Student Number:________________

St. Catherine's School
Waverley

24 February 2009

Assessment Task 1

Extension II Mathematics

Time allowed: 60 minutes *HSC assessment weighting: 15%*

INSTRUCTIONS

- Write your STUDENT NUMBER on each page
- Marks for each part of a question are indicated
- All questions should be attempted on your own paper
- All necessary working should be shown
- Start each question on a NEW page
- Approved scientific calculators and drawing templates may be used

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1},\quad n\neq -1;\quad x\neq 0,\text{ if } n<0$$

$$\int \frac{1}{x}\,dx = \ln x,\quad x>0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax},\quad a\neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax,\quad a\neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax,\quad a\neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax,\quad a\neq 0$$

$$\int \sec ax\,\tan ax\,dx = \frac{1}{a}\sec ax,\quad a\neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a},\quad a\neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a},\quad a>0,\quad -a<x<a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right),\quad x>a>0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE : $\ln x = \log_e x,\quad x>0$

Question 1 (12 marks)

(a) Let $w_1 = 8 - 2i$ and $w_2 = -5 + 3i$. Find in the form $x + iy$:

(i) $w_1 + \overline{w_2}$ **1**

(ii) $\dfrac{1}{w_1 w_2}$ **2**

(b) (i) Show that $(1-2i)^2 = -3-4i$ **1**

(ii) Hence solve the equation $z^2 - 5z + (7+i) = 0$ **2**

(c) (i) Express $1 - i\sqrt{3}$ in modulus-argument form. **2**

(ii) Express $\left(1 - i\sqrt{3}\right)^5$ in modulus-argument form **2**

(iii) Hence express $\left(1 - i\sqrt{3}\right)^5$ in the form $x + iy$ **1**

Question 2 (12 Marks)

a) Find all solutions to the equation $z^3 = -1$ in modulus-argument form. **3**

b) Sketch the region in the Argand diagram where the two inequalities $|z - i| \le 2$ and $0 \le \arg(z+1) \le \frac{\pi}{4}$ hold simultaneously. **3**

c) Describe the locus of Z on the Argand diagram if

$\arg(z-1) - \arg(z+1) = \dfrac{\pi}{2}$, giving its Cartesian equation. **3**

d) Sketch the region in the Argand diagram that satisfies the inequality **3**

$z\overline{z} + 2(z + \overline{z}) \le 0$

Question 3 (12 marks)

(a) If $z = \cos\theta + i\sin\theta$

(i) Show that $z^n + \dfrac{1}{z^n} = 2\cos n\theta$ **2**

(ii) Hence show that $\cos^4\theta = \dfrac{1}{8}(\cos 4\theta + 4\cos 2\theta + 3)$ **3**

(b) The roots of $x^3 + 5x^2 + 11 = 0$, are α, β and γ.

(i) Find the polynomial equation whose roots are α^2, β^2 and γ^2. **2**

(ii) Find the polynomial equation whose roots are $\dfrac{1}{\alpha}$, $\dfrac{1}{\beta}$ and $\dfrac{1}{\gamma}$ **2**

(c) Show that $\dfrac{x^3 - 4x - 10}{x^2 - x - 6} = x + 1 + \dfrac{3x - 4}{x^2 - x - 6}$. **2**

(d) Using part (c) express $\dfrac{x^3 - 4x - 10}{x^2 - x - 6}$ as the sum of partial fractions. **2**

Question 4 (12 marks)

a) Determine the complex roots of $z^6 = 1$ in the form $\cos\theta + i\sin\theta$ and hence factorise $z^6 - 1$ over:

(i) The complex field **2**

(ii) The real field using linear and quadratic factors **4**

b) When a polynomial $P(x)$ is divided by $(x-2)$ and $(x-3)$ the respective remainders are 4 and 9.

Determine what the remainder must be when $P(x)$ is divided by $(x-2)(x-3)$ **3**

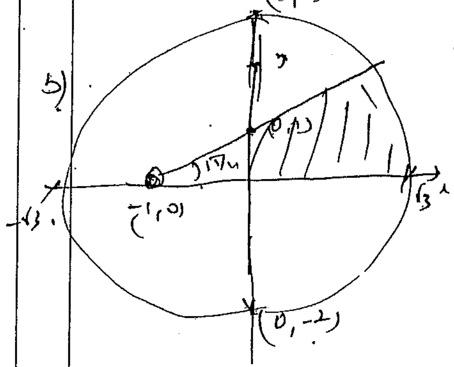
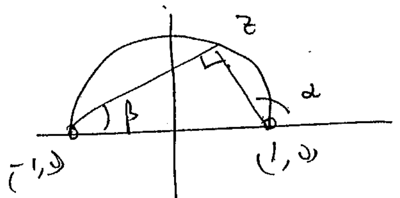
c) If ω is a complex root of $z^3 = 1$

(i) Show that $1 + \omega + \omega^2 = 0$ **1**

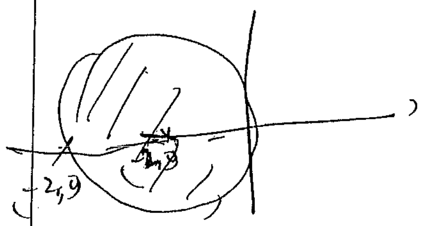
(ii) If k is a positive integer, find two possible values of $1 + \omega^k + \omega^{2k}$ **2**

Qn	Solutions	Marks	Comments+Criteria
1.	$\omega_1 = 8-2i \;;\; \omega_2 = -5+3i$		
(i)	$\omega_1 + \overline{\omega_2} = 8-2i+-5-3i$ $= 3-5i$		
(ii)	$\frac{1}{\omega_1\omega_2}$; $\omega_1\omega_2 = (8-2i)(-5+3i)$ $= (-40-6i^2)+i(24+10)$ $= -34+34i$ $\frac{1}{\omega_1\omega_2} = \frac{1}{34(-1+i)} = \frac{1}{34}\cdot\frac{-1-i}{}$	(2M)	$\frac{1}{34(-1+i)}\cdot\frac{-1-i}{-1-i}$ $= \frac{-1-i}{34(1+1)}$ ~~$\frac{(-1)^2-(-i3)^2}{1+(-1)}$~~ $= -\frac{1}{68}(1+i)$
b)	$(1-2i)^2 = 1+4i^2-4i$ $= -3-4i$ $z^2-5z+7+i = 0$ $z = \frac{5\pm\sqrt{25-4(7+i)}}{2}$ $= \frac{5\pm\sqrt{-3-4i}}{2}$ $= \frac{5\pm(1-2i)}{2}$ $= \frac{6-2i}{2} \;;\; \frac{4+2i}{2}$ $= 3-i \;;\; 2+i$		
c)	$1-i\sqrt{3} =$ $2\,cis(-\pi/3)$ $32\,cis(-5\pi/3) = 32\,cis\,\pi/3$	2 2	$1\frac{1}{2}$ if left as $cis(-5\pi/3)$

$32(\cos 60 + i\sin 60)$

Qn	Solutions	Marks	Comments+Criteria
2.	$z^3 = -1$ Let $z = r\,cis\,\theta$; $\|z\| = r$ $\|z^3\| = \|-1\|$ $\|z\|^3 = 1$ $r^3 = 1$ $r = 1$; r is Real $(cis\,\theta)^3 = -1$ $\cos 3\theta = -1$; $\sin 3\theta = 0$ $3\theta = \pi, 3\pi, 5\pi$ $\theta = \frac{\pi}{3}, \pi, \frac{5\pi}{3}$ The roots are $cis\,\pi/3$, $cis\,\pi = -1$, $cis\,5\pi/3$.		
b)	 $x^2+(y-1)^2 = 4$ $y=0$ $x = \pm\sqrt{3}$		
c)	 α: $Arg(z-1)$; $\beta = Arg(z+1)$ $\alpha-\beta = \frac{\pi}{2}$ given. ∴ Locus is a semi circle, Centre (0,0) a radius 1, whose equation $y = \sqrt{1-x^2}$.		

Qn	Solutions	Marks	Comments+Criteria
d)	Let $z = x + iy$		
	$\bar{z} = x - iy$		
	$z\bar{z} = x^2 + y^2$; $z + \bar{z} = 2x$		
	$\therefore z\bar{z} + 2(z + \bar{z}) \le 0$		
	$x^2 + y^2 + 4x \le 0$		
	$(x+2)^2 + y^2 \le 4$		



Qn	Solutions	Marks	Comments+Criteria
Q.3	Let $z = \cos\theta + i\sin\theta$		
	$\frac{1}{z}$ = [illegible]		
	$z^n = (\cos\theta + i\sin\theta)^n$		
	$= \cos n\theta + i\sin n\theta$ (De Moivre's thm)		
	$\frac{1}{z^n} = \frac{1}{\cos n\theta + i\sin n\theta} \times \frac{\cos n\theta - i\sin n\theta}{\cos n\theta - i\sin n\theta}$		
	$= \cos n\theta - i\sin n\theta$		
	$\therefore z^n + \frac{1}{z^n} = 2\cos n\theta$		
	So $z + \frac{1}{z} = 2\cos\theta$		
	Consider $(z + \frac{1}{z})^4 = z^4 + 4z^3 \cdot \frac{1}{z} + 6z^2 \cdot \frac{1}{z^2} + 4 \cdot z \cdot \frac{1}{z^3} + \frac{1}{z^4}$		

Qn	Solutions	Marks	Comments+Criteria
	$(2\cos\theta)^4 = (z^4 + \frac{1}{z^4}) + 4(z^2 + \frac{1}{z^2}) + 6$		
	$16\cos^4\theta = 2\cos4\theta + 4(2\cos2\theta) + 6$		
	$\cos^4\theta = \frac{1}{8}(\cos4\theta + 4\cos2\theta + 3)$		
b)	$x^3 + 5x^2 + 11 = 0$		
	The roots are α, β, γ		
	Let $P(x)$: $x^3 + 5x^2 + 11$		
	The polynomial with roots $\alpha^2, \beta^2, \gamma^2$ is $P(\sqrt{x}) = 0$ ✓	1	
	$(\sqrt{x})^3 + 5(\sqrt{x})^2 + 11 = 0$		
	$x\sqrt{x} = -5x - 11$ ✓	1	
	$x^3 = (5x - 11)^2$		
	or $x^3 - 25x^2 - 110x - 121 = 0$	1M	
	The polynomial with roots $\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}$ is $P(\frac{1}{x}) = 0$		
	$\frac{1}{x^3} + \frac{5}{x^2} + 11 = 0$		
	$1 + 5x + 11x^3 = 0$	1M.	
c)	$x^2 - x - 6 \overline{)\, x^3 - 4x - 10}$, quotient $x + 1$		
	$x^3 - x^2 - 6x$		
	$x^2 + 2x - 10$		
	$x^2 - x - 6$		
	$3x - 4$		

Qn	Solutions	Marks	Comments+Criteria
	$\therefore \dfrac{x^3-4x-10}{x^2-x-6} = x+1 + \dfrac{3x-4}{x^2-x-6}$		
	Consider $\dfrac{3x-4}{x^2-x-6} \equiv \dfrac{A}{x-3} + \dfrac{B}{x+2}$		
	$3x-4 \equiv A(x+2) + B(x-3)$		
	sub $x=3$; $5 = 5A$ $\therefore A = 1$		
	sub $x=-2$ $-10 = -5B$ $\therefore B = 2$		
	$\therefore \dfrac{x^3-4x-10}{x^2-x-6} = x+1 + \dfrac{1}{x-3} + \dfrac{2}{x+2}$		
Q.4	$z^6 = 1$		
	Let $z = r\,cis\,\theta$; $\lvert z\rvert = r$		
	$\lvert z^6\rvert = \lvert 1\rvert$; $\lvert z\rvert^6 = 1$; $r^6 = 1$; $r = 1$ (r is Real)		
	$(cis\,\theta)^6 = 1$; $cis\,6\theta = 1$; $\cos 6\theta = 1$; $\sin 6\theta = 0$		
	$\therefore 6\theta = 0, 2\pi, 4\pi, 6\pi, 8\pi, 10\pi$		
	$\therefore \theta = 0, \frac{\pi}{3}, \frac{2\pi}{3}, \pi, \frac{4\pi}{3}, \frac{5\pi}{3}$		
	$\therefore z^6 - 1 = 0$ has as roots		
	$cis\,0 = 1$; $cis\frac{\pi}{3}$; $cis\frac{2\pi}{3}$; $cis\,\pi = -1$		
	$cis\frac{4\pi}{3} = \overline{cis\frac{2\pi}{3}}$; $cis\frac{5\pi}{3} = \overline{cis\frac{\pi}{3}}$		

Qn	Solutions	Marks	Comments+Criteria
	$\therefore z^6 - 1 = (z-1)(z+1)(z - cis\frac{\pi}{3})(z - \overline{cis\frac{\pi}{3}})(z - cis\frac{2\pi}{3})(z - \overline{cis\frac{2\pi}{3}})$		in Complex.
	Consider $(z - cis\frac{\pi}{3})(z - \overline{cis\frac{\pi}{3}})$		
	$= z^2 - z(cis\frac{\pi}{3} + \overline{cis\frac{\pi}{3}}) + cis\frac{\pi}{3}\,\overline{cis\frac{\pi}{3}}$		
	$= z^2 - 2z\cos\frac{\pi}{3} + 1$		
	Similarly		
	$(z - cis\frac{2\pi}{3})(z - \overline{cis\frac{2\pi}{3}})$		
	$= z^2 - 2z\cos\frac{2\pi}{3} + 1$.		
	$\therefore z^6 - 1 = (z-1)(z+1)(z^2 - 2z\cos\frac{\pi}{3} + 1)(z^2 - 2z\cos\frac{2\pi}{3} + 1)$		
(b)	When $P(x)$ is divided by $(x-2)(x-3)$, a quadratic expression, the remainder is atmost a linear expression		
	$\therefore P(x) = Q(x)(x-2)(x-3) + ax + b$		
	$P(2) = 4$; $P(3) = 9$		
	$4 = 2a + b$; $9 = 3a + b$		
	$\therefore a = 5$; $b = -6$		
	The remainder is $5x - 6$.		
c)			

Qn	Solutions	Marks	Comments+Criteria
c)	$z^3 = 1$ $(z-1)(z^2+z+1) = 0$ ω is complex $\therefore$ is a root of $z^2+z+1=0$ $\therefore \omega^2+\omega+1=0$ when k is a multiple of 3 $\omega^k = 1$ $\omega^{2k} = 1$ $\therefore 1+\omega^k+\omega^{2k} = 1+1+1 = 3$ when k is not a multiple of 3, $\omega^k = \omega$ or ω^2 $\omega^{2k} = \omega^2$ or ω $\therefore 1+\omega^k+\omega^{2k} =$ ~~1+0+0~~ $1+\omega+\omega^2 = 0$. If ω^k: $\omega^4 = \omega$ $\omega^5 = \omega^3 \times \omega^2 = \omega^2$ if $\omega^k = \omega$: $\omega^{2k} = \omega^2$ if $\omega^k = \omega^2$: $\omega^{2k} = (\omega^2)^2 = \omega^4 = \omega^3 \times \omega = \omega$.	1M	$\omega^6 = 1$ $\omega^7 = \omega^6 \times \omega = \omega$.