Student Number:.

Teacher:..

St. Catherine's School

Year 11 Extension 1 Mathematics

End of Preliminary Examination

Task #3

September 2006

Time allowed: *2 hours*

INSTRUCTIONS

- There are 6 questions of equal value.
- Marks for each part of a question are indicated
- All questions should be attempted.
- All necessary working should be shown
- Start each question on a new page
- Approved scientific calculators and drawing templates may be used

Question 1 **Marks**

(a) Given the points $A(-4, 3)$ and $B(2, -1)$, find the point which divides the interval AB in the ratio 2 : 3 internally. 2

(b) Solve the inequality $\frac{4}{x+3} \le 1$ 3

(c) Find the derivative of the following functions in simplest form: 4

(i) $y = \frac{1}{\sqrt{x}}$

(ii) $f(x) = \frac{x^2 - 2}{x - 1}$

(d) A polynomial $P(x)$ is given by $P(x) = x^3 - 2x^2 - x + 2$

(i) Show that $P(x)$ has a zero at $x = 2$ 1

(ii) Hence or otherwise write $P(x)$ in the form $P(x) = (x-a)(x-b)(x-c)$ 2

Question 2 **Marks**

(a) Give a quick sketch of the polynomial $P(x)=x^3(x-2)(x+2)^2$ **2**
You do not need to use calculus or find any stationary/turning points or points of inflexion

(b) Solve the equation $|x^2-5| = 5x+9$ **4**
(make sure you check your solutions)

(c) Students in an examination were asked to express $2.2\dot{3}$ as a fraction by first expressing it as the sum of a geometric sequence. **3**
The following solution was presented:

$$\begin{aligned}2.2\dot{3} &= 2.2+0.0\dot{3}\\ &= 2.2+0.03+0.003+0.0003+\ldots\\ &= \frac{2.2\left(1-\left(\frac{1}{10}\right)^4\right)}{1-\frac{1}{10}}\\ &= \frac{\frac{11}{5}\left(\frac{9999}{10000}\right)}{\frac{9}{10}}\\ &= \frac{24442}{10000} = \frac{12221}{5000}\end{aligned}$$

Comment on the accuracy of this solution. If it is incorrect, write the correct solution.

(d) Solve $2\sin^2 x+\sin x-1=0$ for $0\le x\le 360°$ **3**

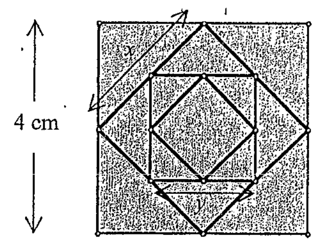
Question 3 **Marks**

(a) (i) Find $\displaystyle\lim_{x\to\infty}\frac{2x^4+x^3-5x-1}{3x^3+x-4x^4-3}$ **1**

(ii) Use trig identities to simplify $\displaystyle\lim_{x\to 0}\frac{\cos^2 x+\sin x\cos x-1}{\sin x}$ **2**

(b) The normal to the curve $y=2x^3-5x^2+9x-10$ at $(1,-4)$ crosses the y-axis at P. **3**
Find the equation of the normal and hence the coordinates of P.

(c) The midpoints of a square of side length 4 cm are joined to give a second square inside the original. The midpoints of the sides of this second square are joined to give a third square and this process is repeated indefinitely. **3**

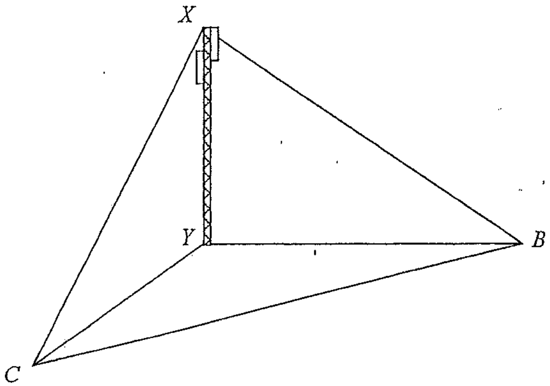


(i) Show that $x=2\sqrt{2}$ and $y=2$

(ii) Hence or otherwise calculate the limit of the sum of the areas of all the squares

Question 3 continues on the next page

(d) From a point C due south of a mobile phone tower XY, Constance notices that the angle of elevation of the tower is 20°. Her great friend Beatrice is due east of the tower at point B and notices that the angle of elevation is 35°. Constance and Beatrice are 40 metres apart.



(i) Show that $CY = h\tan 70°$ and $BY = h\tan 55°$ **1**

(ii) Hence or otherwise show that the height of the tower is given by **2**

$$h = \frac{40}{\sqrt{\tan^2 70° + \tan^2 55°}}$$

Question 4 **Marks**

(a) The tangent to the curve $y = (x-1)^3$ at the point $(2, 1)$ meets the curve again at the point P.

(i) Show that the equation of the tangent is given by $y = 3x - 5$ **1**

(ii) Hence or otherwise find the coordinates of P. **3**

(b) The curves $y = x^3$ and $y = (x-2)^2$ intersect at the point $(1, 1)$. **4**
Show that the acute angle between the tangents to the two curves at this point is 45°.

(c) The function $f(x)$ is given by $f(x) = \sqrt{x+1}$. Find $f'(x)$ by first principles. **4**
(Hint: rationalising the numerator will help)

Question 5 **Marks**

(a) Sum the first 10 terms of the series: 2

$$\log_2 3 + \log_2 9 + \log_2 27 + \ldots$$

(b) Prudence is having trouble finishing off her Mathematical Induction problem. She was asked to show that

'the sum of the cubes of three consecutive integers is divisible by 3 for all positive integers $n \geq 1$.*'*

Prudence has written out some of her proof and it is copied below. She needs help with the section of the proof which is missing from her work. (the shaded area)
Complete the correct working for the missing section for her **in your exam booklet.** 3

> Let $n=1$
> $1^3+2^3+3^3 = 1+8+27$
> $= 36$ So proposition true for $n=1$
>
> Assume $n=k$ is true for some positive integer k
> ie: $k^3+(k+1)^3+(k+2)^3 = 3M$ for M a positive integer
>
> Given $n=k$ true prove $n=k+1$ is true.
> ie: $(k+1)^3+(k+2)^3+(k+3)^3 = 3N$ for N a positive integer
> Proof:
>
> If the proposition is true for $n=k$ then it is true for $n=k+1$. As it is true for $n=1$ so it is true for $n=2,3,4,\ldots\ldots$ and all positive integers.
> The proposition is true by Mathematical Induction.

Question 4 continues on the next page

(c) The polynomial $P(x) = 2x^3+ax^2+bx+6$ is divisible by $(x-1)$ and leaves a remainder of -12 when divided by $(x+2)$. 4
Find the values of a and b.

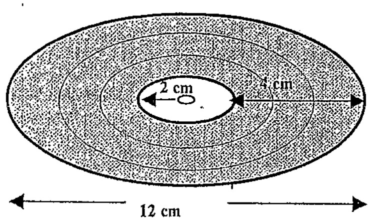
(d) Find in simplest form, a relation in p, q and r such that the following equation has two equal roots: 3

$$(p^2+q^2)x^2+2q(p+r)x+(q^2+r^2)=0$$

Question 6 **Marks**

(a) The section of a CD used to store information starts 2 cm from the centre of the CD and extends 4 cm to the edge of the disk, as in the diagram. As the CD spins, a laser moves along concentric circle 'tracks' on the information section of the disk.

There are 20 of these tracks for every ***millimetre*** of the radius of the information section of the disk.



(i) Find the length of the inside track **1**

(ii) The lengths of the tracks form an arithmetic sequence
Show that the nth concentric track from the centre has length **2**

$$2\pi\left(2+\frac{(n-1)}{200}\right) \text{ cm}$$

(iii) Find the **total** distance of information 'track' on a typical CD **2**

(b) Given $f(x)=ax+\frac{b}{x}$ (where a, $b>0$):

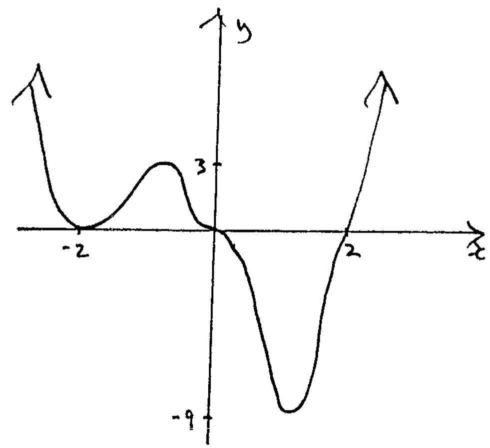
(i) Show that the turning point of the curve is $P\left(\sqrt{\frac{b}{a}}\ ,\ 2\sqrt{ab}\right)$ for $x>0$. **3**

(ii) Sketch the curve for $x>0$, showing all important features. **3**

(iii) The line $y=c$ lies below the curve $f(x)=ax+\frac{b}{x}$ and $c\geq 0$. Using your graph or otherwise, explain why $ab>\frac{c^2}{4}$. **1**

End of examination

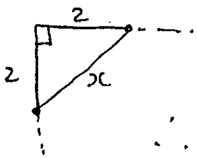
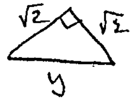
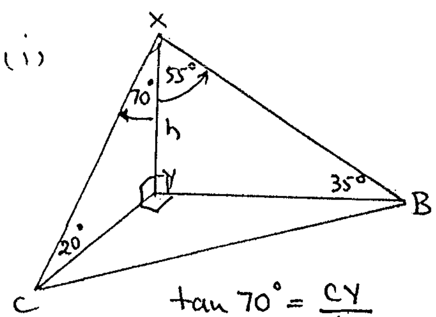
Qn	Solutions	Marks	Comments+Criteria
1(a)	$P \equiv \left(\frac{ax_2+bx_1}{a+b}, \frac{ay_2+by_1}{a+b}\right)$		(1) correct subst^n
	$\equiv \left(\frac{2\cdot 2-3\cdot 4}{2+3}, \frac{2\cdot -1+3\cdot 3}{2+3}\right)$	✓	(1) correct coords
	$\equiv \left(-\frac{8}{5}, \frac{7}{5}\right)$	✓	
(b)	$\frac{4}{x+3} \leq 1$		
	$\frac{4(x+3)^2}{(x+3)} \leq (x+3)^2$	✓	(1) recognition of variable in denom (any technique)
	$4x+12 \leq x^2+6x+9$		
	$x^2+2x-3 \geq 0$	✓	(1) correct inequality as a quadratic.
	$(x+3)(x-1) \geq 0$		(1) correct solution
	$\therefore\ x \geq 1,\ x < -3$	✓	$(-\frac{1}{2})$ for $x \leq -3$
(c)	(i) $y = \frac{1}{\sqrt{x}} = x^{-\frac{1}{2}}$		
	$y' = -\frac{1}{2}x^{-\frac{3}{2}}$	✓	(1) correct derivative in unsimplied form
	$= \frac{-1}{2\sqrt{x^3}}$ $\left[= \frac{-1}{2x\sqrt{x}}\right]$		
	(ii) $f(x) = \frac{x^2-2}{x-1} = \frac{u}{v}$		
	$f'(x) = \frac{(x-1)\cdot 2x - (x^2-2)\cdot 1}{(x-1)^2}$	✓	(1) correct subst^n
	$= \frac{2x^2-2x-x^2+2}{(x-1)^2}$	✓	(1) correct expans^n
	$= \frac{x^2-2x+2}{(x-1)^2}$	✓	(1) correct simpl^n

Qn	Solutions	Marks	Comments+Criteria
(d)	$P(x) = x^3-2x^2-x+2$		
	(i) $P(2) = 8-8-2+2 = 0$	✓	(1) correct use of factor th^m
	$\therefore\ x=2$ is a zero of $P(x)$		
	(ii) $P(x) = (x-2)(x^2-1)$ on div.^{n}	✓	(1) correct division or other
	$P(x) = (x-2)(x+1)(x-1)$	✓	(1) correct factors of $P(x)$
2(a)	$P(x) = x^3(x-2)(x+2)^2$		
		✓✓	(1) correct intercepts and axis cut shape (1) scale and general shape
(b)	$\lvert x^2-5\rvert = 5x+9$		
	(+) $x^2-5 = 5x+9$		(4) full solution $x = 7, -1$ with explanation of excluded values
	$x^2-5x-14 = 0$		
	$(x-7)(x+2) = 0$		
	$x = 7, \cancel{-2}$		(3) full solution (no check, no exclusion)
	check $x=7$ $\quad \lvert 49-5\rvert \overset{?}{=} 35+9$ ✓		
	$x=-2$ $\quad \lvert 4-5\rvert \overset{?}{=} 5\cdot -2+9$ ✗		(2) one case only (correct solutions)

(1) substantial start on one solution

Qn	Solutions	Marks	Comments+Criteria
2(b) cont.	(−) $-x^2+5 = 5x+9$ $x^2+5x+4=0$ $(x+4)(x+1)=0$ $x = \cancel{-4}, -1$ check $x=-4$ $\quad |16-5| \overset{?}{=} -20+9$ ✗ $x=-1$ $\quad |1-5| = -5+9$ ✓ $\therefore x = 7, -1$		
(c)	The solution is incorrect. line 2 $= 2.2+0.03+0.003+\dots$ $= 2.2 + [0.03+0.003+\dots]$ infinite GP with $a=0.03$ $r=0.1$ $= 2.2 + \frac{0.03}{1-0.1}$ $= 2.2 + \frac{0.03}{0.9}$ $= 2\frac{1}{5} + \frac{1}{30}$ $= 2\frac{7}{30}$ $\quad \left[= \frac{67}{30}\right]$	✓ ✓ ✓	(3) correct solution done as a GP and errors noted (2) Correct soln errors not correct (1) start on solutio or note where errors occur
(d)	$2\sin^2 x + \sin x - 1 = 0$ $(2\sin x - 1)(\sin x + 1) = 0$ $\sin x = \frac{1}{2}$ $\quad \sin x = -1$ $x = 30°, 150°$, $\quad x = 270°$	✓ ✓✓	(3) correct solutions (2) solutions to either $\sin x = \frac{1}{2}$ or $\sin x = -1$ or correct without Q2 (1) correct factn or one correct soln for Q1

Qn	Solutions	Marks	Comments+Criteria
3(a)	(i) $\lim\limits_{x\to\infty} \frac{2x^4+x^3-5x-1}{3x^3+x-4x^4-3}$ $= \lim\limits_{x\to\infty} \frac{2+\frac{1}{x}-\frac{5}{x^3}-\frac{1}{x^4}}{\frac{3}{x}+\frac{1}{x^3}-4-\frac{3}{x^4}}$ $= -\frac{1}{2}$	✓	(1) correct respons ($\frac{1}{2}$) correct proces
	(ii) $\lim\limits_{x\to 0} \frac{\cos^2 x + \sin x\cos x - 1}{\sin x}$ $= \lim\limits_{x\to 0} \frac{1-\sin^2 x + \sin x\cos x - 1}{\sin x}$ $= \lim\limits_{x\to 0} \frac{\cancel{\sin x}(\cos x - \sin x)}{\cancel{\sin x}}$ $= \lim\limits_{x\to 0} (\cos x - \sin x)$ $= 1$		(2) correct respons (1) substantial start on simplif ($\frac{1}{2}$) use of $\cos^2 x = 1-\sin^2 x$ ($-\frac{1}{2}$) for $\lim\limits_{x\to 0} \cos x - \sin x = 0$
(b)	$y = 2x^3-5x^2+9x-10$ $y' = 6x^2-10x+9$ at $x=1$ $\quad m_{tan} = 6-10+9$ $= 5$ $\therefore$ $\quad m_{norm} = -\frac{1}{5}$ $\therefore$ eqn norm: $(y+4) = -\frac{1}{5}(x-1)$ at $x=0$ $\quad y = -4+\frac{1}{5}$ $= -3\frac{4}{5}$ $\therefore P \equiv \left(0, -\frac{19}{5}\right)$		(3) Correct eqn of normal and coords of P (2) Correct eqn of normal (1) Correct normal gradient or substantial start on eqn normal

Qn	Solutions	Marks	Comments+Criteria
(c)	(i)  $x^2 = 2^2 + 2^2$ $\therefore x = 2\sqrt{2}$  $y^2 = (\sqrt{2})^2 + (\sqrt{2})^2$ $= 4$ $\therefore y = 2$	✓	(1) show $x = 2\sqrt{2}$ $y = 2$ by any technique
	(ii) Areas of squares are: $4^2, x^2, y^2, \ldots$ $16, 8, 4, 2, \ldots$ $\therefore$ sum of infinite GP $a = 16$ $r = \frac{1}{2}$ $\therefore S_\infty = \frac{16}{1-\frac{1}{2}}$ $= 32 \text{ cm}^2$	✓ ✓	(2) Correct infinite sum (1) correct GP or correct substⁿ into S_∞ but incorrect answer (1) S_∞ of series of lengths $4, 2\sqrt{2}, 2, \ldots$
(d)	(i)  $\tan 70° = \frac{CY}{h}$ $\therefore CY = h \tan 70°$ $\tan 55° = \frac{BY}{h}$ $\therefore BY = h \tan 55°$	✓	(1) correct expressions for CY, BY using tangent of complements

Qn	Solutions	Marks	Comments+Criteria
	(ii) in $\triangle CYB$, $\angle CYB = 90°$ $\therefore 40^2 = CY^2 + BY^2$ $\therefore 40^2 = h^2\tan^2 70° + h^2 \tan^2 55°$ $\therefore h^2 = \frac{40^2}{\tan^2 70 + \tan^2 55}$ $\therefore h = \frac{40}{\sqrt{\tan^2 70 + \tan^2 55}}$ as $h > 0$		(2) correct expression (1) recognition of $\triangle$ and a start on the expression
~~4(a)~~ (c)	$f(x) = \sqrt{x+1}$ $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ $= \lim_{h\to 0} \frac{\sqrt{x+h+1} - \sqrt{x+1}}{h}$ $= \lim_{h\to 0} \frac{(\sqrt{x+h+1} - \sqrt{x+1})(\sqrt{x+h+1} + \sqrt{x+1})}{h(\sqrt{x+h+1} + \sqrt{x+1})}$ $= \lim_{h\to 0} \frac{(x+h+1) - (x+1)}{h(\sqrt{x+h+1} + \sqrt{x+1})}$ $= \lim_{h\to 0} \frac{\cancel{h}}{\cancel{h}(\sqrt{x+h+1} + \sqrt{x+1})}$ $= \frac{1}{2\sqrt{x+1}}$	✓ ✓ ✓ ✓	(4) correct process with correct derivative (any technique for simplification) (3) limit process with correct method incorrect simplifⁿ (2) Correct substⁿ and a start on the simplifⁿ ( ) correct substⁿ

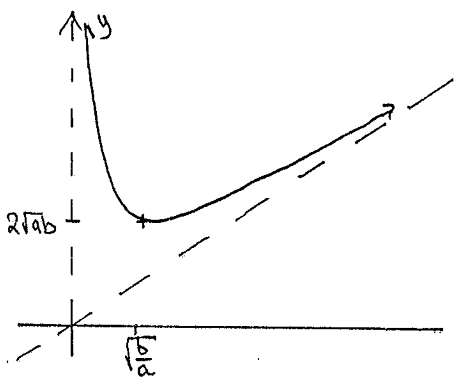
Qn	Solutions	Marks	Comments+Criteria
4(b)	$y=x^3 \cap y=(x-2)^2$ @ $(1,1)$ $y'=3x^2$ | $y'=2(x-2)$ m_{tan} at $x=1$ gives $m_{tan_1}=3$ | m_{tan} at $x=1$ gives $m_{tan_2}=-2$	✓✓	(4) Correct ∡ with correct substⁿ and technique (3) Correct process, gradients etc but incorrect ∡ (2) Correct gradients or one incorrect gradient + correct process leading to incorrect answer (1) Substantial start on process
	$\tan\theta = \left\lvert\frac{m_1-m_2}{1+m_1m_2}\right\rvert$ $= \left\lvert\frac{3+2}{1-6}\right\rvert = 1$	✓	
	$\therefore \theta = 45^\circ$ if acute.	✓	
(a)	(i) $y=(x-1)^3$ at $(2,1)$ $y'=3(x-1)^2 \cdot 1$ $m_{tan}=3$ $\therefore$ equation of tangent is: $y-1=3(x-2)$ $y=3x-5$	✓	(1) Correct gradient and intercept shown
	(ii) $y=3x-5 \cap y=(x-1)^3$ $(x-1)^3=3x-5$ $x^3-3x^2+\cancel{3x}-1=\cancel{3x}-5$ $x^3-3x^2+4=0=P(x)$ $x=2$ gives $P(x)=0$ $\therefore P(x)=(x-2)(x^2-x-2)$ $=(x-2)(x-2)(x+1)$	✓ ✓	(3) Correct soln. with any technique appropriate

Qn	Solutions	Marks	Comments+Criteria
	$\therefore x=2,-1$ the tangent meets the curve again at $x=-1$ $\therefore P \equiv (-1,-8)$		(2) Correct polynomial and an attempt to factorize (1) Correct polynomi
5(a)	$\log_2 3+\log_2 9+\log_2 27+\dots$ $=\log_2 3+2\log_2 3+3\log_2 3+\dots$ AP with $a=\log_2 3$ $d=\log_2 3$	✓	(2) Correct answer (1) Correct a, d or simplⁿ of log expressions
	$\therefore S_{10}=\frac{10}{2}(2\log_2 3+9\log_2 3)$ $=55\log_2 3$	✓	
(b)	$LHS=(k+1)^3+(k+2)^3+(k+3)^3$ $=(3M-k^3)+(k+3)^3$ (using $n=k$ assumption) $=3M-k^3+k^3+9k^2+27k+27$ $=3M+9k^2+27k+27$ $=3(M+3k^2+9k+9)$ which is $=3N$ (where $N=M+3k^2+9k+9$) which is divisible by 3 QED	✓	(3) Correct soln in correct forma (2) Correct substⁿ of $n=k$ and substanti start on proof (1) substitⁿ of $n=k$ into $n=k+1$ expressio correct OR

Qn	Solutions	Marks	Comments+Criteria
(c)	$P(x) = 2x^3 + ax^2 + bx + 6$ $P(1) = 0$ $\therefore\ 2 + a + b + 6 = 0$ $a + b = -8 \quad (i)$ $P(-2) = -12$ $\therefore\ -16 + 4a - 2b + 6 = -12$ $4a - 2b = -2$ $2a - b = -1 \quad (ii)$ $\therefore$ (i) + (ii) gives $3a = -9$ $a = -3$ $b = -5$	✓ ✓ ✓ ✓	(4) Correct values of a, b by any appropriate technique (3) Correct technique with incorrect value of a or b (2) Correct simult. eqns, incorrect solution (1) use of factor th^m or remainder th^m only
(d)	$(p^2+q^2)x^2 + 2q(p+r)x + (q^2+r^2) = 0$ has 2 equal roots: $\therefore\ \Delta = 0$ $\therefore\ 4q^2(p+r)^2 - 4(p^2+q^2)(q^2+r^2) = 0$ $\therefore\ 4q^2(p^2+2pr+r^2) - 4(p^2q^2+p^2r^2+q^4+q^2r^2) = 0$ $\therefore\ 8pq^2r - 4p^2r^2 - 4q^4 = 0$ $q^4 - 2q^2pr + p^2r^2 = 0$ $(q^2 - pr)^2 = 0$ $\therefore\ q^2 = pr$	✓ ✓ ✓	(3) correct relation between p, q, r by any appropriate technique (could use roots α, β etc) (2) Correct applic^n of correct technique with incorrect expression (1) Substantial start on one of the appropriate technique

accept
$q^4 - 2q^2pr + p^2r^2 = 0$ (2)

Qn	Solutions	Marks	Comments+Criteria
6(a)	(i) $C_1 = 2\pi r$ $= 4\pi$	✓	correct exact value (1)
	(ii) $C_2 = 2\pi\left(2 + \frac{1}{200}\right)$ $= 4\pi + \frac{\pi}{100}$ $C_3 = 2\pi\left(2 + \frac{2}{200}\right)$ $= 4\pi + \frac{2\pi}{100}$ $\therefore$ AP with $a = 4\pi$ $d = \frac{\pi}{100}$ $\therefore\ C_n = a + (n-1)d \quad [= T_n]$ $= 4\pi + \frac{(n-1)\pi}{100}$ $= 2\pi\left(2 + \frac{(n-1)}{200}\right)$		(2) correct 'show' of nth track (1) substantial start on 'show'
	(iii) There are 40mm × 20 = 800 track 'spaces' $\therefore$ 801 tracks $\therefore\ S_n = \frac{n}{2}(2a + (n-1)d)$ $= \frac{801}{2}\left(8\pi + 800 \cdot \frac{\pi}{100}\right)$ $= 6408\pi$ cm $[= 20\,131.325\ldots$ cm$]$ $\doteqdot 0.2013$ km OR first track is 4π cm last track is 12π cm $\therefore\ S_{801} = \frac{801}{2}(4\pi + 12\pi)$ $= 6408\pi$ cm		accept 800 or 801 tracks (2) Correct sum given their value of n (1) Correct subst^n incorrect length

Qn	Solutions	Marks	Comments+Criteria
(b)	(i) $f(x) = ax + \frac{b}{x} \quad (a, b > 0)$ $f'(x) = a - \frac{b}{x^2}$ let $f'(x) = 0$ for SP's ie $a - \frac{b}{x^2} = 0$ $x^2 = \frac{b}{a}$ $x = \pm\sqrt{\frac{b}{a}}$ and $x > 0$ $\therefore x = \sqrt{\frac{b}{a}}$ $\therefore f\left(\sqrt{\frac{b}{a}}\right) = a\sqrt{\frac{b}{a}} + \frac{b}{\sqrt{\frac{b}{a}}}$ $= a\sqrt{\frac{b}{a}} + b\sqrt{\frac{a}{b}}$ $= \sqrt{\frac{a^2b}{a}} + \sqrt{\frac{b^2a}{b}} = 2\sqrt{ab}$ $\therefore$ SP at $\left(\sqrt{\frac{b}{a}}, 2\sqrt{ab}\right)$		(3) Correct 'show' with appropriate working (2) Correct derivative and correct process for SP with a mistake (1) correct derivative ignore 'determine nature' processes... −1 incorrect $f\left(\sqrt{\frac{a}{b}}\right)$
	(ii) vertical asymptote at $x=0$ oblique asymptote at $y = ax$  		(3) correct curve for $x>0$ with both asymptotes and SP labelled or indicated (2) correct curve with one asymptote only (1) valid and substantial start on curve (working or curve)

Qn	Solutions	Marks	Comments+Criteria
(b)	(iii) If $y=c$ lies beneath the curve and $c>0$ then $f\left(\sqrt{\frac{b}{a}}\right) > c$ ie $2\sqrt{ab} > c$ $4ab > c^2 \quad (c>0)$ $\therefore ab > \frac{c^2}{4}$	✓	(1) use $2\sqrt{ab} > c$ ie state $f\left(\sqrt{\frac{b}{a}}\right) > c$