Student Number:____________________

St. Catherine's School Waverley

2012

HIGHER SCHOOL CERTIFICATE

ASSESSMENT TASK 1 – 15%

Wednesday 22nd FEBRUARY 2012

Extension 1 Mathematics

General Instructions

- Working time – 55 minutes
- Write using black or blue pen
- Board approved calculators may be used
- All necessary working should be shown
- Start each question on a new booklet.

Total marks - 45

Attempt questions 1-2
The value of all parts is indicated

STUDENT NUMBER/NAME: ....

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE: $\ln x = \log_e x, \quad x > 0$

Question 1 **(23 marks)**

a) Is $(x-4)$ a factor of x^3-3x^2+x-12? Give reasons for your answer. 1

b) i) Fully factorise $x^3+3x^2-10x-24$ given that $x=-2$ is a zero. 3

ii) Hence or otherwise sketch $y=x^3+3x^2-10x-24$ and use it to solve 2

$$x^3+3x^2-10x-24\le 0$$

c) The equation $2x^3-4x^2+7x-6=0$ has roots α, β *and* γ. Find the value of:

i) $\alpha+\beta+\gamma$ 1

ii) $\alpha\beta+\alpha\gamma+\beta\gamma$ 1

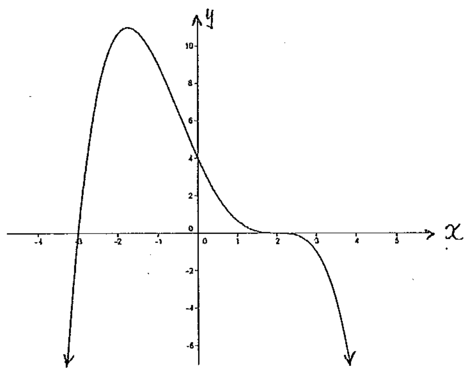
iii) $\alpha^{-1}+\beta^{-1}+\gamma^{-1}$. 2

d) When $P(x)$ is divided by $(x-3)(x+4)$ the quotient is $Q(x)$ and the remainder is $R(x)$.

i) Explain why $R(x)=ax+b$ 1

ii) When $P(x)$ is divided by $(x-3)$ the remainder is 6, and when $P(x)$ is divided by $(x+4)$ the remainder is -2. Find $R(x)$. 3

e) The graph of $y=f(x)$ is shown below.



Write down a possible equation for this function. 2

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f) Solve the equation $3x^3-17x^2-8x+12=0$, given that the product of two of the roots is 4. 3

g) Prove by mathematical induction that $n^3+(n+1)^3+(n+2)^3$ is divisible by 9 for $n\ge 1$. 4

Question 2 **(22 marks) (Start a new page)**

a) If $sin\alpha=\frac{3}{5}$, $0°<\alpha<90°$ and $cos\beta=\frac{2}{3}$, $270°<\beta<360°$, write down the exact value of $sin(\alpha+\beta)$. 3

b) Solve $cos2x=sinx$ for $0\le x\le 2\pi$. 3

c) Write down the general solution to $cos\left(2\theta-\frac{\pi}{3}\right)=\frac{1}{2}$ 2

d) i) Express $sin3t-cos3t$ in the form $rsin(3t-\alpha)$. 2

ii) Hence or otherwise solve $sin3t-cos3t=1$ for $0\le t\le\pi$. 2

e) i) Use the expansion for $cos(\theta+2\theta)$ to show that 2

$$cos3\theta=4cos^3\theta-3cos\theta$$

ii) Hence solve the equation 2

$$cos3\theta+2cos\theta=0 \text{ for } 0\le\theta\le 2\pi.$$

f) Using the result $tan\theta=\frac{2t}{1-t^2}$, where $t=tan\frac{\theta}{2}$ 3.

find the exact value of $tan15°$.

g) Prove the following: 3

$$tan\left(\frac{\pi}{4}+x\right)=sec2x+tan2x$$

End of paper

3unit.

SOLUTIONS

1 a) SINCE $P(4) = 4^3 - 3(4)^2 + 4 - 12$

$= 8 \neq 0$

$\therefore$ ACCORDING TO THE FACTOR THEOREM $(x-4)$ IS NOT A FACTOR. (IF IT WAS $P(4)=0$)

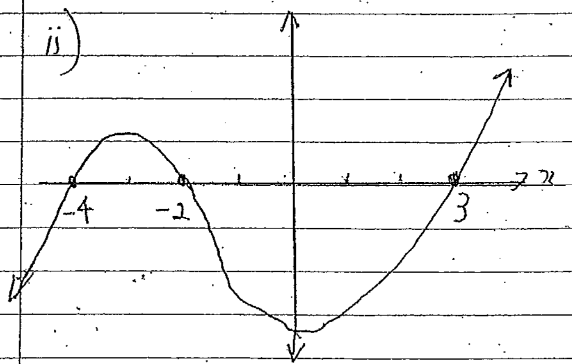
b) i)

$$\begin{array}{r} x^2+x-12 \\ (x+2)\overline{)x^3+3x^2-10x-24} \\ \underline{x^3+2x^2} \\ x^2-10x \\ \underline{x^2+2x} \\ -12x-24 \\ \underline{-12x-24} \\ 0 \end{array}$$

$\therefore x^3+3x^2-10x-24 = (x+2)(x^2+x-12)$

$= (x+2)(x+4)(x-3)$

ii)



FOR $x^3+3x^2-10x-24 \leq 0$

$x \leq -4, \ -2 \leq x \leq 3$

c) $2x^3-4x^2+7x-6=0$

i) $\Sigma\alpha = -\frac{b}{a}$

$= 2$

ii) $\Sigma\alpha\beta = \frac{c}{a}$

$= \frac{7}{2}$

iii) $\alpha\beta\gamma = -\frac{d}{a}$

$= 3$

$\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma} = \frac{\alpha\beta+\alpha\gamma+\beta\gamma}{\alpha\beta\gamma}$

$= \frac{7/2}{3}$

$= \frac{7}{6}$

d) $P(x) = (x-3)(x+4)Q(x) + R(x)$.

i) The degree of $R(x)$ must be less than the degree of the divisor, $(x-3)(x+4)$. Since the divisor is of degree 2, $R(x)$ must be of degree one.

ie $R(x) = ax+b$.

$P(x) = (x-3)(x+4)Q(x) + ax + b$

$P(3) = 6$

$3a + b = 6$ (1)

$P(-4) = -2$

$-4a + b = -2$ (2)

$7a = 8$ (1)-(2)

$a = \frac{8}{7}$

SUB IN (1)

$\frac{24}{7} + b = 6$

$b = \frac{18}{7}$

$\therefore R(x) = \frac{8x}{7} + \frac{18}{7}$

e) $y = -\frac{1}{6}(x+3)(x-2)^3$

Has roots $x = -3, 2, 2, 2$ and passes through $(0, 4)$

f) $3x^3 - 17x^2 - 8x + 12 = 0$

ROOTS $x = \alpha, \frac{4}{\alpha}, \beta$

$\Sigma\alpha = -\frac{b}{a}$ $\qquad$ $\Sigma\alpha\beta = \frac{c}{a}$

$\alpha + \frac{4}{\alpha} + \beta = \frac{17}{3}$ (1) $\qquad$ $4 + \alpha\beta + \frac{4\beta}{\alpha} = \frac{-8}{3}$

$\alpha\beta\gamma = -\frac{d}{a}$

$4\beta = \frac{-12}{3}$

$\therefore \beta = -1$

IN (1)

$\alpha + \frac{4}{\alpha} - 1 = \frac{17}{3}$

$\alpha + \frac{4}{\alpha} - \frac{20}{3} = 0$

$3\alpha^2 - 20\alpha + 12 = 0$

$3\alpha \quad -2$

$\alpha \quad -6$

$(3\alpha - 2)(\alpha - 6) = 0$

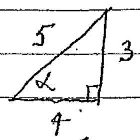
$\alpha = \frac{2}{3}, 6$

$\therefore$ ROOTS ARE

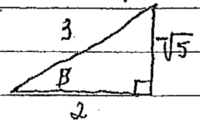
$x = \frac{2}{3}, 6, -1$.

(2)

a) $\sin\alpha = \frac{3}{5}$



$\cos\beta = \frac{2}{3}$



$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$

$= \frac{3}{5}\cdot\frac{2}{3} + \frac{4}{5}\cdot\left(\frac{-\sqrt{5}}{3}\right)$

$= \frac{6-4\sqrt{5}}{15}$

b) $\cos 2x = \sin x$

$1 - 2\sin^2 x = \sin x$

$2\sin^2 x + \sin x - 1 = 0$

$(2\sin x - 1)(\sin x + 1) = 0$

$\sin x = \frac{1}{2}$ $\quad \sin x = -1$

$x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{3\pi}{2}$

c) $\cos\left(2\theta - \frac{\pi}{3}\right) = \frac{1}{2}$

$2\theta - \frac{\pi}{3} = 2n\pi \pm \frac{\pi}{3}$

$2\theta = \frac{\pi}{3} + 2n\pi \pm \frac{\pi}{3}$

$\theta = \frac{\pi}{6} + n\pi \pm \frac{\pi}{6}$

$a=1,\ b=1$

d) i) $\sin 3t - \cos 3t = r\sin(3t - \alpha)$

$r = \sqrt{a^2+b^2}$ $\quad \tan\alpha = \frac{b}{a}$

$= \sqrt{1^2+1^2}$ $\quad = 1$

$= \sqrt{2}$ $\quad \alpha = \frac{\pi}{4}$

$\therefore \sin 3t - \cos 3t = \sqrt{2}\sin\left(3t - \frac{\pi}{4}\right)$

ii) $\sqrt{2}\sin\left(3t - \frac{\pi}{4}\right) = 1$

$\sin\left(3t - \frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$

$3t - \frac{\pi}{4} = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{9\pi}{4}, \frac{11\pi}{4}, \cancel{\frac{17\pi}{4}}$

$3t = \frac{\pi}{2}, \pi, \frac{5\pi}{2}, 3\pi$

$t = \frac{\pi}{6}, \frac{\pi}{3}, \frac{5\pi}{6}, \pi$

e) $\cos(\theta+2\theta) = \cos\theta\cos2\theta - \sin\theta\sin2\theta$

$= \cos\theta(2\cos^2\theta - 1) - \sin\theta(2\sin\theta\cos\theta)$

$= 2\cos^3\theta - \cos\theta - 2(1-\cos^2\theta)\cos\theta$

$= 2\cos^3\theta - \cos\theta - 2\cos\theta + 2\cos^3\theta$

$= 4\cos^3\theta - 3\cos\theta$

ii) $\cos 3\theta + 2\cos\theta = 0$

$4\cos^3\theta - 3\cos\theta + 2\cos\theta = 0$

$4\cos^3\theta - \cos\theta = 0$

$\cos\theta(4\cos^2\theta - 1) = 0$

$\cos\theta = 0$ $\quad \cos\theta = \pm\frac{1}{2}$

$\theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{\pi}{3}, \frac{2\pi}{3}, \frac{5\pi}{3}, \frac{4\pi}{3}$

q) $n^3 + (n+1)^3 + (n+2)^3$ IS DIVISIBLE BY 9.

PROVE TRUE FOR $n=1$

$1^3 + 2^3 + 3^3 = 36$

$= 9\times 4$

$\therefore$ DIVISIBLE BY 9 AND TRUE FOR $n=1$

ASSUME TRUE FOR $n=k$

$k^3 + (k+1)^3 + (k+2)^3 = 9Q$

Q SOME INTEGER

PROVE TRUE FOR $n=k+1$

$(k+1)^3 + (k+2)^3 + (k+3)^3 = 9M$

M SOME INTEGER

$LHS = 9Q - k^3 + (k+3)^3$

$= 9Q - k^3 + k^3 + 9k^2 + 27k + 27$

$= 9(Q + k^2 + 3k + 3)$

$= 9M$ WHERE $M = Q + k^2 + 3k + 3$ SOME INTEGER

$= RHS$

IF TRUE FOR $n=k$, THEN PROVE TRUE FOR $n=k+1$. BUT PROVED TRUE FOR $n=1$, $\therefore$ BY MATHEMATICAL INDUCTION TRUE FOR $n=2,3,4,\ldots$ & TRUE FOR ALL $n\geq 1$

f) $\tan\theta = \frac{2t}{1-t^2}$

$\tan 30^\circ = \frac{2t}{1-t^2}$ where $t = \tan 15^\circ$

$\frac{1}{\sqrt{3}} = \frac{2t}{1-t^2}$

$1-t^2 = 2\sqrt{3}t$

$t^2+2\sqrt{3}t-1=0$

$t = \frac{-2\sqrt{3} \pm \sqrt{12+4}}{2}$

$= \frac{-2\sqrt{3} \pm 4}{2}$

$= -\sqrt{3} \pm 2$

Since $\tan 15^\circ > 0$

$\tan 15^\circ = 2-\sqrt{3}$

g) $\tan\left(\frac{\pi}{4}+x\right) = \sec 2x + \tan 2x$

ie $\frac{1+\tan x}{1-\tan x} = \sec 2x + \tan 2x$

RHS $= \frac{1}{\cos 2x} + \frac{2\tan x}{1-\tan^2 x}$

$= \frac{1}{\cos^2 x - \sin^2 x} + \frac{2\tan x}{1-\tan^2 x}$

DIVIDE BY $\cos^2 x$

$= \frac{\sec^2 x}{1-\tan^2 x} + \frac{2\tan x}{1-\tan^2 x}$

$= \frac{1+\tan^2 x + 2\tan x}{1-\tan^2 x}$

$= \frac{(\tan x + 1)^2}{(1-\tan x)(1+\tan x)}$

$= \frac{1+\tan x}{1-\tan x}$

$=$ LHS.

(Note: there are other ways of proving this.)