St Catherine's School

Year: 12

Subject: Extension 1 Mathematics

Time allowed: 2 hours plus 5 minutes reading time

Date: April 2006

Student number 16361275

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Directions to candidates:

- All questions are to be attempted.(Q.1 to Q.7)
- Marks may be deducted for careless or badly arranged work
- All necessary **working** must be shown
- Approved calculators may be used

Marks:

12	Q 1	10½
12	Q 2	11
12	Q 3	10
12	Q 4	10½
12	Q.5	11
12	Q.6	12
13	Q.7	10
	Total	75/85

Question 1.

a) What are the two possible gradients of lines that make an angle of 45^0 with the line $2x-y+1=0$ — 2 3

b) Find the coordinates of the point, which divides externally the interval joining A (1,2) and B (5,8) in the ratio 2 : 1 — 2 2

c) Use the principle of Mathematical Induction to show that n(n+1)(n+2) is divisible by 3 for all $n \geq 1$ — 4 4

d) Solve for x, the inequality $\dfrac{3x}{x-2} \leq 1$ — 2.5 3

Question 2

a) (i) Show that $\sqrt{3}\sin x + \cos x = 2\ \sin(x + \dfrac{\pi}{6})$ — 2 2

(ii) Hence or otherwise find the solution to $\sqrt{3}\sin x + \cos x = 1 \qquad 0 \leq x \leq 2\pi$ — 1.5 2

b) Find the general solution to : $3\tan 2x - 2\tan x = 0$ — 2.5 3

c) Find the general solution to the equation $\cos 2x = \cos x$ — 3 3

d) Show that $\dfrac{\sin x}{\cos x} + \dfrac{\cos x}{\sin x} = 2\cos ec 2x$ — 2 2

Question 3.

a) P $(2p, p^2)$ and Q $(2q, q^2)$ are two variable points in the parabola $x^2 = 4y$

(i) Show that the equation of the tangent at P is given by $y = px - p^2$. 2 2

(ii) Show that the coordinates of the point of intersection of the tangents at P and Q is given by x =(p+q) and y=pq R $\frac{2}{10}$ 2pt intersection

(iii) If PQ passes through the point (3,1),

show that $\dfrac{p+q}{pq+1} = \dfrac{2}{3}$ 3

(iv) Find the Cartesian equation of the locus of R. 0 1

b) Find the equation of the chord of contact of the tangents to the parabola $x^2 = 4y$ from the point (3,-1) 3 4

Question 4.

a) Evaluate without the use of calculators $\cos(2\sin^{-1}\frac{3}{5})$ 2 2

b) The polynomial $P(x) = ax^3 + bx^2 + 2ax + c = 0$ has real roots

$\sqrt{p}, \frac{1}{\sqrt{p}}$ *and* α

(i) Explain why $\alpha = -\frac{c}{a}$ 1 1

(ii) Show that $a^2 + c^2 = bc$. 0.5 2

c) (i) Divide the polynomial $P(x) = x^3 + x^2 + 3x + 4$ by $A(x) = x^2 + 3$ and express the result in the form

$$\frac{P(x)}{A(x)} = Q(x) + \frac{R(x)}{A(x)}$$

2 2

(ii) Hence evaluate $\int_0^1 \frac{P(x)}{A(x)}\,dx$ 2

d) Find the value of k for which $y = 2x + k$ is a tangent to the parabola $x^2 = 4y$ 2 3

Question 6

a) (i) On the same set of axes, sketch the graphs of 2
$y = \tan^{-1} x$ and $y = 1 - x$

(ii) On your diagram indicate the root say α of the equation 1
$x + \tan^{-1} x - 1 = 0$

(iii) Show that $0.5 < \alpha < 1$ 1

(iv) Use one application of Newton's method to find a better approximation for α 2
(Take $x_1 = 0.75$ as the first approximation)

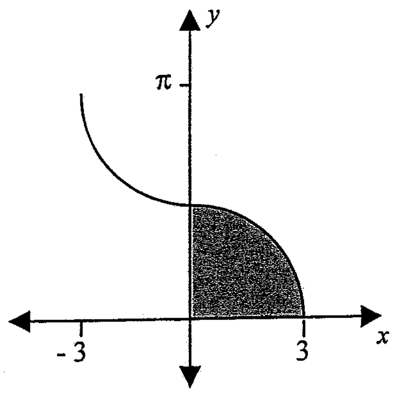
b) (i) Show that $\dfrac{1 - \cos x}{\sin x} = \tan \dfrac{x}{2}$ 2

(ii) use the result to find the exact value of $\tan \dfrac{\pi}{12}$ 2

c) Integrate $\displaystyle\int \frac{dx}{\sqrt{4 - 9x^2}}$ 2

Question 7.

a) (i) The sketch shows the function $y = \cos^{-1}\dfrac{x}{3}$. Find the shaded area.



3

b) (i) Sketch the parabola $y = x(x-4)$ and state the coordinates of its vertex. 1

(ii) Suggest **one** suitable restriction to its domain so that the above function may have an inverse function 1

(iii) Find this inverse function 2

(iv) State the domain and the range of the inverse function. 2

(v) On the same set of axes draw the original function with the restriction on the domain and its inverse function. 2

(vi) Find the point(s) of intersection between the function and its inverse 2

End of Paper

Ext 1 Maths April 06

Q1

$m_1 = 2 \quad m_2 = ?$

a) $\tan\theta = \left|\dfrac{m_1 - m_2}{1 + m_1 m_2}\right|$

$\tan 45^\circ = \left|\dfrac{2 - m}{1 + 2m}\right|$ (1)

$\dfrac{2-m}{1+2m} = 1 \qquad \dfrac{2-m}{1+2m} = -1$

$3m = 1 \qquad m = -3$ (1)

$m = \frac{1}{3}$ (1) ($\perp$ as expected)

b) $A(1,2)\ B(5,8)\quad -2 : -1$

$M = (9, 14)$ (2)

c) $P(1): 1 \times 2 \times 3 = 3R$ True $R=2$

$P(k): k(k+1)(k+2) = 3R \quad R \in \mathbb{Z}$ assume true

$P(k+1): (k+1)(k+2)(k+3) = 3Q \quad Q \in \mathbb{Z}$ prove true

$LHS = k(k+1)(k+2) + 3(k+1)(k+2)$

$= 3R + 3(k+1)(k+2)$

$= 3(R + k^2 + 3k + 2)$

$= 3Q$ ($R, k \in \mathbb{Z}$ $\therefore Q \in \mathbb{Z}$)

$\therefore P(k+1)$ true if $P(k)$ true

$P(1)$ true $\therefore$ by principle of MI

$P(n)$ is true for all $n \geq 1$

d) $\dfrac{3x}{x-2} \times (x-2)^2 \leq (x-2)^2$

$3x(x-2) - (x-2)^2 \leq 0$

$(3x-1)(x-2) \leq 0$

$\frac{1}{3} \leq x \leq 2$

Q2

i) Show $\sqrt{3}\sin x + \cos x = 2\sin(x + \frac{\pi}{6})$

$RHS = 2(\sin x \cos\frac{\pi}{6} + \sin\frac{\pi}{6}\cos x)$

$= 2(\sin x \cdot \frac{\sqrt{3}}{2} + \frac{1}{2}\cos x)$

$= \sqrt{3}\sin x + \cos x$

(ii) $2\sin(x + \frac{\pi}{6}) = 1$

$\sin(x + \frac{\pi}{6}) = \frac{1}{2}$

2b) $3\tan 2x - 2\tan x = 0$

$3 \times \dfrac{2t}{1-t^2} - 2t = 0 \qquad t = \tan$

$6t - 2t + 2t^3 = 0$

$2t^3 + 4t = 0$

$2t(t^2 + 2) = 0$

$2t = 0$ or $t^2 = -2$ (no soln)

$\therefore \tan\theta = 0$

$\theta = 0, \pi, 2\pi \ldots$

$\theta = n\pi \qquad n \in \mathbb{Z}$

c) $\cos 2x = \cos x$

$2\cos^2 x - 1 = \cos x$

$2\cos^2 x - \cos x - 1 = 0$

$(2\cos x + 1)(\cos x - 1)$

$\cos x = -\frac{1}{2} \qquad \cos x = 1$

soln: $x = 2\pi n \pm \frac{\pi}{3}$

and $x = 2\pi n$

d) $LHS = \dfrac{\sin x}{\cos x} + \dfrac{\cos x}{\sin x} = \dfrac{\sin^2 x + \cos^2 x}{\sin x \cos x}$

$= \dfrac{1}{\sin x} \cdot \dfrac{1}{\cos x}$

$= \dfrac{2}{2\sin x \cos x}$

$= \dfrac{2}{\sin 2x} = 2\operatorname{cosec} 2x = RHS$.

3a) i) $y = \dfrac{x^2}{4}$

$y' = \dfrac{2x}{4} = \dfrac{x}{2}$ at $x = 2p$, $y' = p$

$\therefore$ eq $y - p^2 = p(x - 2p)$

$y = px - p^2$

ii) Sim tang at Q $y = qx - q^2$

solving, $px - p^2 = qx - q^2$

$px - qx = p^2 - q^2$

$x = \dfrac{p^2 - q^2}{p - q} = \dfrac{(p+q)(p-q)}{p-q}$

when $x = p+q$, $y = p^2 + pq - p^2 = pq$

$\therefore x = p+q, \ y = pq$.

iii) Grad $PQ = \dfrac{p^2 - q^2}{2p^2 - 2q} = \dfrac{(p+q)(p-q)}{2(p-q)}$

$\therefore$ eq $PQ \quad y - p^2 = \dfrac{p+q}{2}(x - 2p)$

3(iii) $1 = \frac{p+q}{2} \times 3 - pq$

$1+pq = \frac{3}{2}(p+q)$

$\frac{1+pq}{p+q} = \frac{3}{2}$

$\therefore \frac{p+q}{1+pq} = \frac{2}{3}$

iv) Locus of R:

$R\,(p+q,\ pq)$

$x = p+q, \quad y = pq$

so $\frac{x}{1+y} = \frac{2}{3}$ from iii)

$\therefore\ 3x - 2y - 2 = 0$

b) $4y = x^2$

Pt of intersection of tangents is $(p+q, pq)$

from a) This is $(3,-1)$

$\therefore\ p+q = 3, \quad pq = -1$

Eq PQ from a) is

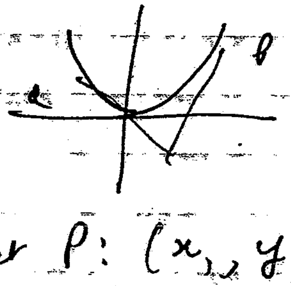
$y = \frac{p+q}{2}x - pq$

so $y = \frac{3}{2}x + 1$

$\therefore\ 3x - 2y + 2 = 0.$

or

Let the tgts from T meet the parabola at P & Q



Let $P: (x_1, y_1)$ & $Q\,(x_2, y_2)$

$x^2 = 4y \quad y' = \frac{x}{2}$

$y'_{\text{at } P} = \frac{x_1}{2}$

tgt. at P is

$y - y_1 = \frac{x_1}{2}(x - x_1)$

$2y - 2y_1 = xx_1 - x_1^2$

$2y - 2y_1 = xx_1 - 4y_1$

$xx_1 - 2y - 2y_1 = 0$ —(1)

IIIly $xx_2 - 2y - 2y_2 = 0$ —(2)

They (1) & (2) pass through $(3,-1)$

$3x_1 + 2 - 2y_1 = 0$

$3x_2 + 2 - 2y_2 = 0$

(x_1, y_1) & (x_2, y_2) lie on $3x - 2y + 2 = 0$

Hence the chord of contact is

$3x - 2y + 2 = 0.$

Q 4

a) $\cos\left(2\sin^{-1}\frac{3}{5}\right)$

Let $\alpha = \sin^{-1}\frac{3}{5}$

$\therefore \sin\alpha = \frac{3}{5}$

$\therefore \cos 2\alpha$

$= 1 - 2\sin^2\alpha$

$= 1 - 2\left(\frac{3}{5}\right)^2$

$= 1 - \frac{18}{25}$

$= \frac{7}{25}$.

b)

$P(x) = ax^3 + bx^2 + 2ax + c = 0$.

$\sqrt{p} + \frac{1}{\sqrt{p}} + \alpha = -\frac{b}{a}$ — (1)

$1 + \alpha\left(\sqrt{p} + \frac{1}{\sqrt{p}}\right) = \frac{2a}{a}$ — (2)

$\alpha \times \sqrt{p} \times \frac{1}{\sqrt{p}} = -\frac{c}{a}$

$\therefore \boxed{\alpha = -\frac{c}{a}}$

(1) $\Rightarrow$ $\sqrt{p} + \frac{1}{\sqrt{p}} = -\frac{b}{a} + \frac{c}{a}$.

Sub in (2)

$1 - \frac{c}{a}\left(\frac{c-b}{a}\right) = 2$

$a^2 - c^2 + bc = 2a^2$

$\therefore a^2 + c^2 = bc$

c) $P(x) = x^3 + x^2 + 3x + 4$.

$A(x) = x^2 + 3$.

$$\begin{array}{r} x + 1 \\ x^2+3 \,\big)\, \overline{x^3 + x^2 + 3x + 4} \\ \underline{x^3 \quad + 3x \quad} \\ x^2 \quad + 4 \\ \underline{x^2 \quad + 3} \\ 1 \end{array}$$

$\therefore \frac{P(x)}{A(x)} = x + 1 + \frac{1}{x^2+3}$.

$\therefore \int_0^1 \frac{P(x)}{A(x)} = \int_0^1 (x+1)\,dx + \int_0^1 \frac{1}{x^2+3}\,dx$

$= \left(\frac{x^2}{2} + x\right)_0^1 + \frac{1}{\sqrt{3}}\left(\tan^{-1}\frac{x}{\sqrt{3}}\right)_0^1$

$= \frac{3}{2} + \frac{1}{\sqrt{3}}\frac{\pi}{6}$.

d) $y = 2x + k$ meets $x^2 = 4y$ at

pts. given by $x^2 = 4(2x + k)$

$x^2 - 8x - 4k = 0$ — (1)

tgt. when Δ of (1) $= 0$

i.e. $64 + 16k = 0$

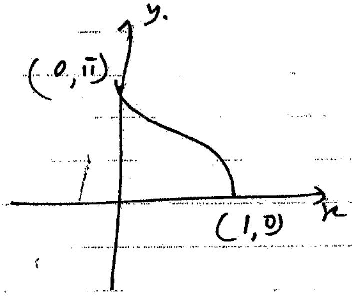
$\therefore k = -4$.

Q.5

$y = 2\cos^{-1}\sqrt{x}$

D: $0 \le \sqrt{x} \le 1$

$0 \le x \le 1$



$\cos^{-1}1 \le \cos^{-1}\sqrt{x} \le \cos^{-1}0$

$0 \le \cos^{-1}\sqrt{x} \le \frac{\pi}{2}$

R $\quad 0 \le 2\cos^{-1}\sqrt{x} \le \pi$

b) $y = x\cos^{-1}x - \sqrt{1-x^2}$

$y' = x\left(\frac{-1}{\sqrt{1-x^2}}\right) + \cos^{-1}x - \frac{1}{2\sqrt{1-x^2}}(-2x)$

$= \cos^{-1}x$

$\therefore \int_0^1 \cos^{-1}x\,dx = \left(x\cos^{-1}x - \sqrt{1-x^2}\right)_0^1$

$= \cos^{-1}1 - (-1)$

$= \cos^{-1}1 + 1 = 1$

c) $y = \operatorname{cosec} x$

Inv. fn. is given by $x = \operatorname{cosec} y$

$= \frac{1}{\sin y}$

$x . \sin y = 1$

$\sin y = \frac{1}{x} \quad \therefore y = \sin^{-1}\frac{1}{x}$

D: $-1 \le \frac{1}{x} \le 1$

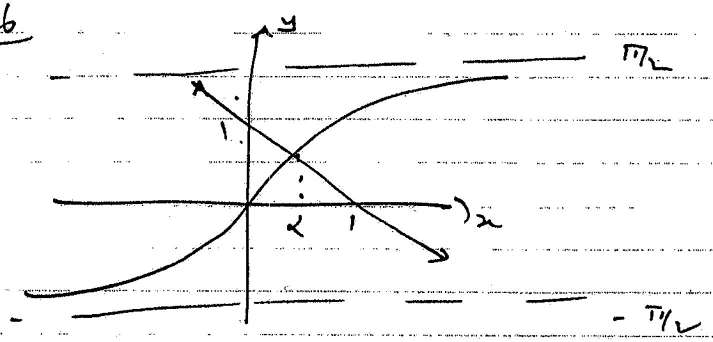
or $x \le -1$ or $x \ge 1$

(ii) $y = \sin^{-1}\left(\frac{1}{x}\right)$

$y' = \frac{1}{\sqrt{1-\left(\frac{1}{x}\right)^2}}\left(-\frac{1}{x^2}\right)$

$= \frac{-1}{x\sqrt{x^2-1}}$

Q.6



$y = \tan^{-1}x$ meets $y = 1 - x$ at pts given by

$\tan^{-1}x = 1 - x$

$x + \tan^{-1}x - 1 = 0$

Let $p(x) = x + \tan^{-1}x - 1$

$p(0.5) < 0$

$p(1) > 0$

$\therefore 0.5 < \alpha < 1$

$p(0.75) = 0.3935$

$p'(x) = 1 + \frac{1}{1+x^2}$

$p'(0.75) = 1.64$

$x_2 = 0.75 - \frac{p(0.75)}{p'(0.75)}$

$= 0.51$

b) $\dfrac{1-\cos x}{\sin x}$

$= \dfrac{1-(1-2\sin^2 x)}{2\sin\frac{x}{2}\cos\frac{x}{2}}$

$= \tan\frac{x}{2}.$

$\therefore \tan\frac{\pi}{12} = \dfrac{1-\cos\pi/6}{\sin\pi/6}$

$= \dfrac{1-\frac{\sqrt3}{2}}{\frac12}$

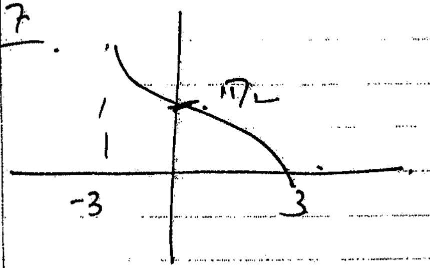
$= \underline{2-\sqrt3}.$

c) $\displaystyle\int \frac{dx}{\sqrt{4-9x^2}}.$

$= \dfrac13 \displaystyle\int \frac{dx}{\sqrt{\frac49 - x^2}}$

$= \dfrac13 \cdot \sin^{-1}\dfrac{3x}{2} + C.$

Q.7.



$\displaystyle\int_0^{\pi/2} x\,dy$

$y = \cos^{-1}\frac{x}{3}.$

$\cos y = \frac{x}{3}$

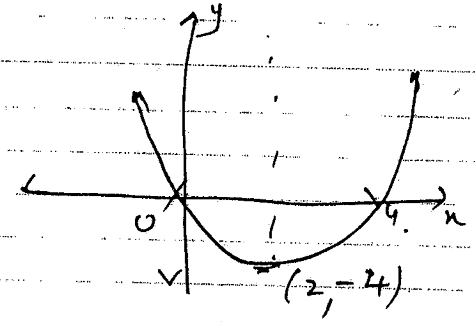
$x = 3\cos y$

$= \displaystyle\int_0^{\pi/2} 3\cos y\,dy$

$= 3\left(\sin y\right)_0^{\pi/2}$

$= 3 \cdot \text{units}^2.$

b)



(i) $\underline{x \geq 2}$ or $x \leq 2$

Inv. fn. is given by:

$x = y(y-4)$

$x = y^2-4y$

or $y^2-4y-x=0$

$y = \dfrac{4 \pm \sqrt{16+4x}}{2}$

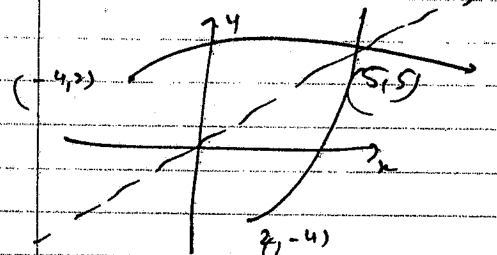
$= \dfrac{4 \pm 2\sqrt{4+x}}{2}.$

$= 2 \pm \sqrt{4+x}.$

~~for~~ $x\geq2$ Inv fn.

$y = 2+\sqrt{4+x}$

Dom: $\underline{x \geq -4}$

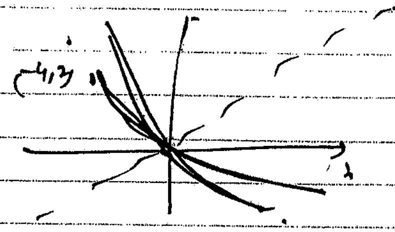


Range: $y \geq 2$

Inv. fn.

$y = 2-\sqrt{4+x}$

Dom: $x \geq -4$.



R: $y \leq 2$.

A fu. & its inv. meet on $y = x$

$x = x(x-4)$; $x(1-(x-4)) = 0$

$x = 0$; $x = 5$

$y = 0$, $y = 5$

$(0,0)$ or $(5;5)$

b) $\dfrac{1-\cos x}{\sin x}$

$= \dfrac{1-(1-2\sin^2\frac{x}{2})}{2\sin\frac{x}{2}\cos\frac{x}{2}}$

$= \tan\frac{x}{2}$.

$\therefore \tan\frac{\pi}{12} = \dfrac{1-\cos\pi/6}{\sin\pi/6}$

$= \dfrac{1-\frac{\sqrt{3}}{2}}{\frac{1}{2}}$

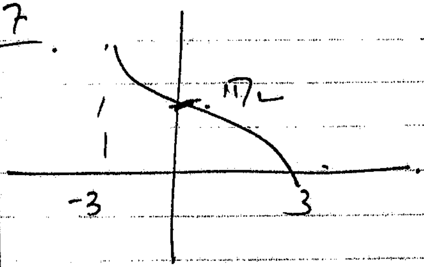
$= \underline{2-\sqrt{3}}$.

c) $\displaystyle\int \frac{dx}{\sqrt{4-9x^2}}$.

$= \dfrac{1}{3}\displaystyle\int \frac{dx}{\sqrt{\frac{4}{9}-x^2}}$

$= \dfrac{1}{3}\cdot \sin^{-1}\dfrac{3x}{2} + c$.

Q.7.



$\displaystyle\int_0^{\pi/2} x\,dy$

$y = \cos^{-1}\frac{x}{3}$.

$\cos y = \frac{x}{3}$

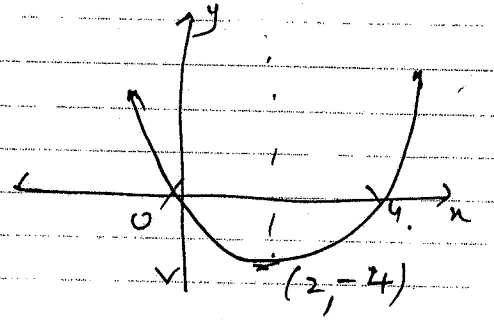
$x = 3\cos y$

$= \displaystyle\int_0^{\pi/2} 3\cos y\,dy$

$= 3(\sin y)_0^{\pi/2}$

$= 3 \cdot \text{units}^2$.

b)



(i) $\underline{x \geq 2}$. or $x \leq 2$

Inv. fn. is given by

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$x = y^2 - 4y$

or $y^2 - 4y - x = 0$

$y = \dfrac{4 \pm \sqrt{16+4x}}{2}$

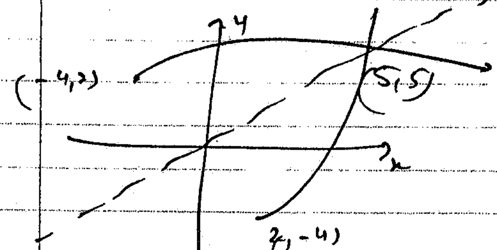
$= \dfrac{4 \pm 2\sqrt{4+x}}{2}$.

$= 2 \pm \sqrt{4+x}$.

~~for $x \geq 2$~~ Inv fn.

$y = 2 + \sqrt{4+x}$

Dom: $\underline{x \geq -4}$



inv. fn.

$y = 2 - \sqrt{4+x}$

Dom: $x \geq -4$.