 = 0$

$x^2+y^2+3(x^2-y^2)-4 = 0$

$x^2+y^2+3x^2-3y^2-4 = 0$

$4x^2-2y^2-4 = 0$

$\therefore x^2 - \frac{y^2}{2} = 1$

(iii) Locus of z is an hyperbola

$b^2 = a^2(e^2-1)$

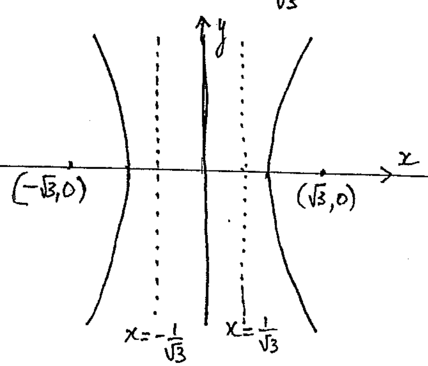
$2 = 1(e^2-1)$

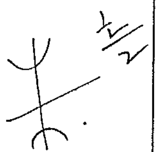
$e = \sqrt{3}$

$\therefore$ foci are $(-\sqrt{3},0)$ and $(\sqrt{3},0)$

directrices $x = \pm\frac{a}{e}$

$\therefore x = \pm\frac{1}{\sqrt{3}}$





Question 1:

c) $0 < z-\bar{z} < 4$

$\therefore 0 < x+iy - (x-iy) < 4$ error

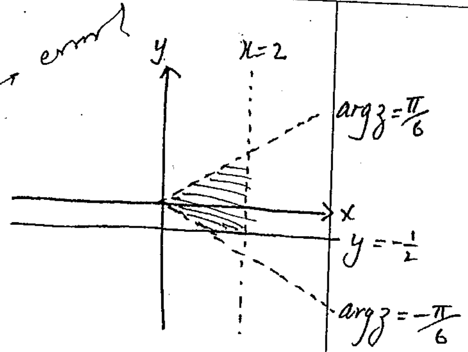
$\therefore 0 < iy < 2$

$|z| \le |z+i|$

$\therefore \sqrt{x^2+y^2} \le \sqrt{x^2+(y+1)^2}$

$x^2+y^2 \le x^2+y^2+2y+1$

$-\frac{1}{2} \le y$



d) (i) $r(\cos\theta + i\sin\theta) = \sqrt{3}-i$

$r\cos\theta = \sqrt{3}$ — ①

$r\sin\theta = -1$ — ②

$r^2 = 3+1$

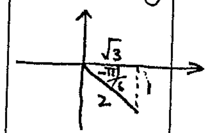
$\therefore r = 2$

② ÷ ① ⇒ $\tan\theta = -\frac{1}{\sqrt{3}}$

$\therefore \theta = -\frac{\pi}{6}$

$\therefore \sqrt{3}-i = 2\left[\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)\right]$

quite acceptable to achieve the answer from a diagram.



(ii) $(\sqrt{3}-i)^n + (\sqrt{3}+i)^n$

$= 2^n\left[\cos\left(\frac{-\pi}{6}\right) + i\sin\left(\frac{-\pi}{6}\right)\right]^n + 2^n\left[\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right]^n$

$= 2^n\left[\cos\frac{n\pi}{6} - i\sin\frac{n\pi}{6} + \cos\frac{n\pi}{6} + i\sin\frac{n\pi}{6}\right]$

$= 2^n\left[2\cos\frac{n\pi}{6}\right]$

$= 2^{n+1}\cdot\cos\frac{n\pi}{6}$ which is real.

Question 2 a) $\int \frac{2x}{\sqrt{1+x^2}}\,dx$ let $u = 1+x^2$, $du = 2x\,dx$

$= \int \frac{du}{\sqrt{u}}$

$= \int u^{-1/2}\,du$

$= 2u^{1/2} + c$

$= 2\sqrt{1+x^2} + c$

b) $\int_0^{\pi/6} x\cos 3x\,dx$ $u = x$, $v = \frac{\sin 3x}{3}$, $du = 1$, $v' = \cos 3x$

$= \left[\frac{x\sin 3x}{3}\right]_0^{\pi/6} - \int_0^{\pi/6} \frac{\sin 3x}{3}\,dx$

$= \left(\frac{\pi}{18} - 0\right) + \frac{1}{3}\left[\frac{\cos 3x}{3}\right]_0^{\pi/6}$

$= \frac{\pi}{18} + \frac{1}{3}\left[0 - \frac{1}{3}\right]$

$= \frac{\pi}{18} - \frac{1}{9} = \frac{\pi - 2}{18}$

c) $\int_2^5 \frac{1}{\sqrt{5+4x-x^2}}\,dx$ Note: $5+4x-x^2 = 9-4+4x-x^2 = 9-(x^2-4x+4) = 9-(x-2)^2$

$= \int_2^5 \frac{1}{\sqrt{9-(x-2)^2}}\,dx$ 1

$= \left[\sin^{-1}\left(\frac{x-2}{3}\right)\right]_2^5$ 1

$= \sin^{-1}1 - \sin^{-1}0$

$= \frac{\pi}{2}$ 1

Question 2

d) (i) $\frac{5x^2+3x+13}{(x+1)(x^2+4)} \equiv \frac{A}{x+1} + \frac{Bx+C}{x^2+4}$

$= \frac{Ax^2+4A+Bx^2+Cx+Bx+C}{(x+1)(x^2+4)}$

$= \frac{(A+B)x^2+(B+C)x+4A+C}{(x+1)(x^2+4)}$

$\therefore A+B = 5$ — ①

$B+C = 3$ — ②

$4A+C = 13$ — ③

③ − ② $4A - B = 10$ — ④

① + ④ $5A = 15$

$A = 3$

$\therefore B = 2$

$\therefore C = 1$

$\therefore \int \frac{5x^2+3x+13}{(x+1)(x^2+4)}\,dx = \int\left(\frac{3}{x+1} + \frac{2x+1}{x^2+4}\right)dx$

$= \int \frac{3}{x+1}\,dx + \int \frac{2x}{x^2+4}\,dx + \int \frac{1}{x^2+4}\,dx$

$= 3\ln(x+1) + \ln(x^2+4) + \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + c$

e). $\int_0^{\pi/3} \frac{dx}{1-\sin x}$ If $t = \tan\frac{x}{2}$ $dx = \frac{2dt}{1+t^2}$ *

$\sin x = \frac{2t}{1+t^2}$ 1

$x = 0$ $t = 0$

$x = \frac{\pi}{3}$ $t = \frac{1}{\sqrt{3}}$

$= \int_0^{1/\sqrt{3}} \frac{\cancel{1+t^2}}{1+t^2-2t} \cdot \frac{2dt}{\cancel{1+t^2}}$

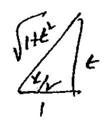
$= 2\int_0^{1/\sqrt{3}} \frac{1}{(t-1)^2}\,dt$

$= 2\int_0^{1/\sqrt{3}} (t-1)^{-2}\,dt$

$= -2\left[\frac{1}{t-1}\right]_0^{1/\sqrt{3}}$ 1

$= -2\left[\frac{\sqrt{3}}{1-\sqrt{3}} + 1\right] = -2\left[\frac{1}{1-\sqrt{3}}\right] = -\frac{2}{1-\sqrt{3}} \cdot \frac{1+\sqrt{3}}{1+\sqrt{3}} = \frac{-\cancel{2}(1+\sqrt{3})}{-\cancel{2}} = 1+\sqrt{3}$ 1

* accepted, but some will calculate



$\frac{dt}{dx} = \frac{1}{2}\sec^2\frac{x}{2}$

$\frac{dt}{dx} = \frac{1+t^2}{2}$

$\therefore dx = \frac{2dt}{1+t^2}$

Question 3.

a). If $x = 2+i$ is a root then $x = 2-i$ is also (coefficients of polynomial are real).

$\therefore [x-(2+i)][x-(2-i)]$ is a factor

i.e. $x^2-(2+i)x-(2-i)x+(2-i)(2+i)$ is a factor

(x^2-4x+5) is a factor.

$\therefore P(x) = (x^2-4x+5)(x^2+2x-4)$ (by division). 1

$\therefore$ roots of $P(x)=0$ are

$x = 2+i, 2-i, \frac{-2\pm\sqrt{20}}{2}$

ie $x = 2+i, 2-i, -1\pm\sqrt{5}$ 1

$2-i+2+i+\alpha+\beta = 2$

$\alpha+\beta = -2$

$(2-i)(2+i)\alpha\beta = -20$

$\alpha\beta = -4$

$\alpha - \frac{4}{\alpha} = -2$

$\alpha^2+2\alpha-4=0$

$\alpha = -\frac{2\pm\sqrt{4+16}}{2}$

$= -1\pm\sqrt{5}$

$\beta = \frac{-4}{(-1\pm\sqrt{5})}\cdot\frac{(-1\mp\sqrt{5})}{(-1\mp\sqrt{5})}$

$=$

b). $x^3+5x^2+11=0$ has roots α, β, γ

If α, β, γ satisfy $x^3+5x^2+11=0$

then $\alpha^2, \beta^2, \gamma^2$ satisfy $(\sqrt{x})^3+5(\sqrt{x})^2+11=0$ 1

$\therefore x^{3/2}+5x+11=0$ 1

$\therefore x^{3/2} = -5x-11$

$x^3 = [-(5x+11)]^2$

$x^3 = (5x+11)^2$ 1

$x^3 = 25x^2+110x+121$

$\therefore$ the required polynomial is

$x^3-25x^2-110x-121=0$

Question 3:

c) (i) $P(x) = x^3+3px+q$ has double root at $x=k$.

$\therefore x=k$ is a root of $P'(x)$

now $P'(x) = 3x^2+3p$

$\therefore 3k^2+3p=0$

$\therefore k^2=-p$

and. $p=-k^2$ 1

(ii) Since k is a root of $P(x)$

$k^3+3pk+q=0$

but $p=-k^2$ $\therefore k^3-3k^3+q=0$

$\therefore q=2k^3$ 1

(iii) $4p^3+q^2 = 4(-k^2)^3+(2k^3)^2$

$= -4k^6+4k^6$

$= 0$

d) (i) If $1, \omega, \omega^2$ are the roots of $x^3-1=0$

then $1+\omega+\omega^2=0$ since the sum of the roots of $x^3-1=0$ is zero.

(ii) If $x^3-1=0$ has roots $1, \omega, \omega^2$

If $x=1$ is a common root $p+q+r=0$

If $x=\omega$ is a common root $p\omega^5+q\omega+r=0$

If $x=\omega^2$ is a common root $p\omega^{10}+q\omega^2+r=0$

$\therefore (p+q+r)(p\omega^5+q\omega+r)(p\omega^{10}+q\omega^2+r)$ must equal zero.

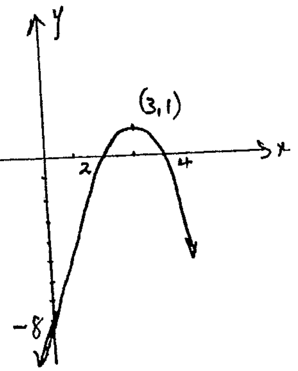
Question 4: $y = f(x) = -x^2+6x-8$

$= -(x^2-6x+8)$

$= -(x-2)(x-4)$

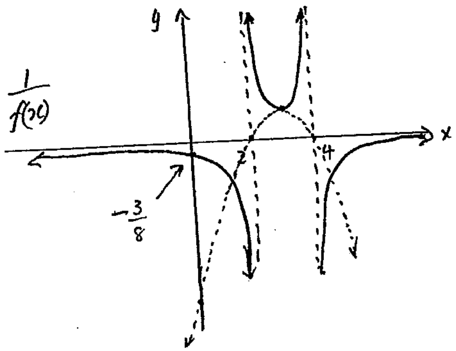
a) (i)

$y = f(x)$



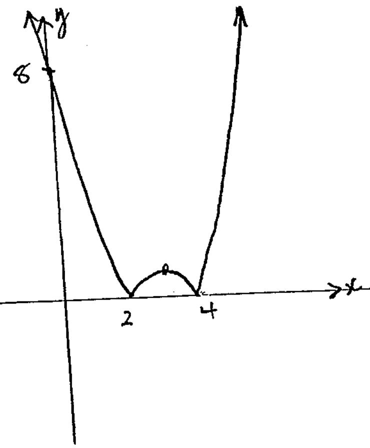
(ii)

$y = \dfrac{1}{f(x)}$



(iii)

$y = |f(x)|$



Question 4 a)

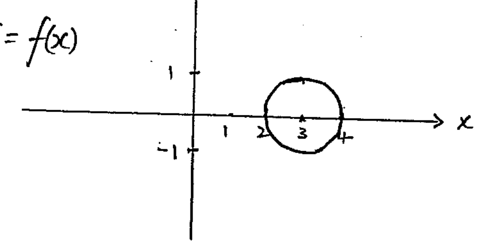
(iv)

$y^2 = -x^2+6x-8$

$x^2+y^2-6x = -8$

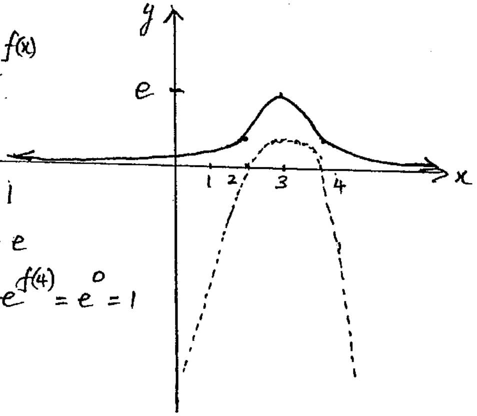
$(x-3)^2+y^2 = 1$

$y^2 = f(x)$



(iv)

$y = e^{f(x)}$



$f(3) = 1$

$\therefore e^{f(3)} = e$

$e^{f(2)} = e^{f(4)} = e^0 = 1$

(b)

(over page)

Qn	Solutions	Marks	Comments+Criteria

(b) Define statement $S(n)$: $u_n = 2^{n-1} - 3n + 5 \quad n = 1, 2, 3 \ldots$

Consider $S(1)$ $\quad 2^0 - 3 + 5 = 3 = u_1 \quad \therefore S(1)$ true

$S(2) \quad 2^1 - 6 + 5 = 1 = u_2 \quad \therefore S(2)$ true

$S(3) \quad 2^2 - 9 + 5 = 0 = u_3 \quad \therefore S(3)$ true

Let k be a positive integer $k \geq 3$

If $S(n)$ is true for all integers $n \leq k$

then $u_n = 2^{n-1} - 3n + 5$ for $n = 1, 2, 3, \ldots, k$

Consider $S(k+1)$

$$u_{k+1} = 4u_k - 5u_{k-1} + 2u_{k-2} \quad (\text{since } k+1 \geq 4)$$

$$\therefore u_{k+1} = 4(2^{k-1} - 3k + 5) - 5(2^{k-2} - 3(k-1) + 5) + 2(2^{k-3} - 3(k-2) + 5)$$

$$= 2^{k-2}(8 - 5 + 1) - 3k(4 - 5 + 2) + (20 - 40 + 22)$$

$$= 2^k - 3(k+1) + 5 \quad \text{if } S(n) \text{ is true } n = 1, 2, 3 \ldots k$$

for $k = 3, 4, \ldots$, $S(n)$ true for all positive integers $n \leq k$

implies $S(k+1)$ is true

But $S(1)$ $S(2)$ $S(3)$ are true

$\therefore$ by induction $S(n)$ true for all $n \geq 1$

St Catherines's School Waverley

Course:

Marking Scheme for Task:

Academic Year: 2008-9

Question 5 a). $P\,(a\cos\theta, b\sin\theta)$

$S\,(ae, 0)$

$S'(-ae, 0)$

(i) $SP = e\,PM$ (by definition)

$S'P = e\,PM'$ (" ").

adding

$$SP + SP' = e(PM + PM')$$

$$= e\left(2 \times \frac{a}{e}\right)$$

$$= 2a$$

(ii) differentiating $\quad \dfrac{2x}{a^2} + \dfrac{2y}{b^2} \cdot \dfrac{dy}{dx} = 0$

$$\therefore \frac{dy}{dx} = -\frac{b^2 x}{a^2 y}$$

at P $\quad \dfrac{dy}{dx} = -\dfrac{b^2 a\cos\theta}{a^2 b\sin\theta}$

$$= -\frac{b\cot\theta}{a}$$

$\therefore$ gradient of normal at $P = \dfrac{a\tan\theta}{b}$

$\therefore$ equation of normal:

$$y - b\sin\theta = \frac{a\tan\theta}{b}(x - a\cos\theta)$$

(iii) now at G $\quad y = 0$

$$\therefore -b^2\sin\theta = a\tan\theta\, x - a^2\sin\theta$$

$$\therefore x = \frac{(a^2 - b^2)\sin\theta}{a\tan\theta}$$

$$= \frac{(a^2 - b^2)\cos\theta}{a}$$

$\therefore$ coordinates of G are $\left[\dfrac{(a^2 - b^2)\cos\theta}{a}, 0\right]$

Question 5 a)

(iv) $$\frac{GS}{GS'} = \frac{-\frac{(a^2-b^2)\cos\theta}{a} + ae}{\frac{(a^2-b^2)\cos\theta}{a} + ae}$$

$$= -\frac{(a^2-b^2)\cos\theta + a^2e}{(a^2-b^2)\cos\theta + a^2e}$$

$$= -\frac{(a^2-a^2+a^2e^2)\cos\theta + a^2e}{(a^2-a^2+a^2e^2)\cos\theta + a^2e}$$

$$= \frac{a^2e(1-e\cos\theta)}{a^2e(1+e\cos\theta)}$$

$$= \frac{1-e\cos\theta}{1+e\cos\theta}$$

$$\frac{PS}{PS'} = \frac{PM}{PM'} = \frac{\frac{a}{e} - a\cos\theta}{\frac{a}{e} + a\cos\theta}$$

$$= \frac{a(1-e\cos\theta)}{a(1+e\cos\theta)}$$

$$= \frac{1-e\cos\theta}{1+e\cos\theta}$$

$$= \frac{GS}{GS'}$$

b)

$P(x) = (x-2)\,Q(x) + 4$ or $P(2) = 4$

$P(x) = (x-3)\,R(x) + 7$ or $P(3) = 7$

now when $P(x)$ is divided by x^2-5x+6

$P(x) = (x^2-5x+6)\,G(x) + (ax+b)$ note: linear remainder

now $P(2) = 4$ $\therefore 2a+b = 4$ — ①

$\therefore P(3) = 7$ $\therefore 3a+b = 7$ — ②

② − ① $a = 3$

$b = -2$

$\therefore$ remainder is $(3x-2)$

Question 5 c)

Given $1+2+3+\dots+(k-1)+k = \frac{k(k+1)}{2}$

$\therefore$ for $1+2+3+\dots+15n$

$T_N = a+(N-1)d$ $\quad S_{15n} = \frac{15n}{2}[1+15n]$

$15n = 1+(N-1)1$

$\therefore N = 15n$

for $3+6+9+\dots+15n$ (numbers divisible by 3)

$T_N = 3+(N-1)3$ $\quad S_{5n} = \frac{5n}{2}[3+15n]$

$15n = 3N$ $\quad = \frac{15n}{2}(1+5n)$

$\therefore N = 5n$

$\therefore$ Sum of integers not divisible by 3

$= S_{15n} - S_{5n}$

$= \frac{15n}{2}(1+15n) - \frac{15n}{2}(1+5n)$

$= \frac{225n^2}{2} - \frac{75n^2}{2}$

$= \frac{150n^2}{2}$

$= 75n^2$

Question 6 a)

(i) distance $PR = \left|\dfrac{x+y-\sqrt2}{\sqrt2}\right|$

distance $PQ = \sqrt{(x-\sqrt2)^2+(y-\sqrt2)^2}$

$PQ^2 = 2 \times PR^2$

$(x-\sqrt2)^2+(y-\sqrt2)^2 = 2\left[\dfrac{x+y-\sqrt2}{\sqrt2}\right]^2$

$\cancel{x^2} - \cancel{2\sqrt2 x} + 2 + \cancel{y^2} - \cancel{2\sqrt2 y} + \cancel{2} = \cancel{x^2} + xy - \cancel{x\sqrt2} + xy + \cancel{y^2} - \cancel{\sqrt2 y} - \cancel{\sqrt2 x} + \cancel{\sqrt2 y} + \cancel{2}$

$2 = 2xy$

$\therefore xy = 1$

(ii) $y = \frac{1}{x}$ $\quad \frac{dy}{dx} = -\frac{1}{x^2}$

at $\left(t, \frac{1}{t}\right)$ $\frac{dy}{dx} = -\frac{1}{t^2}$

$\therefore$ equation of tangent at P is

$y - \frac{1}{t} = -\frac{1}{t^2}(x-t)$

$yt^2 + x - 2t = 0$

(iii) at A $y=0$ $\therefore x = 2t$ $\therefore A(2t, 0)$

at B $x = 0$ $\therefore y = \frac{2}{t}$ $\therefore B\left(0, \frac{2}{t}\right)$

(iv) A $(2t, 0)$ $\quad a \quad \overset{C}{\times} \quad b \quad$ B $\left(0, \frac{2}{t}\right)$

$\therefore$ Coordinates of C are. $\left(\dfrac{2bt}{a+b}, \dfrac{2a}{t(a+b)}\right)$ (division of a line internally)

i.e. $x = \dfrac{2bt}{a+b}$ $\quad y = \dfrac{2a}{t(a+b)}$

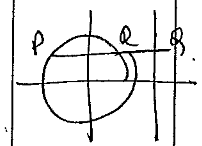
$x \times y = \dfrac{2bt}{a+b} \times \dfrac{2a}{t(a+b)}$

$\therefore xy = \dfrac{4ab}{(a+b)^2}$ is the locus of C

Question 6 b)

(i) let the points on the circle at height y be P and Q

produce PQ to R on the line $x = 9$



Q has coordinates $x_1 = \sqrt{16-y^2}$

P has coordinates $-x_1 = -\sqrt{16-y^2}$

$|PR| = 9 + \sqrt{16-y^2}$

$|QR| = 9 - \sqrt{16-y^2}$

Area of the annulus $= \pi PR^2 - \pi \cancel{PQ^2}\, QR^2$

$= \pi(PR + \cancel{PQ}\,QR)(PR - \cancel{PQ}\,QR)$

$= \pi\left[18 \cdot 2\sqrt{16-y^2}\right]$

$A(y) = 36\pi\sqrt{16-y^2}$

(ii) $\delta V = \displaystyle\lim_{\delta y \to 0} \sum_{y=-4}^{4} A(y)\,\delta y$

$= \displaystyle\lim_{\delta y \to 0} \sum_{y=-4}^{4} 36\pi\sqrt{16-y^2}\,\delta y$

$= \displaystyle\int_{-4}^{4} 36\pi\sqrt{16-y^2}\,dy$

$= 36\pi \displaystyle\int_{-4}^{4} \sqrt{16-y^2}\,dy$

$= 36\pi \cdot \frac{1}{2}\pi \cdot 4^2$ $\quad \left(\displaystyle\int_{-4}^{4}\sqrt{16-y^2}\,dy \text{ is area of semi circle}\right)$

$= 36\pi \cdot 8\pi$

$= 288\pi^2$ units^3

alternative for finding $\int_{-4}^{4}\sqrt{16-y^2}\,dy$

let $y = 4\sin\theta \quad \therefore dy = 4\cos\theta\, d\theta$

$\theta = \sin^{-1}\left(\frac{y}{4}\right)$ when $y=-4 \quad \theta = -\frac{\pi}{2}$

$y = 4 \quad \theta = \frac{\pi}{2}$

$$\therefore \int_{-4}^{4}\sqrt{16-y^2}\,dy = \int_{-\pi/2}^{\pi/2}\sqrt{16-16\sin^2\theta}\,.\,4\cos\theta\, d\theta$$

$$= \int_{-\pi/2}^{\pi/2} 4\sqrt{1-\sin^2\theta}\,.\,4\cos\theta\, d\theta$$

$$= 16\int_{-\pi/2}^{\pi/2}\cos^2\theta\, d\theta$$

$$= 32\int_{0}^{\pi/2}\cos^2\theta\, d\theta \qquad (\cos^2\theta \text{ is even function})$$

$$= 32\int_0^{\pi/2}\frac{1}{2}(1+\cos 2\theta)\, d\theta$$

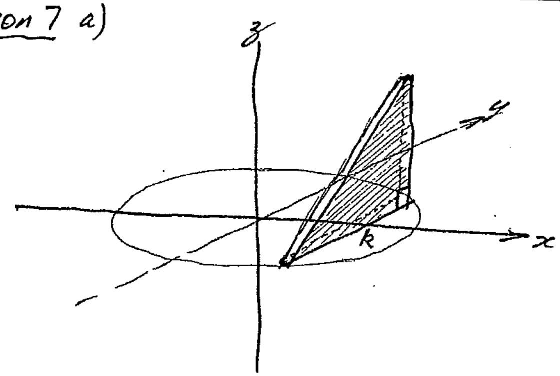
$$= 16\left[\theta + \frac{\sin 2\theta}{2}\right]_0^{\frac{\pi}{2}}$$

$$= 16\left[\left(\frac{\pi}{2}+\frac{\sin\pi}{2}\right)-\left(0+\frac{\sin 0}{2}\right)\right]$$

$$= 16\left[\frac{\pi}{2}\right]$$

$$= 8\pi$$

Question 7 a)



(i) $\frac{x^2}{16}+\frac{y^2}{9}=1$

When $x=k \quad \frac{y^2}{9} = 1-\frac{k^2}{16}$

$y^2 = 9\left(\frac{16-k^2}{16}\right)$

$\therefore y = \frac{3}{4}\sqrt{16-k^2}$ 1

now area of Δ is $\frac{1}{2}\cdot 2y\cdot 2y$

$= \frac{1}{2}\cdot 4y^2$

$= \frac{1}{2}\cdot 4\cdot\frac{9}{16}(16-k^2)$

$= \frac{9}{8}(16-k^2)$ 1

(ii) Volume of typical slice:

$\delta V = \frac{9}{8}(16-x^2)\,.\,\delta x$

$\therefore V = 2\int_0^4 \frac{9}{8}(16-x^2)\,dx$ 1

$= \frac{9}{4}\left[16x-\frac{x^3}{3}\right]_0^4$ 1

$= \frac{9}{4}\left[64-\frac{64}{3}\right] = \frac{\cancel{9}^3}{4}\times\frac{\cancel{128}^{32}}{\cancel{3}}$ 1

$= 288 \text{ units}^3$

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Question 7 b)

(i) $f(x) = \dfrac{1}{1+(\tan x)^k} \quad 0 \le x \le \frac{\pi}{2}$

$f\left(\frac{\pi}{2}\right) = 0$ let $u = (\tan x)^k$

$\therefore \left[\tan\left(\frac{\pi}{2} - x\right)\right]^k = (\cot x)^k$

$= \dfrac{1}{u}$

$\therefore f(x) + f\left(\frac{\pi}{2} - x\right) = \dfrac{1}{1+u} + \dfrac{1}{1+\frac{1}{u}}$

$= \dfrac{1}{1+u} + \dfrac{u}{1+u}$

$= \dfrac{1+u}{1+u}$

$= 1$ for $0 \le x \le \frac{\pi}{2}$

(ii) Since $f(x) + f\left(\frac{\pi}{2} - x\right) = 1$

$f'(x) - f'\left(\frac{\pi}{2} - x\right) = 0$

$\therefore f'(x) = f'\left(\frac{\pi}{2} - x\right)$

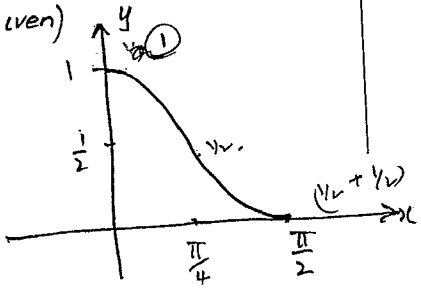
(iii) $f\left(\frac{\pi}{2}\right) = 0$ (given)

$f'(0) = 0$ (given)

$f(0) = 1$ substituting in $f(x)$

$f'\left(\frac{\pi}{2}\right) = 0$ (given)

$f\left(\frac{\pi}{4}\right) = \frac{1}{2}$



Question 7 b(iv)

let $I = \int_0^{\frac{\pi}{2}} f(x)\,dx \qquad f(x) = \dfrac{1}{1+(\tan x)^k}$

let $u = \frac{\pi}{2} - x \quad \therefore \int_0^{\frac{\pi}{2}} f\left(\frac{\pi}{2} - x\right)dx$

$du = -dx \quad = -\int_{\frac{\pi}{2}}^{0} f(u)\,du$

$= \int_0^{\frac{\pi}{2}} f(u)\,du$

$= \int_0^{\frac{\pi}{2}} f(x)\,dx$

$= I$

Now from (i) $f(x) + f\left(\frac{\pi}{2} - x\right) = 1$

Integrating: $\int_0^{\frac{\pi}{2}} f(x)\,dx + \int_0^{\frac{\pi}{2}} f\left(\frac{\pi}{2} - x\right)dx = \int_0^{\frac{\pi}{2}} 1\,dx$

$\therefore I + I = \frac{\pi}{2}$

$I = \frac{\pi}{4}$

Question 8: a) (i) $DM^2 = a^2 - \frac{a^2}{4}$ (pythagoras)

In ΔDMC; $\therefore DM^2 = \frac{3a^2}{4}$ ($DM \perp BC$)

$\therefore DM = \frac{\sqrt{3}a}{2}$ $\frac{1}{2}$

now $DH = \frac{2}{3} \times \frac{\sqrt{3}a}{2}$ (given)

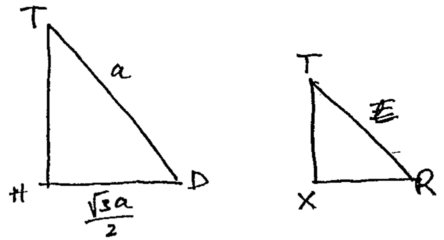
$\therefore DH = \frac{\sqrt{3}a}{3}$ $\frac{1}{2}$

Now $TH^2 = a^2 - DH^2$ (Pythagoras)

$TH^2 = a^2 - \frac{3a^2}{9}$

$\therefore TH^2 = \frac{6a^2}{9} = \frac{2a^2}{3}$ 1

(ii)



$\angle T$ is common

$\angle TXR = \angle THD = 90°$ 1

$\therefore \Delta TXR \,|||\, \Delta THD$

(iii) $\frac{TX}{TH} = \frac{PQ}{BC}$

$\frac{x}{\sqrt{\frac{2}{3}}a} = \frac{PQ}{a}$ 1

$\therefore PQ = \sqrt{\frac{3}{2}}x$

Question 8 a) (iv) if δV is the volume of typical slice of thickness δx units

$\delta V = \frac{3\sqrt{3}}{8}x^2\delta x$ units^3 $\frac{1}{2} \times \sqrt{\frac{3}{2}}x \times \sqrt{\frac{3}{2}}x \times \sin 60°$

$\therefore V = \lim_{\delta x \to 0} \sum_{x=0}^{\sqrt{\frac{2}{3}}a} \frac{3\sqrt{3}}{8}x^2 \delta x$ $= \frac{3\sqrt{3}}{8}x^2$.

$= \int_0^{\sqrt{\frac{2}{3}}a} \frac{3\sqrt{3}}{8}x^2\,dx$

$= \frac{3\sqrt{3}}{8}\left[\frac{x^3}{3}\right]_0^{\sqrt{\frac{2}{3}}a}$

$= \frac{\cancel{3}\cancel{\sqrt{3}}}{\cancel{8}\,4} \cdot \frac{\cancel{2}}{\cancel{3}} \frac{\sqrt{2}}{\cancel{\sqrt{3}}} \cdot \frac{a^3}{3}$ 3.

$= \frac{a^3\sqrt{2}}{12}$ units^3

b) (i) $\frac{x^{n-2} - x^n}{\sqrt{1-x^2}} = x^{n-2}\sqrt{1-x^2}$

LHS. $= \frac{x^n\left(\frac{1}{x^2} - 1\right)}{\sqrt{1-x^2}}$ $x^{n-2}\sqrt{1-x^2}\sqrt{1-x^2}$ $= x^{n-2}(1-x^2)$ $= x^{n-2} - x^n$.

$= \frac{x^n\left(\frac{1-x^2}{x^2}\right)}{(1-x^2)^{1/2}}$

$= \frac{x^{n-2}(1-x^2)}{(1-x^2)^{1/2}}$

$= x^{n-2}(1-x^2)^{1/2}$

$= x^{n-2}\sqrt{1-x^2}$

Question 8 b) ii). $I_n = \int_0^1 \frac{x^n}{\sqrt{1-x^2}}$

$\therefore I_n = \int_0^1 x^{n-1} \cdot \frac{x}{\sqrt{1-x^2}}\,dx$

I.B.P. $u = x^{n-1}$ $\quad v = -\sqrt{1-x^2}$

$u' = (n-1)x^{n-2}$ $\quad v' = \frac{x}{\sqrt{1-x^2}}$

$\therefore I_n = \left[-x^{n-1}\sqrt{1-x^2}\right]_0^1 + \int_0^1 (n-1)x^{n-2}\sqrt{1-x^2}\,dx$

$= 0 + (n-1)\int_0^1 \frac{x^{n-2} - x^n}{\sqrt{1-x^2}}$ [from (i)]

$= (n-1)\left[\int_0^1 \frac{x^{n-2}}{\sqrt{1-x^2}}dx - \int_0^1 \frac{x^n}{\sqrt{1-x^2}}dx\right]$

$= n-1\left[I_{n-2} - I_n\right]$

$(n-1)I_n + I_n = (n-1)I_{n-2}$

$nI_n = n-1\,I_{n-2}$

$I_n = \left(\frac{n-1}{n}\right)I_{n-2}$

(iii) $\int_0^1 \frac{x^5}{\sqrt{1-x^2}} = \frac{4}{5} \cdot \frac{2}{3} \cdot I_1$

now $I_1 = \int_0^1 \frac{x}{\sqrt{1-x^2}}$ $\quad$ let $u = 1-x^2$

$du = -2x\,dx$

$= -\frac{1}{2}\int_0^1 \frac{-2x}{\sqrt{1-x^2}}dx$ $\quad$ $x=0$ $u=1$

$x=1$ $u=0$

Q8 b) (cont)

$= -\frac{1}{2}\int_1^0 \frac{du}{\sqrt{u}}$

$= \frac{1}{2}\int_0^1 u^{-1/2}\,du$

$= \frac{1}{2}\left[2\sqrt{u}\right]_0^1$

$= \frac{1}{2} \times 2$

$= 1$

$\therefore \int_0^1 \frac{x^5}{\sqrt{1-x^2}}dx = \frac{4}{5} \cdot \frac{2}{3} \cdot 1$

$= \frac{8}{15}$

END OF PAPER