Student Number ....

St Catherine's School

Waverley, Sydney

An Anglican Day and Boarding School for Girls, Kindergarten to Year 12. Founded in 1856.

2007

TRIAL HIGHER SCHOOL

CERTIFICATE EXAMINATION

Mathematics Extension 2

General Instructions

- Reading Time- 5 minutes
- Working Time – 3 hours
- Write using a blue or black pen
- Approved calculators may be used
- A table of standard integrals is provided at the back of this paper.
- All necessary working should be shown for every question.

Total marks (**120**)

- Attempt Questions 1-8
- All questions are of equal value
- Start a fresh page for each question.
- Put your student number at the top of this page and on each writing booklet used.

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1},\quad n\neq -1;\ x\neq 0,\ \text{if } n<0$$

$$\int \frac{1}{x}\,dx = \ln x,\quad x>0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax},\quad a\neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax,\quad a\neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax,\quad a\neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax,\quad a\neq 0$$

$$\int \sec ax\tan ax\,dx = \frac{1}{a}\sec ax,\quad a\neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a},\quad a\neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a},\quad a>0,\ -a<x<a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right)\quad x>a>0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE: $\ln x = \log_e x, \qquad x>0$

Question 1 (15 marks) **Marks**

(a) Two complex numbers are given by:

$z = 3 - 4i$ and $w = 2 - 2i$

(i) Find the value of the product $\bar{z}w$ **1**

(ii) Find the two square roots of z **2**

(iii) Express w in modulus argument form and hence find the value of w^4 **2**

(b) What is the locus of Z if $W = \dfrac{Z-i}{Z-2}$ is purely imaginary? Sketch the locus of Z. **3**

(c) On an Argand diagram, show the region where the inequalities **2**

$1 \le |Z| \le 3$ and $\dfrac{\pi}{4} \le \arg Z \le \dfrac{\pi}{2}$ hold simultaneously.

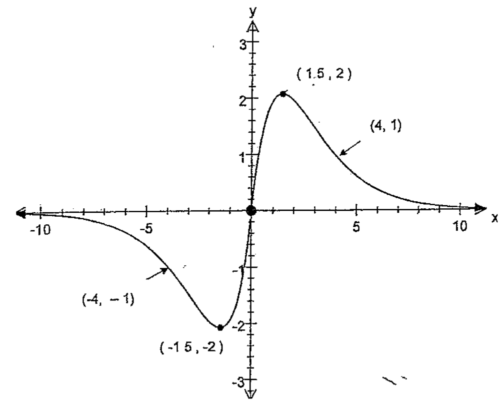
(d) (i) Find all the solutions to the equation $z^6 = 1$ in the form $x + iy$. **3**

(ii) If ω is a non-real solution to the equation $z^6 = 1$, show that $\omega^4 + \omega^2 = -1$. **2**

(iii) By choosing one particular value of ω, explain with the aid of a diagram, or otherwise, why $\omega^4 + \omega^2 = -1$. **1**

Question 2 (15 marks) **Marks**

(a) The diagram shows the graph of $y = f(x)$



Draw separate sketches of the following:

(i) $y = \dfrac{1}{f(x)}$ **2**

(ii) $y = [f(x)]^2$ **2**

(iii) $y^2 = f(x)$ **2**

(iv) $y = x + f(x)$ **2**

(b) Find the equation of the tangent to the curve $x^2 + x - xy + y + y^2 = 12$ at the point (0, 3). **3**

(c) If $u_1 = 8$, $u_2 = 20$ and $u_n = 4u_{n-1} - 4u_{n-2}$ for $n \ge 3$.

(i) Determine u_3 and u_4. **1**

(ii) Prove by induction that $u_n = (n+3)2^n$ for $n \ge 1$. **3**

Question 3 **(15 marks)** **Marks**

(a) Find $\int x\sin(x^2+3)dx$ 2

(b) Show that $e^{-(x-2\log_e\sqrt{x})}$ can be expressed as xe^{-x} 3

Hence using integration by parts, or otherwise, find $\int e^{(\log_e x - x)}dx$

(c) Use the substitution $u=\sqrt{x}$ to evaluate $\int_4^9 \frac{x}{\sqrt{x}(1+x)}dx$. 3

(d) (i) Find the real numbers a, b and c such that 2

$$\frac{2x^2+2x+5}{(x^2+2)(1-x)} \equiv \frac{ax+b}{x^2+2}+\frac{c}{1-x}.$$

(ii) Hence find $\int \frac{2x^2+2x+5}{(x^2+2)(1-x)}dx$. 2

(e) By completing the square, prove that $\int_0^1 \frac{4}{4x^2+4x+5}dx = \tan^{-1}\left(\frac{4}{7}\right)$ 3

Question 4 **(15 marks)** **Marks**

(a) Given that $z=\cos\theta+i\sin\theta$

(i) Show that $z^n+\frac{1}{z^n}=2\cos n\theta$ 1

(ii) Hence express $\cos^4\theta$ in terms of $\cos n\theta$. 3

(b) (i) Given that $1-\sqrt{3}i$ is a root of $P(x)=0$ where $P(x)=x^4-2x^3+5x^2-2x+4$, write down two of the linear factors of $P(x)$. 2

(ii) Hence factorise $P(x)$ completely into real factors. 2

(c) The cubic equation $x^3-5x^2+5=0$ has roots α,β and γ.

(i) Find the equation whose roots are $\alpha-1$, $\beta-1$ and $\gamma-1$. 2

(ii) Find the value of $\alpha^3+\beta^3+\gamma^3$. 2

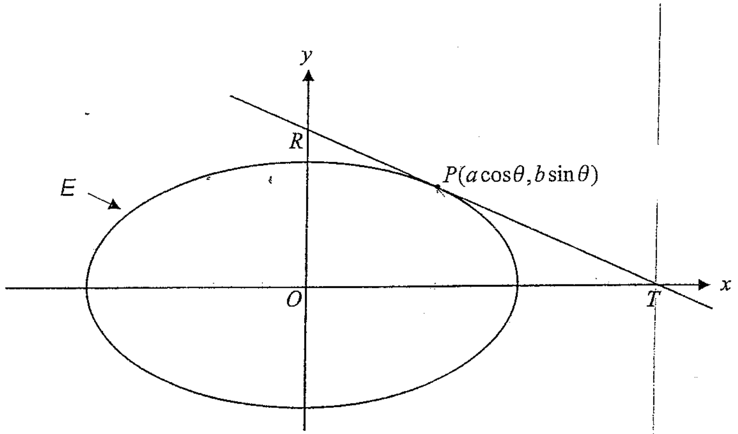
(d) The roots of the equation $x^3-px^2+q=0$ are α,β and γ. 3

If $S_n=\alpha^n+\beta^n+\gamma^n$ where n is a positive integer, prove that

$$pS_{n+2}-qS_n=S_{n+3}.$$

Question 5 **(15 marks)** **Marks**

(a)



The ellipse E, with equation $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ shown in the diagram above, has a tangent at the point $P(a\cos\theta, b\sin\theta)$. The tangent cuts the x-axis at T and the y-axis at R.

(i) Show that the equation of the tangent at the point P is **2**

$$\frac{x\cos\theta}{a}+\frac{y\sin\theta}{b}=1.$$

(ii) If T, the point of intersection of the tangent at P with the x-axis, also lies on one of the directrices of the ellipse, show that $\cos\theta = e$. **3**

(iii) Hence find the angle that the focal chord through P makes with the x-axis. **1**

(iv) Using similar triangles, or otherwise, show that $RP = e^2RT$. **3**

(b) The normal at $P\left(ct, \frac{c}{t}\right)$ on the rectangular hyperbola $xy=c^2$ meets the curve again at Q

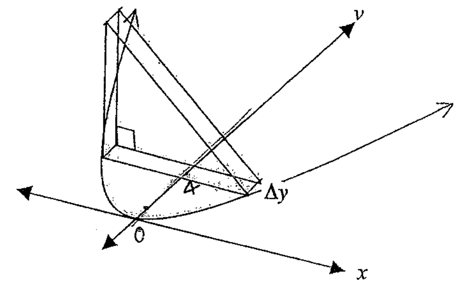
(i) Show that the normal to the hyperbola at P has the equation $t^3x - ty = ct^4 - c$. **2**

(ii) Find the coordinates of Q. **2**

(iii) A line from P through the origin meets the hyperbola again at R. Prove that $PQ^2 = PR^2 + RQ^2$ **2**

Question 6 **(15 marks)** **Marks**

(a) A solid shape is formed as shown at right. Its base is in the xy plane and is in the shape of a parabola $y=x^2$. The vertical cross section is in the shape of a right angled isosceles triangle. By using the method of slicing, calculate the volume of the solid between the values $y=0$ and $y=4$. **5**



(b) The length of a curve between the points where $x=a$ and $x=b$ is given by **4**

$$L=\left|\int_b^a \sqrt{1+[f'(x)]^2}\,dx\right|$$

By considering $f(x)=\sqrt{r^2-x^2}$ and letting $a=r$ and $b=0$ show that the formula for L gives the correct length for one quarter of the circumference of a circle.

(c) The ellipse $\frac{x^2}{4}+\frac{y^2}{3}=1$ is revolved about the line $x=4$.

(i) Use the method of cylindrical shells to show that the volume of the solid of revolution is given by

$$V = 8\sqrt{3}\,\pi\int_{-2}^{2}\sqrt{4-x^2}\,dx - 2\sqrt{3}\,\pi\int_{-2}^{2}x\sqrt{4-x^2}\,dx$$ **4**

(ii) Prove that the volume $V=16\sqrt{3}\,\pi^2$ **2**

Question 7 (15 marks) **Marks**

(a) A body of mass 1Kg is projected vertically upwards from the ground at a speed of 20m per second. The particle is under the effect of both gravity and a resistance which, at any time, has a magnitude of $\frac{1}{40}v^2$, where v is the magnitude of the particle's velocity at that time. Acceleration due to gravity is taken as 10 ms^{-2}

While the body is travelling upwards the equation of motion is

$$\ddot{x} = -(10 + \frac{1}{40}v^2).$$

(i) Calculate the greatest height reached by the body. **2**

(ii) Calculate the time taken to reach this greatest height. **3**

(iii) Write the equation of motion as the body falls after reaching its greatest height. **1**

(iv) Find the speed of the particle when it returns to its starting point. **3**

(b) Let $I_n = \int_0^1 x(x^2-1)^n \, dx$ for $n = 0,1,2,...$

(i) Use integration by parts to show that $I_n = \frac{-n}{n+1}I_{n-1}$ for $n \ge 1$ **3**

(ii) Hence or otherwise show that $I_n = \frac{(-1)^n}{2(n+1)}$ for $n \ge 0$ **2**

(iii) Explain why $I_{2n} > I_{2n-1}$ for $n \ge 0$ **1**

Question 8 (15 marks) **Marks**

(a) (i) If $x \ge 0$, show that $\frac{x}{x^2+4} \le \frac{1}{4}$. **2**

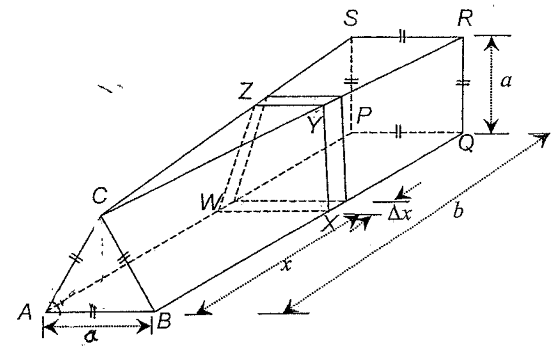
(ii) By integrating both sides of this inequality with respect to x between the limits $x = 0$ and $x = \alpha$, show that **2**

$$e^{\frac{1}{2}\alpha} \ge \frac{1}{4}\alpha^2 + 1 \text{ for } \alpha \ge 0.$$

(b) The diagram shows a sandstone solid with rectangular base ABQP of length b metres and width a metres.

The end PQRS is a square, and the other end ABC is an equilateral triangle.
Both ends are perpendicular to the base.

Consider the slice of the solid with face WXYZ and thickness Δx metres, as shown in the diagram. The slice is parallel to the ends and $AW = BX = x$ metres.



(i) Find the height of the equilateral triangle ABC. **1**

(ii) Given that triangles CRS and CYZ are similar, find YZ in terms of a, b and x. **2**

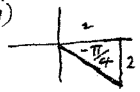
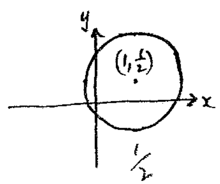
(iii) Let the perpendicular height of the trapezium WXYZ be h metres. **3**

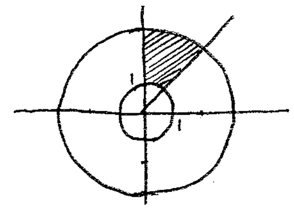
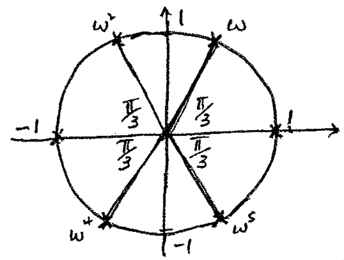
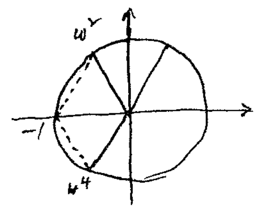
Show that $h = \frac{a}{2}\left[\sqrt{3} + \left(2-\sqrt{3}\right)\frac{x}{b}\right]$

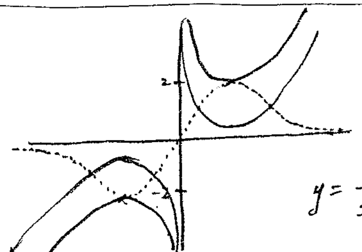
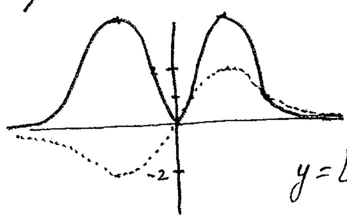
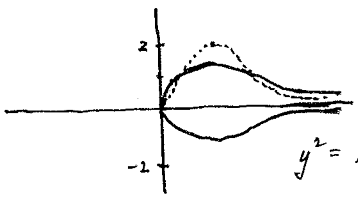
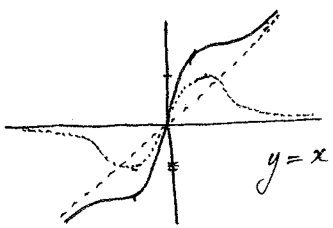
(iv) Hence show that the cross-sectional area of WXYZ is given by **2**

$$\frac{a^2}{4b^2}\left[\left(2-\sqrt{3}\right)x + b\sqrt{3}\right](b+x)$$

(v) Find the volume of the solid **3**

Solutions	Marks	Comments
Question 1: $z = 3-4i \quad \omega = 2-2i$		
a) (i) $\bar{z}\omega = (3+4i)(2-2i)$ $= 6+2i+8$ $= 14+2i$	1	
(ii) $(x+iy)^2 = 3-4i$ $x^2-y^2 = 3$ — ① $2xy = -4$ — ② also $x^2+y^2 = 5$ — ③	1	
①+③ $2x^2 = 8$ $x = \pm 2$ sub in ② $y = \mp 1$ $\therefore \sqrt{3-4i} = \pm(2-i)$	1	
(iii)  $\omega = 2\sqrt{2}\,\text{cis}\left(-\frac{\pi}{4}\right)$	1	
$\omega^4 = \left[2\sqrt{2}\,\text{cis}\left(-\frac{\pi}{4}\right)\right]^4$ $= 64\,\text{cis}(-\pi)$ $= 64\left[\cos(-\pi) + i\sin(-\pi)\right]$ $= 64(-1)$ $= -64$	1	
b) $w = \frac{z-i}{z-2}$ let $z = x+iy$ $= \frac{x+iy-i}{x+iy-2}$ $= \frac{x+i(y-1)}{(x-2)+iy} \times \frac{(x-2)-iy}{(x-2)-iy}$	$1\frac{1}{2}$	
real part $= \frac{x^2-2x+y^2-y}{(x-2)^2+y^2} = 0$ $\therefore x^2-2x+y^2-y = 0$ $x^2-2x+1+y^2-y+\frac{1}{4} = \frac{5}{4}$ $(x-1)^2+\left(y-\frac{1}{2}\right)^2 = \left(\frac{\sqrt{5}}{2}\right)^2$ 1 	$\frac{1}{2}$ 1	

Solutions	Marks	Comments
Question 1: c) 	2	
d). (i) $z^6 - 1 = 0$ $z^6 = 1$ $z = 1$ and -1 other roots equally spaced around unit circle	1	
	1	
Solutions: $z = \pm 1,\ \pm\text{cis}\frac{\pi}{3},\ \pm\text{cis}\frac{2\pi}{3}$ $z = \pm 1,\ \frac{1}{2} \pm \frac{\sqrt{3}}{2}i,\ -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$	1	
(ii) $z^6 - 1 = (z^2-1)(z^4+z^2+1) = 0$ if ω is a non real solution then $\omega^4 + \omega^2 + 1 = 0$ $\therefore \omega^4 + \omega^2 = -1$	2	
(iii)  vector addition of ω^2 and ω^4 results in -1 $\therefore \omega^2 + \omega^4 = -1$	1	

Solutions	Marks	Comments
Question 1: d) iii) take $\omega = cis\frac{\pi}{3}$		
$\omega^2 = cis\frac{2\pi}{3} = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$		
$\omega^4 = cis\frac{4\pi}{3} = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$		
Now $\omega^2 + \omega^4 = cis\frac{2\pi}{3} + cis\frac{4\pi}{3}$	1	
$= -1$ (by addition)		
Question 2: a)(i)		
 $y = \frac{1}{f(x)}$	2	
(ii)  $y = [f(x)]^2$	2	
(iii)  $y^2 = f(x)$	2	
(iv)  $y = x + f(x)$	2	

Solutions	Marks	Comments
Question 2: b) $x^2 + x - xy + y + y^2 = 12$		
$2x + 1 - y - x\frac{dy}{dx} + \frac{dy}{dx} + 2y\frac{dy}{dx} = 0$	1	
$(2x + 1 - y) + \frac{dy}{dx}(1 + 2y - x) = 0$		
$\frac{dy}{dx} = \frac{y - 2x - 1}{1 + 2y - x}$		
at $(0,3)$ $\frac{dy}{dx} = \frac{3-1}{1+6}$	1	
$= \frac{2}{7}$		
$\therefore$ tangent has equation		
$y - 3 = \frac{2}{7}(x - 0)$		
$7y - 21 = 2x$	1	
$2x - 7y + 21 = 0$		
c) $u_1 = 8$ $u_2 = 20$ $u_n = 4u_{n-1} - 4u_{n-2}$ $n \geq 3$		
(i) $u_3 = 4u_2 - 4u_1$ $= 80 - 32$ $= 48$; $u_4 = 4u_3 - 4u_2$ $= 192 - 80$ $= 112$	1	
(ii) Prove $u_n = (n+3)2^n$ $n \geq 1$		
$n = 1$ $u_n = 4.2^1 = 8$ true	1	
$n = 2$ $u_n = 5.2^2 = 20$ true.		
let k be integer $k \geq 2$		
assume $u_k = (k+3)2^k$ + $u_{k-1} = (k+2)2^{k-1}$	1	
aim to prove true for $n = k+1$		
now $u_{k+1} = 4u_k - 4u_{k-1}$		
$= 4\left((k+3).2^k - (k+2)2^{k-1}\right)$		
$= 4(k+3)2^k - 2(k+2)2^k$		
$= (2k+8)2^k$		
$= (k+4)2^{k+1}$		
$= (k+1+3)2^{k+1}$		
$\therefore$ true for $n = k+1$		

Solutions	Marks	Comments
Hence for $k \geq 2$ true for all positive integers $n \leq k$ implies true for $n = k+1$		
But U_1 and U_2 are true therefore by induction true for all $n \geq 1$		
ie. $U_n = (n+3)2^n$ for all $n \geq 1$	1	
Question 3: a) $\int x \sin(x^2+3)\,dx$ $\quad u = x^2+3$, $du = 2x\,dx$	1	
$= \frac{1}{2}\int \sin u \, du$		
$= -\frac{1}{2}\cos u + c$		
$= -\frac{1}{2}\cos(x^2+3) + c$	1	
b) $e^{-(x - 2\log_e\sqrt{x})} = e^{-x} \cdot e^{2\log_e\sqrt{x}}$		
$= e^{-x} \cdot e^{\log_e x}$ $\quad * \; e^{\log x - x}$		
$= e^{-x} \cdot x$		
$= xe^{-x}$	1	
Now $\int e^{(\log_e x - x)}\,dx$		
$= \int x \cdot e^{-x}\,dx$ $\quad u = x$, $v = -e^{-x}$, $u' = 1$, $v' = e^{-x}$	1	
$= -x \cdot e^{-x} + \int e^{-x}\,dx$		
$= -xe^{-x} - e^{-x} + c$		
$= -e^{-x}(x+1) + c$	1	
c) $\int_4^9 \frac{x}{\sqrt{x}(1+x)}\,dx$ $\quad u = \sqrt{x}$, $x = u^2$, $du = \frac{1}{2\sqrt{x}}\,dx$		
$= 2\int_2^3 \frac{u^2}{1+u^2}\,du$ $\quad x = 9$, $u = 3$; $x = 4$, $u = 2$	1	

Solutions	Marks	Comments
$= 2\int_2^3 \left(\frac{1+u^2}{1+u^2} - \frac{1}{1+u^2}\right) du$	1	
$= 2\int_2^3 \left(1 - \frac{1}{1+u^2}\right) du$		
$= 2\left[u - \tan^{-1}u\right]_2^3$		
$= 2\left[(3 - \tan^{-1}3) - (2 - \tan^{-1}2)\right]$		
$= 2 - 2\tan^{-1}3 + 2\tan^{-1}2$	1	
d) (i) $\frac{2x^2+2x+5}{(x^2+2)(1-x)} = \frac{ax+b}{x^2+2} + \frac{c}{1-x}$		
$= \frac{(ax+b)(1-x) + c(x^2+2)}{(x^2+2)(1-x)}$		
true iff $2x^2+2x+5 = (ax+b)(1-x) + c(x^2+2)$	1	
let $x = 1$ $\quad 9 = 3c \Rightarrow c = 3$		
let $x = 0$ $\quad 5 = b + 2c \Rightarrow b = -1$		
let $x = -1$ $\quad 5 = (-a-1)2 + 9$		
$5 = -2a + 7 \Rightarrow a = 1$		
$\therefore a = 1 \quad b = -1 \quad c = 3.$	1	
(ii) $\int \frac{2x^2+2x+5}{(x^2+2)(1-x)}\,dx = \int \frac{x-1}{x^2+2} + \frac{3}{1-x}\,dx$		
$= \int \frac{x}{x^2+2}\,dx - \int \frac{1}{x^2+2}\,dx + \int \frac{3}{1-x}\,dx$	1	
$= \frac{1}{2}\ln(x^2+2) - \frac{1}{\sqrt{2}}\tan^{-1}\frac{x}{\sqrt{2}} - 3\ln\lvert 1-x\rvert + c$		
$= \ln\left\lvert\frac{\sqrt{x^2+2}}{(1-x)^3}\right\rvert - \frac{1}{\sqrt{2}}\tan^{-1}\frac{x}{\sqrt{2}} + c$	1	

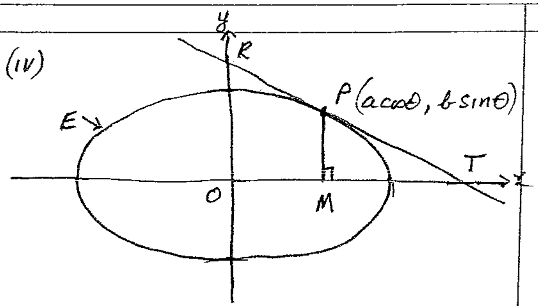
Solutions	Marks	Comments
Question 3 e) $\int_0^1 \frac{4}{4x^2+4x+5}\,dx$		
$= \int_0^1 \frac{4}{(2x+1)^2+4}\,dx$ let $u = 2x+1$, $du = 2dx$, $x=0$ $u=1$, $x=1$ $u=3$	1	
$= \int_1^3 \frac{2\,du}{u^2+4}$		
$= \left[\tan^{-1}\frac{u}{2}\right]_1^3$	1	
$= \tan^{-1}\frac{3}{2} - \tan^{-1}\frac{1}{2}$		
$= \tan^{-1}\left(\frac{\frac{3}{2}-\frac{1}{2}}{1+\frac{3}{2}\cdot\frac{1}{2}}\right)$		
$= \tan^{-1}\left(\frac{4}{7}\right)$	1	
Question 4 a) $z = \cos\theta + i\sin\theta$		
(i) $z^n = \cos n\theta + i\sin n\theta$ (De Moivre's)		
$\frac{1}{z^n} = \cos(-n\theta) + i\sin(-n\theta)$ (De Moivre's)		
$= \cos n\theta - i\sin n\theta$		
$\therefore z^n + \frac{1}{z^n} = 2\cos n\theta$	1	
(ii) from (i) $2\cos\theta = z + \frac{1}{z}$	1	
$\therefore 16\cos^4\theta = \left(z+\frac{1}{z}\right)^4$		
$= z^4 + 4z^3\cdot\frac{1}{z} + 6z^2\frac{1}{z^2} + 4z\frac{1}{z^3} + \frac{1}{z^4}$		
$= \left(z^4+\frac{1}{z^4}\right) + 4\left(z^2+\frac{1}{z^2}\right) + 6$	1	
$\therefore 16\cos^4\theta = 2\cos 4\theta + 8\cos 2\theta + 6$		
$\cos^4\theta = \frac{1}{8}\cos 4\theta + \frac{1}{2}\cos 2\theta + \frac{3}{8}$	1	

Solutions	Marks	Comments
Question 4 b) if $\alpha = 1-\sqrt{3}i$ is one root		
$\bar{\alpha} = 1+\sqrt{3}i$ is also a root (coefficients of $P(x)$ are real)	1	
(i) $[x-(1-\sqrt{3}i)]$ and $[x-(1+\sqrt{3}i)]$ are linear factors		
ie. $(x-1+\sqrt{3}i)$ and $(x-1-\sqrt{3}i)$	1	
(ii) Since $(x-1+\sqrt{3}i)$ and $(x-1-\sqrt{3}i)$ are factors		
then $(x-1+\sqrt{3}i)(x-1-\sqrt{3}i)$ is factor		
ie. $x^2 - x - \cancel{\sqrt{3}xi} - x + 1 + \cancel{\sqrt{3}i} + \cancel{\sqrt{3}xi} - \cancel{\sqrt{3}i} + 3$		
$= x^2 - 2x + 4$ is a factor	1	
$\therefore P(x) = (x^2-2x+4)(x^2+1)$	1	
c). $x^3 - 5x^2 + 5 = 0$ ——— (1)		
let $y = x-1$		
$\therefore x = y+1$	1	
Sub in (1) $(y+1)^3 - 5(y+1)^2 + 5 = 0$		
$y^3 + 3y^2 + 3y + 1 - 5y^2 - 10y = 0$		
$y^3 - 2y^2 - 7y + 1 = 0$		
$\therefore$ required polynomial is $x^3 - 2x^2 - 7x + 1 = 0$	1	

Solutions	Marks	Comments
Question 4 d) (ii) If $\alpha\ \beta\ \gamma$ are roots then		
$\alpha^3-5\alpha^2+5=0$		
$\beta^3-5\beta^2+5=0$		
$\gamma^3-5\gamma^2+5=0$		
$\therefore\ (\alpha^3+\beta^3+\gamma^3)-5(\alpha^2+\beta^2+\gamma^2)+15=0$ (adding)	1	
now $\alpha^2+\beta^2+\gamma^2=(\alpha+\beta+\gamma)^2-2(\alpha\beta+\alpha\gamma+\beta\gamma)$		
$=25-0$		
$=25$		
$\therefore\ \alpha^3+\beta^3+\gamma^3-5(25)+15=0$		
$\therefore\ \alpha^3+\beta^3+\gamma^3=110$	1	
d) $S_n=\alpha^n+\beta^n+\gamma^n$ $\quad x^3-px^2+q=0$		
$\therefore\ \alpha+\beta+\gamma=p$		
$LHS=pS_{n+2}-qS_n$ $\quad \alpha\beta\gamma=-q$		
$=p(\alpha^{n+2}+\beta^{n+2}+\gamma^{n+2})-q(\alpha^n+\beta^n+\gamma^n)$		
$=p(\alpha^2\alpha^n+\beta^2\beta^n+\gamma^2\gamma^n)-q\alpha^n-q\beta^n-q\gamma^n$		
$=\alpha^n(p\alpha^2-q)+\beta^n(p\beta^2-q)+\gamma^n(p\gamma^2-q)$	1	
now if α is a root of $x^3-px^2+q=0$ then $\alpha^3=p\alpha^2-q$		
Similarly for β + γ	1	
$\therefore\ LHS=\alpha^n\alpha^3+\beta^n\beta^3+\gamma^n\gamma^3$		
$=\alpha^{n+3}+\beta^{n+3}+\gamma^{n+3}$		
$=S_{n+3}$		
$=RHS.$	1	

Solutions	Marks	Comments
Question 5: a) (i) at P. $x=a\cos\theta$ $\quad y=b\sin\theta$		
$\frac{dx}{d\theta}=-a\sin\theta$ $\quad \frac{dy}{d\theta}=b\cos\theta$		
$\therefore\ \frac{dy}{dx}=-\frac{b\cos\theta}{a\sin\theta}$	1	
$\therefore$ Equation of tangent is		
$y-b\sin\theta=-\frac{b\cos\theta}{a\sin\theta}(x-a\cos\theta)$		
$ay\sin\theta-ab\sin^2\theta=-bx\cos\theta+ab\cos^2\theta$		
$bx\cos\theta+ay\sin\theta=ab(\sin^2\theta+\cos^2\theta)$		
$\div ab$ $\quad \frac{x\cos\theta}{a}+\frac{y\sin\theta}{b}=1$ as required	1	
(ii) If T is on directrix		
T has coordinates $\left(\frac{a}{e},0\right)$	1	
gradient of tangent at P is $-\frac{b\cos\theta}{a\sin\theta}$		
gradient of PT $=\frac{b\sin\theta-0}{a\cos\theta-\frac{a}{e}}$		
$=\frac{eb\sin\theta}{ae\cos\theta-a}$	1	
$\therefore\ \frac{eb\sin\theta}{ae\cos\theta-a}=-\frac{b\cos\theta}{a\sin\theta}$		
$\therefore\ abe\sin^2\theta=-abe\cos^2\theta+ab\cos\theta$		
$\therefore\ abe(\sin^2\theta+\cos^2\theta)=ab\cos\theta$	1	
$\div ab$ $\quad e=\cos\theta$ as required		
(iii) coordinates of $P\ (ae,b\sin\theta)$ $[\cos\theta=e]$		
$\therefore$ focal chord through P makes an angle of 90° with x axis as focus $(ae,0)$	1	

Solutions	Marks	Comments
Questions a) (iv)		



Join P to M so that M is foot of perpendicular to x axis from P.

Coordinate of M $(a\cos\theta, 0)$ — 1

Coordinate of T $(\frac{a}{e}, 0)$

now $\frac{RP}{RT} = \frac{OM}{OT}$ (ratio of intercepts) — 1

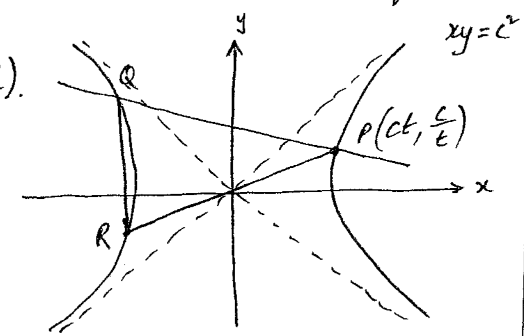
$\frac{RP}{RT} = \frac{a\cos\theta}{\frac{a}{e}}$

$\frac{RP}{RT} = e\cos\theta$

* but $\cos\theta = e$ (from (ii))

$\therefore \frac{RP}{RT} = e^2 \Rightarrow RP = e^2 RT$ as required. — 1

b).



Solutions	Marks	Comments
Qs b) (i)		

(i) $y = \frac{c^2}{x}$ $\quad \frac{dy}{dx} = -\frac{c^2}{x^2}$

at P $\frac{dy}{dx} = \frac{-c^2}{c^2t^2}$

$= -\frac{1}{t^2}$

$\therefore$ gradient of normal at $P = t^2$ — 1

$\therefore$ equation of normal $y - \frac{c}{t} = t^2(x - ct)$

$ty - c = t^3x - ct^4$ — 1

i.e. $t^3x - ty = ct^4 - c$

(ii) Sub $y = \frac{c^2}{x}$ into normal

$t^3x - \frac{tc^2}{x} = ct^4 - c$

$t^3x^2 - tc^2 = ct^4x - cx$

$t^3x^2 - (ct^4 - c)x - tc^2 = 0$ — 1

$x = \frac{(ct^4-c) \pm \sqrt{(ct^4-c)^2 + 4t^4c^2}}{2t^3}$

$= \frac{(ct^4-c) \pm \sqrt{(ct^4+c)^2}}{2t^3}$

$= ct, -\frac{c}{t^3}$

$\therefore y = \frac{c}{t}, ct^3$

$\therefore Q\left(-\frac{c}{t^3}, -ct^3\right)$ — 1

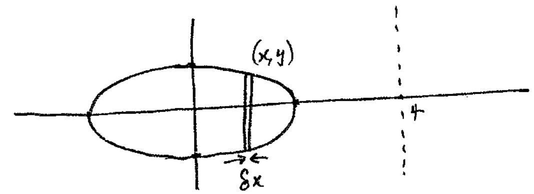
(iii) gradient $PR = \frac{c/t}{ct} = \frac{1}{t^2}$

gradient $QR = \frac{-ct^3 + \frac{c}{t}}{-\frac{c}{t^3} + ct}$

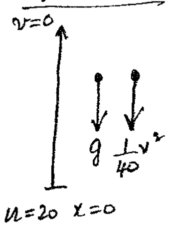
$= \frac{-ct^4 + c}{t} \times \frac{t^3}{ct^4 - c}$ — 1

$= -t^2$

Solutions	Marks	Comments
Questions 6 b) (iii) $\therefore PR \perp QR \quad (m_1 m_2 = -1)$		
$\therefore \triangle PQR$ is right angled at R		
By Pythagoras $PQ^2 = PR^2 + RQ^2$	1	
Question 6 a) $y = x^2$		
length of base of isosceles $\triangle = 2x$	1	
$\therefore$ height of isosceles $\triangle = 2x$	1	
$\therefore$ area of typical slice $= 2x^2$	1	
Now $\delta V = 2x^2 \delta y$		oops! a bit easy for 5 marks
$\therefore V = \sum_{y=0}^{4} 2x^2 \delta y$ but $y = x^2$	1	
$= \int_0^4 2y \, dy$		
$= \left[y^2 \right]_0^4$		
$= 16 \, U^3$	1	
b). $f(x) = (r^2 - x^2)^{\frac{1}{2}}$		
$\therefore f'(x) = \frac{1}{2}(r^2 - x^2)^{-\frac{1}{2}} \cdot -2x$		
$= \frac{-x}{\sqrt{r^2 - x^2}}$	1	
$\therefore L = \int_0^r \sqrt{1 + \frac{x^2}{r^2 - x^2}} \, dx$		
$= \int_0^r \sqrt{\frac{r^2}{r^2 - x^2}} \, dx$		
$= \int_0^r \frac{r}{\sqrt{r^2 - x^2}} \, dx$	1	

Solutions	Marks	Comments
Question 6 b) $= r\left[\sin^{-1} \frac{x}{r} \right]_0^r$		
$= r\left[\sin^{-1} 1 - \sin^{-1} 0 \right]$	1	
$= r \frac{\pi}{2}$		
$\therefore$ length $= \frac{\pi r}{2}$		
$= \frac{2\pi r}{4}$ $\therefore$ quarter circle	1	
c) 		
(i) $\delta V = \pi\left[R^2 - r^2 \right] h \quad \left[\text{or } 2\pi x y \, \delta x \right]$	1	
$= \pi\left[(4-x)^2 - (4-x-\delta x)^2 \right] 2y$		
$= 2\pi\left[(4-x)^2 - (4-x)^2 + 2(4-x)\delta x - \delta x^2 \right] y$		
$= 2\pi\left[2(4-x)\delta x \right] y$ (ignoring $(\delta x)^2$)	1	
$= 4\pi (4-x) y \, \delta x$		
$\therefore V = 4\pi \int_{-2}^{2} (4-x) \frac{\sqrt{3}}{2} \sqrt{4-x^2} \, dx$ — Note: $y^2 = 3\left(1 - \frac{x^2}{4}\right) = \frac{3}{4}(4 - x^2)$, $y = \frac{\sqrt{3}}{2}\sqrt{4-x^2}$	1	
$= 2\pi\sqrt{3} \int_{-2}^{2} (4-x)\sqrt{4-x^2} \, dx$		
$= 8\sqrt{3}\pi \int_{-2}^{2} \sqrt{4-x^2} \, dx - 2\sqrt{3}\pi \int_{-2}^{2} x\sqrt{4-x^2} \, dx$	1	

Solutions	Marks	Comments
Question 6 c) ii) $\therefore V = 8\sqrt{3}\pi \int_{-2}^{2} \sqrt{4-x^2}\, dx$	1	
* Note $\int_{-2}^{2} x\sqrt{4-x^2}\, dx = 0$ (odd function)		
$\therefore V = 8\sqrt{3}\pi \cdot \frac{1}{2}\pi 2^2$ (semi circle)		
$= 16\sqrt{3}\pi^2\ m^3$	1	
Question 7: a) (i) $\ddot{x} = -\left(10 + \frac{1}{40}v^2\right)$		
$v\frac{dv}{dx} = -\left(10+\frac{1}{40}v^2\right)$		
$\therefore v\frac{dv}{dx} = -\left(\frac{400+v^2}{40}\right)$		
$\therefore -dx = \frac{40v}{400+v^2}\, dv$	1	
at greatest height $v=0$ $-\int_0^x dx = \int_{20}^{0} \frac{40v}{400+v^2}\, dv$		
$-x = \left[20\ln(400+v^2)\right]_{20}^{0}$		
$-x = 20\ln 400 - 20\ln 800$		
$x = 20\ln 2$	1	
$= 20\log_e 2$		
(ii) $\frac{dv}{dt} = -\left(\frac{400+v^2}{40}\right)$	1	
$\frac{dt}{dv} = -\frac{40}{400+v^2}$		
integrating $\therefore t = -\int_{20}^{0} \frac{40}{400+v^2}\, dv$	1	



Solutions	Marks	Comments
Question 7 a)(ii) $\therefore t = -\left[2\tan^{-1}\frac{v}{20}\right]_{20}^{0}$		
$t = -\left[0 - \frac{\pi}{2}\right]$		
$t = \frac{\pi}{2}$	1	
(iii) $\ddot{x} = 10 - \frac{1}{40}v^2$	1	
(iv) $v\frac{dv}{dx} = \frac{400-v^2}{40}$		
$\frac{dv}{dx} = \frac{400-v^2}{40v}$		
$\frac{dx}{dv} = \frac{40v}{400-v^2}$	1	
$\int_0^x dx = \int_0^v \frac{40v}{400-v^2}\, dv$		
but from part (i) body falls distance $20\ln 2$		
$\therefore \int_0^{20\ln 2} dx = \int_0^v \frac{40v}{400-v^2}\, dv$	1	
$\therefore 20\ln 2 = -20\int_0^v \frac{-2v}{400-v^2}\, dv$		
$20\ln 2 = -20\left[\ln(400-v^2)\right]_0^v$		
$20\ln 2 = -20\ln(400-v^2) + 20\ln 400$		
$(\div 20)$ $\therefore \ln 2 = -\ln(400-v^2) + \ln 400$		
$\ln(400-v^2) = \ln 400 - \ln 2$		
$\ln(400-v^2) = \ln 200$		
$\therefore 400 - v^2 = 200$		
$v^2 = 200$		
$v = \sqrt{200}$	1	

Solutions	Marks	Comments
Question 7 b)(i) $I_n = \int_0^1 x(x^2-1)^n\,dx$		
$u = (x^2-1)^n \qquad v = \frac{x^2}{2}$		
$u' = 2nx(x^2-1)^{n-1} \qquad v' = x$		
$\therefore I_n = \left[\frac{x^2}{2}(x^2-1)^n\right]_0^1 - \int_0^1 \frac{x^2}{2}\cdot 2nx(x^2-1)^{n-1}\,dx$	1	
$= 0 \quad - n\int_0^1 x^3(x^2-1)^{n-1}\,dx$		
$= -n\int_0^1 \frac{x^3(x^2-1)^n}{x^2-1}\,dx$		
$= -n\int_0^1 \left(x+\frac{x}{x^2-1}\right)(x^2-1)^n\,dx$	1	
$\therefore I_n = -n\int_0^1 x(x^2-1)^n\,dx - n\int x(x^2-1)^{n-1}\,dx$		
$= -nI_n - nI_{n-1}$		
$\therefore (n+1)I_n = -nI_{n-1}$		
$\therefore I_n = \frac{-n}{n+1}I_{n-1}$ for $n \ge 1$	1	
(ii) $I_n = \frac{-n}{n+1}I_{n-1}$		
$\therefore I_n = \frac{-n}{n+1}\cdot\frac{-n+1}{n}\cdot\frac{-n+2}{n-1}\cdots\frac{-3}{4}\cdot\frac{-2}{3}\cdot\frac{-1}{2}I_0$	1	
$= (-1)^n\cdot\frac{1}{2(n+1)} \quad n \ge 0$		
Note $I_0 = \int_0^1 x(x^2-1)^0\,dx$	1	
$= \left[\frac{x^2}{2}\right]_0^1$		
$= \frac{1}{2}$		

Solutions	Marks	Comments
Question 7 b)(iii) $I_0 = \frac{1}{2}$		
$I_1 = -\frac{1}{4}$		
$I_2 = \frac{1}{6}$		
$I_3 = -\frac{1}{8}$		
$I_4 = \frac{1}{10}$		
Clearly (even) $I_{2n} > 0$		
(odd) $I_{2n+1} < 0$		
$\therefore I_{2n} > I_{2n+1}$		
OR From (ii)		
$I_{2n} = \frac{(-1)^{2n}}{2(2n+1)}$		
$= \frac{1}{2(2n+1)}$		
> 0 for $n \ge 0$		
$I_{2n+1} = \frac{(-1)^{2n+1}}{2(2n+2)}$		
$= \frac{-1}{4(n+1)}$		
< 0 for $n \ge 0$		
$\therefore I_{2n} > I_{2n+1}$		

Solutions	Marks	Comments

Question 8: a) (i) $\dfrac{x}{x^2+4} \le \dfrac{1}{4}$

i.e. Prove $\dfrac{x}{x^2+4} - \dfrac{1}{4} \le 0$ — **1**

$$\begin{aligned} LHS &= \frac{4x - x^2 - 4}{4(x^2+4)} \\ &= \frac{-(x^2-4x+4)}{4(x^2+4)} \\ &= \frac{-(x-2)^2}{4(x^2+4)} \\ &\le 0 \text{ for all } x \ge 0 \end{aligned}$$ — **1**

(ii) $$\int_0^{\alpha} \frac{x}{x^2+4}\,dx \le \int_0^{\alpha} \frac{1}{4}\,dx$$

$$\therefore \left[\frac{1}{2}\ln(x^2+4)\right]_0^{\alpha} \le \left[\frac{x}{4}\right]_0^{\alpha}$$ — **1**

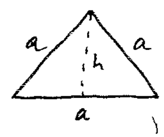
$$\begin{aligned} \frac{1}{2}\ln(\alpha^2+4) - \frac{1}{2}\ln 4 &\le \frac{\alpha}{4} \\ \frac{1}{2}\ln\left(\frac{\alpha^2+4}{4}\right) &\le \frac{\alpha}{4} \\ \ln\left(\frac{\alpha^2+4}{4}\right) &\le \frac{\alpha}{2} \\ \frac{\alpha^2+4}{4} &\le e^{\alpha/2} \\ \alpha^2+4 &\le 4e^{\alpha/2} \\ \therefore e^{\alpha/2} &\ge \frac{\alpha^2}{4}+1 \end{aligned}$$ — **1**

$\alpha \ge 0$

Solutions	Marks	Comments

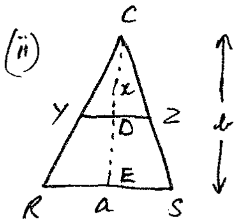
Question 8: b.

(i)



$$\begin{aligned} h^2 &= a^2 - \left(\frac{a}{2}\right)^2 \\ h^2 &= a^2 - \frac{a^2}{4} \\ h^2 &= \frac{3a^2}{4} \\ h &= \frac{\sqrt{3}a}{2} \end{aligned}$$ — **1**

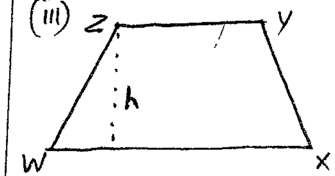
(ii)



Since $\Delta CYZ \parallel\!\mid \Delta CRS$

$$\begin{aligned} \frac{YZ}{RS} &= \frac{CD}{CE} \\ \therefore YZ &= \frac{RS \cdot CD}{CE} \\ &= \frac{ax}{b} \end{aligned}$$ — **2**

(iii)



as the edges of the solid are linear then

$h = mx + c$ — **1**

when $x = 0$ $h = \dfrac{\sqrt{3}a}{2}$ (part (i))

$x = b$ $h = a$

$$\therefore \frac{\sqrt{3}a}{2} = m.0 + c$$
$$\therefore c = \frac{\sqrt{3}a}{2}$$
$$a = mb + \frac{\sqrt{3}a}{2}$$
$$\therefore m = \frac{a - \frac{\sqrt{3}a}{2}}{b} = \frac{2a-\sqrt{3}a}{2b}$$ — **1**

$$\begin{aligned} \therefore h &= \frac{2a-\sqrt{3}a}{2b}x + \frac{\sqrt{3}a}{2} \\ &= \frac{a}{2}\left[(2-\sqrt{3})\frac{x}{b} + \sqrt{3}\right] \\ &= \frac{a}{2}\left[\sqrt{3} + (2-\sqrt{3})\frac{x}{b}\right] \end{aligned}$$ — **1**

Solutions	Marks	Comments
Question 8: (iv) Area of trapezium $= \frac{h}{2}[a+b]$		
$\therefore A = \frac{a}{4}\left[\sqrt{3} + (2-\sqrt{3})\frac{x}{b}\right]\left[a + \frac{ax}{b}\right]$	1	
$= \frac{a}{4}\left[\frac{b\sqrt{3} + (2-\sqrt{3})x}{b}\right]\left[\frac{ab+ax}{b}\right]$		
$= \frac{a}{4b}\left[b\sqrt{3} + (2-\sqrt{3})x\right]\frac{a}{b}\left[b+x\right]$		
$= \frac{a^2}{4b^2}\left[(2-\sqrt{3})x + b\sqrt{3}\right]\left[b+x\right]$	1	
(v) $V = \int_0^b \frac{a^2}{4b^2}\left[(2-\sqrt{3})x + b\sqrt{3}\right]\left[b+x\right]dx$	1	
$= \frac{a^2}{4b^2}\int_0^b \left(b(2-\sqrt{3})x + (2-\sqrt{3})x^2 + b^2\sqrt{3} + bx\sqrt{3}\right)dx$		
$= \frac{a^2}{4b^2}\int_0^b 2bx - \cancel{b\sqrt{3}x} + (2-\sqrt{3})x^2 + b^2\sqrt{3} + \cancel{b\sqrt{3}x}\Big)dx$		
$= \frac{a^2}{4b^2}\left[bx^2 + (2-\sqrt{3})\frac{x^3}{3} + b^2\sqrt{3}x\right]_0^b$	1	
$= \frac{a^2}{4b^2}\left[b^3 + \frac{2b^3}{3} - \frac{\sqrt{3}b^3}{3} + b^3\sqrt{3}\right]$		
$= \frac{a^2b^3}{4b^2}\left[1 + \frac{2}{3} - \frac{\sqrt{3}}{3} + \sqrt{3}\right]$		
$= \frac{a^2b}{4}\left[\frac{3+2-\sqrt{3}+3\sqrt{3}}{3}\right]$		
$= \frac{a^2b}{4}\left(\frac{5+2\sqrt{3}}{3}\right)u^3$	1	