St Catherine's School
Waverley, Sydney

Student Name/Number:

Class Teacher (if relevant):

HSC Assessment Task 2
March 2007

Extension I Mathematics

Half Yearly Examination

Time allowed: **2 hours**

Reading time: **5 mins**

Course weighting: 30%

General Instructions

- There are 7 questions of equal value
- Marks for each part of a question are indicated
- All questions should be attempted on the separate paper provided
- All necessary working should be shown
- Start each question on a new page
- Approved scientific calculators and drawing templates may be used badly arranged work.

Sections	Marks
Q1	/12
Q2	/12
Q3	/12
Q4	/12
Q5	/12
Q6	/12
Q7	/12

Total marks = **/84**

Question 1 **12 Marks**

a. Solve for x the inequality $\frac{1}{x} < \frac{1}{x+1}$. [3]

b. Find the acute angle between the lines $2x - y = 0$ and $x + 3y = 0$ giving the answer correct to the nearest minute. [2]

c. A is the point $(-2,-1)$, B is the point $(1,5)$. Find the coordinates of the point Q which divides AB externally in the ratio 5 : 2. [2]

d. Evaluate $\sum_{k=2}^{5} (-1)^k k$ [1]

e. Use the principle of Mathematical Induction to prove that $7^n + 2$ is divisible by 3 for all positive integers n. [4]

Question 2 – Start a new page **12 Marks**

a. Evaluate $\int_0^4 x\sqrt{x^2+9}\ dx$ using the substitution $u = x^2 + 9$ [3]

b. Use a first approximation of $x = 0.5$ with one approximation of Newton's method to solve $2x^2 + x - 2 = 0$ correct to 1 decimal place. [3]

Question 2 Continued on Next Page

Question 2 Continued

c. (i) Factorise completely the polynomial $p(x)=x^3-x^2-8x+12$ given that the equation $p(x)=0$ has a double root. [3]

(ii) Consider the polynomial $q(x)=p(x)(x+a)$, where:
- $p(x)$ is the polynomial as in part (2ci); and
- a is such that $q(x)\ge 0$ for all real values of x.

(a) Sketch $y=p(x)$ [1]

(b) Sketch $y=q(x)$ [1]

(c) Hence or otherwise find the possible value(s) of a. [1]

Question 3 – Start a new page **12 Marks**

a. $P(4p,2p^2)$ and $Q(4q,2q^2)$ are two points of the parabola $x^2=8y$. The chord PQ subtends a right angle at the origin O.

(i) Show that $pq=-4$. [1]

(ii) If M is the midpoint of PQ, find the locus of M as P and Q move on the parabola. [3]

b. Consider the parabola $x^2=8y$.

(i) Show that the equation of the tangent at $P(x_1,y_1)$ on $x^2=8y$ is:
$$x_1x-4y=4y_1$$ [2]

(ii) Hence or otherwise find the chord of contact from the point $A(3,-1)$ to $x^2=8y$. [3]

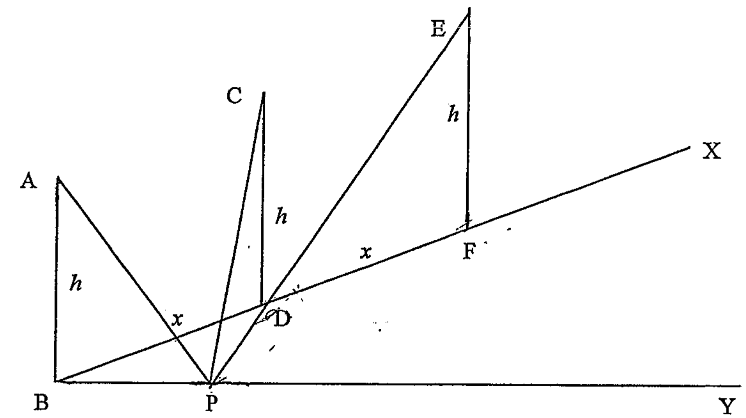
c. (i) Find the derivative of $x\sqrt{x+3}$. [2]

(ii) Hence find $\int \frac{x+2}{\sqrt{x+3}}\,dx$ [1]

Question 4 – Start a new page **12 Marks**

a. If $x=\tan\theta+\sec\theta$ show that $\frac{x^2-1}{x^2+1}=\sin\theta$ [3]

b. In the diagram below, BX and BY represent two roads intersecting at an angle of 60°. On the road BX are situated three telegraph poles AB, CD and EF, all of equal height, the same distance, x metres apart (ie. $BD=DF=x$). P is a point on the road BY. The angles of elevation of A and C from P are 45° and 30° respectively.



(i) Show $BP=h$ and $DP=h\sqrt{3}$ [2]

(ii) By the use of the sine rule in ΔBDP, show that $\angle BDP=30^\circ$ *and* hence that ΔBDP is right angled at P. [2]

(iii) Prove that $x=2h$ [2]

(iv) By the use of the cosine rule in ΔPDF show that $PF=h\sqrt{13}$ *and* hence show that the angle of elevation of E from P is approximately 15.5°. [3]

START NEW BOOKLET

Question 5 – Start a new page **12 Marks**

a. Tangents are drawn at $P(2ap, ap^2)$ and $Q(2aq, aq^2)$ at the ends of a focal chord on $x^2 = 4ay$.

(i) Show that the tangent at P has equation: $y = px - ap^2$ [2]

(ii) Find the equation of the chord PQ. [1]

(iii) Show that $pq = -1$, given that PQ is a focal chord. [1]

(iv) Show that the tangents at P and Q intersect on the directrix. [2]

b. Find the general solution for:

$$2\sin^2 x - \cos x - 1 = 0$$ [3]

c. Noting that $2\cos^2 x \equiv 1 + \cos 2x$ prove that:

$$3 + 4\cos 2x + \cos 4x \equiv 8\cos^4 x.$$ [3]

Question 6 – Start a new page **12 Marks**

a. Consider the function $f(x) = 2\cos^{-1}\frac{x}{3}$. Draw its graph $y = f(x)$ and state its domain and range. [3]

b. Find $\tan[\sin^{-1}(-\frac{2}{3})]$ [2]

c. Find the equation of the tangent to $y = \sin^{-1}(x-1)$ at the point $(\frac{3}{2}, \frac{\pi}{6})$. [3]

d. Find the area under the curve $y = \frac{1}{9+2x^2}$ bounded by the x-axis and the lines $x = \frac{\sqrt{3}}{\sqrt{2}}$ and $x = \frac{3}{\sqrt{2}}$. [3]

Question 7 – Start a new page **12 Marks**

a. Evaluate $\int \frac{\sqrt{1-x^2}}{x^2}\,dx$ using the substitution $x = \cos\theta$. [4]

b. Consider the function: $y = (x-4)^2 + 2$

(i) Sketch $y = f(x)$ and $y = f^{-1}(x)$ for the largest domain containing only positive numbers for which $f(x)$ has an inverse function. [2]

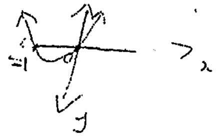
(ii) Find any point(s) of intersection of the graphs of $y = f(x)$ and $y = f^{-1}(x)$ for the domain stated in part b(i) above. [2]

(iii) For $x \geq 4$, find the inverse function of $y = f(x)$. [2]

(iv) Let N be a real number not in the domain, ie., $x < 4$. Show that:

$$f^{-1}(f(N)) = 8 - N$$ [2]

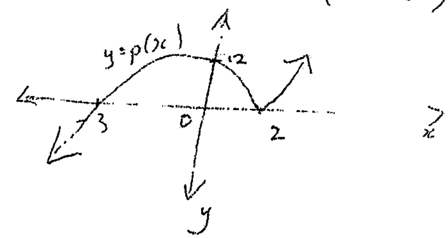
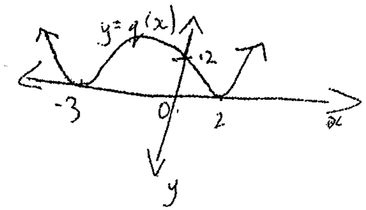
End of Task

	Solutions	Marks	Comments+Criteria
1a	$\frac{1}{x} < \frac{1}{x+1}$ Note $x \neq 0$, $x \neq -1$ ($\times x^2(x+1)^2$)		
	$(x+1)^2 x < (x+1)x^2$	1	
	$x(x+1)[(x+1)-x] < 0$		
	$x(x+1) < 0$	1	
	$-1 < x < 0$ 	1	
b	$2x - y = 0$ $\quad y = 2x \quad \therefore m_1 = 2$ $\qquad x + 3y = 0 \quad y = -\frac{x}{3} \quad m_2 = -\frac{1}{3}$	1	
	$\tan\theta = \left\lvert \frac{m_1 - m_2}{1 + m_1 m_2} \right\rvert$		
	$= \left\lvert \frac{2 + 1/3}{1 + 2 \times -\frac{1}{3}} \right\rvert$	$\frac{1}{2}$	
	$= \left\lvert \frac{2\frac{1}{3}}{1/3} \right\rvert$		
	$= 7$		
	$\therefore \theta = 81°52'$ (to near min)	$\frac{1}{2}$	
c	$A(-2,-1)$, $B(1,5)$		
	m:n $\quad 5:-2$ $\quad Q\left(\frac{mx_2+nx_1}{m+n}, \frac{my_2+ny_1}{m+n}\right)$	1	
	$= \left(\frac{5\times1-2\times-2}{5-2}, \frac{5\times5-2\times-1}{3}\right)$		
	$= \left(\frac{9}{3}, \frac{27}{3}\right)$		
	$= (3, 9)$	1	

Qn	Solutions	Marks	Comments+Crite
d	$\sum_{k=2}^{5}(-1)^k k = (-1)^2 \times 2 + (-1)^3 \times 3 + (-1)^4 \times 4 + (-1)^5 \times 5$	1/2	
	$= 2 - 3 + 4 - 5$		
	$= -2$	1/2	
e	Prove $7^n + 2$ is divisible by 3 for $n \geq 1$		
	Step 1: Test if the statement is correct for $n=1$		
	$7^1 + 2 = 9$		
	$\therefore$ True for $n=1$	1	
	Step 2: Assume the statement is true for $n=k$ ~~terms~~;		
	$\frac{7^k + 2}{3} = M$ (where M is an integer)		
	ie $7^k + 2 = 3M$ * $\therefore 7^k = 3M - 2$	1	An alternative method to step 3.
	Step 3: Prove the statement is true for $(k+1)$ ~~terms~~, $n = k+1$.		from *
	$7^{k+1} + 2 = 7 \times 7^k + 2$	1	$7^{k+1} + 2 = 7.7^k + 2$
	$= 7 \times 7^k + (14 - 12)$		$= 7(3M-2) + 2$
	$= 7(7^k + 2) - 12$		$= 21M - 14 + 2$
	$= 7 \times 3M - 12$ (from step 2)		$= 21M - 12$
	$= 3(7M - 4)$ which is divisible by 3 because M is	1	$= 3(7M - 4)$

an integer from above, $\therefore$ if the statement is true for $n=k$, it is also true for $n=k+1$.

Qn	Solutions	Marks	Comments+Criteria
	Step 4 Since it is true for $n=1$, then by induction it must also be true for $n=1+1=2$. It follows that it must also be true for $n=2+1=3$ & so on. Hence it must be true for all positive integers n.		
2a	$\int_0^4 x\sqrt{x^2+9}\,dx$ $u=x^2+9$, $du=2x\,dx$		
	$=\frac{1}{2}\int_9^{25} u^{\frac{1}{2}}\,du$ At $x=0$, $u=9$; $x=4$, $u=25$	$1\frac{1}{2}$	
	$=\frac{1}{2}\times\frac{2}{3}\left[u^{3/2}\right]_9^{25}$	1/2	
	$=\frac{1}{3}\left(25^{3/2}-9^{3/2}\right)$	1/2	
	$=\frac{1}{3}(125-27)$		
	$=\frac{98}{3}$		
	$=32\frac{2}{3}$	1/2	
b	$f(x)=2x^2+x-2$		
	$f'(x)=4x+1$	1/2	
	$f(.5)=-1$ $f'(.5)=3$	1	
	$x_2=x_1-\frac{f(x_1)}{f'(x_1)}$	1/2	
	$=0.5+1/3$		
	$=0.8$ (correct to 1 d.p)	1	Did not deduct marks for not rounding to 1 d.p.

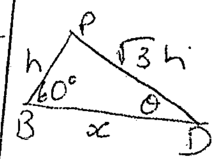
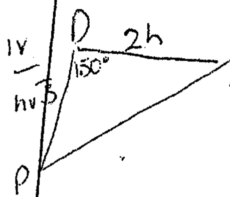
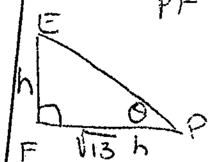
Qn	Solutions	Marks	Comments+Criteria
2ci	$p(x)=x^3-x^2-8x+12$		1 correct root
	$p'(x)=3x^2-2x-8$ $P=-24$, $S=-2$		Correct double root
	$=3x^2-6x+4x-8$		
	$=3x(x-2)+4(x-2)$		
	$=(3x+4)(x-2)$	1	
	$p(2)=0$ & $p'(2)=0$		
	$\therefore p(x)=(x-2)^2(x-a)$	1	
	$=(x^2-4x+4)(x-a)$		
	$\Rightarrow x^3-x^2-8x+12\equiv x^3-4x^2+4x-ax^2+4ax-4a$		
	x^2: $-1=-4-a$		
	$\therefore a=-3$		
	$p(x)=(x-2)^2(x+3)$	1	
c ii a		1	y-int not required
b		1	
c	From part b, $a=3$.		

Qn	Solutions	Marks	Comments+Criteria
3a i	$P(4p, 2p^2)$ $Q(4q, 2q^2)$		Must <u>SHOW</u>
	m_1 of OP: $\frac{2p^2}{4p} = \frac{p}{2}$	$\frac{1}{2}$	
	m_2 of OQ: $\frac{q}{2}$		
	$m_1 m_2 = -1$		
	$\frac{p}{2} \times \frac{q}{2} = -1$	$\frac{1}{2}$	
	$pq = -4$ (1)		
ii	Mdpt. $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$		
	$= \left(\frac{4(p+q)}{2}, \frac{2(p^2+q^2)}{2}\right)$		
	$= (2(p+q), p^2+q^2)$		
	$\therefore x = 2(p+q)$		
	$p+q = \frac{x}{2}$ (2)		
	$y = p^2+q^2$		
	$= (p+q)^2 - 2pq$	1	
	$y = \frac{x^2}{4} + 8$ (from (1) & (2))	1	
	$4y = x^2 + 32$		
	$\therefore x^2 = 4(y-8)$		
b	$x^2 = 8y$		
	$y = \frac{1}{8}x^2$		
	$y' = \frac{x}{4}$		
	$\therefore y'$ at $P = \frac{x_1}{4}$	$\frac{1}{2}$	

Qn	Solutions	Marks	Comments+Criteria
	tangent at P:		
	$y - y_1 = \frac{x_1}{4}(x - x_1)$	$\frac{1}{2}$	
	$4y - 4y_1 = x_1 x - x_1^2$		
	$4y - 4y_1 = x x_1 - 8y_1$	$\frac{1}{2}$	
	$\therefore$ Tangent at P is: $x_1 x - 4y = 4y_1$	$\frac{1}{2}$	
(ii)	$\therefore$ AP $x_1 x - 4y = 4y_1$		
	AQ $x_2 x - 4y = 4y_2$	1	No reasoning 1/3
	AP & AQ both pass thru $(3, -1)$		
	$\therefore$ AP $3x_1 + 4 = 4y_1$		Correct eqn of a tangent(s) 1/3
	AQ $3x_2 + 4 = 4y_2$	1	$+\frac{1}{2}$M if correct format for external pt.
	$P(x_1, y_1)$ & $Q(x_2, y_2)$ lie on the line $3x + 4 = 4y$		
	$\therefore$ The eqn of the chord of contact PQ is:		
	$3x - 4y + 4 = 0$	1	

	Solutions	Marks	Comments+Criteria
c i	$\frac{d}{dx} x\sqrt{x+3} = \frac{d}{dx} x(x+3)^{1/2}$		
	$= \frac{x}{2\sqrt{x+3}} + \sqrt{x+3}$ $\quad u = x \quad v = (x+3)^{1/2}$, $u' = 1 \quad v' = \frac{1}{2}(x+3)^{-1/2}$	1	
ii	$\frac{d}{dx} x\sqrt{x+3} = \frac{x + (x+3)2}{2\sqrt{x+3}}$	2	
	$= \frac{3x+6}{2\sqrt{x+3}}$		
	$= \frac{3(x+2)}{2\sqrt{x+3}}$		
	$\therefore \int \frac{x+2}{\sqrt{x+3}} = \frac{2}{3} x\sqrt{x+3} + c$	2	Did not penalise if no "c".
4a	$x = \tan\theta + \sec\theta$		
	$\frac{x^2-1}{x^2+1} = \sin\theta$		
	$LHS = \frac{\tan^2\theta + 2\tan\theta\sec\theta + \sec^2\theta - 1}{\tan^2\theta + 2\tan\theta\sec\theta + \sec^2\theta + 1}$	1	
	$= \frac{2\tan^2\theta + 2\tan\theta\sec\theta}{2\sec^2\theta + 2\tan\theta\sec\theta}$		
	$= \frac{2\tan\theta(\tan\theta + \sec\theta)}{2\sec\theta(\sec\theta + \tan\theta)}$	1	
	$= \frac{\sin\theta}{\cancel{\cos\theta}} \times \cancel{\cos\theta}$		
	$= \sin\theta$		
	$= RHS.$	1	

$\therefore$ True

Qn	Solutions	Marks	Comments+Criteria
bi	In ΔABP: $\tan 45° = \frac{h}{BP}$		
	$BP = h$	1	
	In ΔDCP: $\tan 30° = \frac{h}{DP}$		
	$DP = \sqrt{3}h$	1	
ii			
	$\frac{\sin\theta}{h} = \frac{\sin 60°}{h\sqrt{3}}$		
	$\sin\theta = \frac{\sqrt{3}}{2} \times \frac{1}{\sqrt{3}}$		
	$\therefore \theta = 30°$		
	$\therefore \angle BPD = 180° - 60° - 30°$	1	
	$= 90°$ ($\angle$ sum Δ, $180°$)		
	$\therefore$ Right angled at P	1	
iii	$h^2 + 3h^2 = x^2$ (By Pythag).	1	
	$x^2 = 4h^2$		
	$\therefore x = 2h$ (only consider +ve solns.)	1	
iv			
	$PF^2 = 4h^2 + 3h^2 + 2 \times 2 \times \sqrt{3} \times h^2 \cos 30°$		
	$= 7h^2 + 2 \times \frac{\sqrt{3}}{2} \times 2\sqrt{3}h^2$		
	$= 13h^2$	1	
	$PF = \sqrt{13}h$	1	
			
	$\tan\theta = \frac{1}{\sqrt{13}}$		
	$\theta = 15.5°$ (to 1 d.p)	1	

Qn	Solutions	Marks	Comments+Criteria

5a $y = \frac{x^2}{4a}$

$y' = \frac{2x}{4a}$

At $x = 2ap$, $m = p$

$y - ap^2 = p(x - 2ap)$

$\therefore y = px - ap^2$

ii $m = \frac{a(p-q)(p+q)}{2a(p-q)}$

$= \frac{p+q}{2}$ Note $p \neq q$

$\therefore y - ap^2 = \frac{p+q}{2}(x - 2ap)$ — 1

$y - ap^2 = \frac{(p+q)x}{2} - ap^2 - apq$

$y = \frac{(p+q)x}{2} - apq$

iii Passes thru $(0, a)$

$a = -apq$

$\therefore pq = -1$ — 1

iv $x = \frac{y}{p} + ap$ & $x = \frac{y}{q} + aq$ from 5(a)

Equate for y: $\frac{y}{p} + ap = \frac{y}{q} + aq$ — 1

$yq + ap^2q = py + apq^2$

$apq(p-q) = y(p-q)$

$\therefore y = apq$ $(p \neq q)$ $\Rightarrow \therefore y = -a$ from (iii) above. $\therefore$ intersect on the directrix. — 1

b $2\sin^2 x - \cos x - 1 = 0$

$2(1 - \cos^2 x) - \cos x - 1 = 0$

$2 - 2\cos^2 x - \cos x - 1 = 0$

$2\cos^2 x + \cos x - 1 = 0$ — 1

$S = 1$ $P = -2$

$2\cos^2 x + 2\cos x - \cos x - 1 = 0$

$2\cos x(\cos x + 1) - (\cos x + 1) = 0$ (1M)

$(2\cos x - 1)(\cos x + 1) = 0$

where n is an integer

Qn	Solutions	Marks	Comments+Criteria

c $3 + 4\cos 2x + \cos 4x \equiv 8\cos^4 x$

$LHS = 3 + 4(2\cos^2 x - 1) + 2\cos^2 2x - 1$ — 1

$= -2 + 8\cos^2 x + 2(2\cos^2 x - 1)^2$ — 1

$= \cancel{-2} + 8\cos^2 x + 2(4\cos^4 x - 4\cos^2 x + 1)$

$= 8\cos^4 x$

$= RHS$ — 1

True

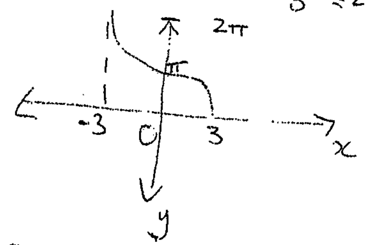
6a $f(x) = 2\cos^{-1}\left(\frac{x}{3}\right)$

Domain: $-1 \leq \frac{x}{3} \leq 1$

$-3 \leq x \leq 3$ — 1

Range: $0 \leq \cos^{-1} x \leq \pi$

$0 \leq 2\cos^{-1}\frac{x}{3} \leq 2\pi$ — 1

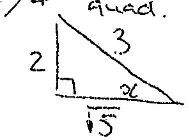


b $\tan\left[\sin^{-1}\left(-\frac{2}{3}\right)\right]$

Let $x = \sin^{-1}(-2/3)$

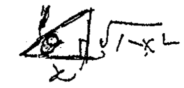
$\sin x = -2/3$ where $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$ $\Rightarrow$ 4th quad. — 1

$\therefore \tan x = -\frac{2}{\sqrt{5}}$ — 1

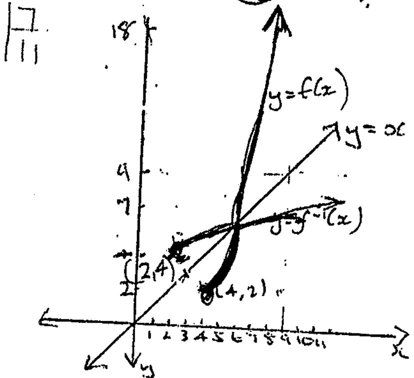


−0.89

Qn	Solutions	Marks	Comments+Criteria
c	$y = \sin^{-1}(x-1)$		
	$\frac{dy}{dx} = \frac{1}{\sqrt{1-(x-1)^2}} \times \frac{d}{dx}(x-1)$		$-1\frac{1}{2}$ if differentiation was TOTALLY wrong!
	$= \frac{1}{\sqrt{2x-x^2}}$	1	
	At $x = 3/2$ $\quad m = \frac{1}{\sqrt{3-9/4}}$		
	$= \frac{1}{\sqrt{3/4}}$		
	$= 2/\sqrt{3}$		
	$\therefore$ Tangent:	1	
	$y - \frac{\pi}{6} = \frac{2}{\sqrt{3}}\left(x - 3/2\right)$	1	
	$6\sqrt{3}y - \sqrt{3}\pi = 12x - 6$		
	$0 = 12x - 6\sqrt{3}y + \sqrt{3}\pi - 6$		
d	$\int_{\frac{\sqrt{3}}{\sqrt{2}}}^{3/\sqrt{2}} \frac{1}{9+2x^2}\,dx = \frac{1}{2}\int_{\frac{\sqrt{3}}{\sqrt{2}}}^{3/\sqrt{2}} \frac{1}{\frac{9}{2}+x^2}$	½	
	$= \frac{\sqrt{2}}{3} \times \frac{1}{2}\left[\tan^{-1}\frac{\sqrt{2}}{3}x\right]_{\sqrt{3}/\sqrt{2}}^{3/\sqrt{2}}$	1	
	$= \frac{\sqrt{2}}{6}\left(\tan^{-1}1 - \tan^{-1}\frac{\sqrt{3}}{3}\times\frac{\sqrt{3}}{\sqrt{3}}\right)$		
	$= \frac{\sqrt{2}}{6}\left(\frac{\pi}{4} - \frac{\pi}{6}\right)$	1	
	$= \frac{\sqrt{2}}{6}\left(\frac{3\pi - 2\pi}{12}\right)$		
	$= \frac{\sqrt{2}\pi}{72}\ u^2$	½	0.06

Qn	Solutions	Marks	Comments+Criteria
7a	$\int_{\frac{1}{2}}^{1} \frac{\sqrt{1-x^2}}{x^2}\,dx$ $\quad$ Let $x = \cos\theta$, $dx = -\sin\theta\, d\theta$		
	$= \int \frac{\sqrt{1-\cos^2\theta}}{\cos^2\theta} \times -\sin\theta\ d\theta$		
	$= \int \frac{-\sin^2\theta}{\cos^2\theta}\,d\theta$		
	$= -\int \tan^2\theta\ d\theta$	$1\frac{1}{2}$	
	$= -\int \sec^2\theta + \int 1\ d\theta$	½	
	$= -\tan\theta + \theta$	½	
	$= -\frac{\sqrt{1-x^2}}{x} + \cos^{-1}x + C$		
b (i)	$y = (x-4)^2 + 2$	1	
		1	ignored if not sketched on same number plane

x	4	5	6	7
y	2	3	6	11



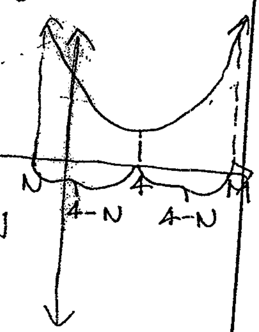
(ii) From TOV in part (i) or alternatively:

$x = x^2 - 8x + 16 + 2$ $\quad$ 1

$0 = x^2 - 9x + 18$

$0 = (x-6)(x-3)$

$\therefore x = 6,\ y = 6$ for the domain $x \geq 4$ $\quad$ 1

Qn	Solutions	Marks	Comments+Criteria
(iii)	$f^{-1}(x)$: $x = (y-4)^2 + 2$ $(y-4)^2 = x - 2$ $\therefore y - 4 = \sqrt{x-2}$ for the domain $y = 4 + \sqrt{x-2}$	1 1	
(iv)	We need to find a no., say M in the domain such that: $f(N) = f(M)$ $f^{-1}(f(N)) = f^{-1}(f(M))$ $= M$ From the sketch, $M = 4 + 4 - N$ $= 8 - N$  $\therefore f^{-1}(f(N)) = 8 - N$	1 1	