Student Number: ____________________

Teacher: ____________________

St Catherine's School

Waverley, Sydney

Year 11 Extension 1 Mathematics

End of Preliminary Examination

Task #3

September 2007

Time allowed: *1.5 hours*
Reading time: *5 minutes*

INSTRUCTIONS

- There are 5 questions of equal value.
- Marks for each part of a question are indicated.
- All questions should be attempted.
- All necessary working should be shown.
- Start each question on a new page.

Approved scientific calculators and drawing templates may be used.

Question 1 **Marks**

(a) Solve $\frac{2x-3}{x} \geq -1$ 3

(b) $A(-4,3)$ and $B(8,-5)$ are two points. Find the coordinates of the point P that divides the interval AB internally in the ratio $3:2$ 2

(c) **(i)** Show that the gradients of the tangents to the curves $y=\frac{4}{x}$ and $y=\frac{\sqrt{x}}{2}$ at the point $P(4,1)$ are $-\frac{1}{4}$ and $\frac{1}{8}$ respectively. 2

(ii) Find the acute angle between these two tangents, giving the answer correct to the nearest degree. 2

(d) Evaluate $\sum_{k=1}^{30}(4k+1)$ 3

Question 2 ***(START A NEW PAGE)*** **Marks**

(a) Find the derivative of the following functions in its simplest form:

(i) $f(x)=3x^2-2x+5$ 1

(ii) $y=\frac{-2}{x^4}$ 2

(iii) $f(x)=(x-2)(4x+3)^3$ 3

(iv) $y=\frac{x+1}{\sqrt{x}}$ 3

(b) If $1,(m+2),(m+2)^2,(m+2)^3,\ldots$ is a geometric sequence and has a limiting sum, find the possible values of m.

3

Question 3 *(START A NEW PAGE)* **Marks**

(a) A parabola has its vertex at the point $(3,1)$ and its directrix has equation $y=-1$.

(i) What is the focal length? **1**

(ii) State the co-ordinates of the focus. **1**

(iii) Find the equation of the parabola. **1**

(b) $A(0,4)$ and $B(5,0)$ are two fixed points. $P(x,y)$ is a variable point which moves such that $\angle APB$ is a right angle.

(i) Show that the locus of P is the equation of the circle $x^2+y^2-5x-4y=0$ **2**

(ii) Find the centre and the radius of the circle in part (i) **2**

(c) A quadratic equation of the form $ax^2+bx+c=0$ has roots α and β such that $\alpha\beta=-6$ and $\alpha+\beta=2$.

(i) Find the value of $\alpha^2+\beta^2$ **1**

(ii) Find a possible set of values for a, b and c. **2**

(d) Find the domain of the function $f(x)=\sqrt{2-\sqrt{x}}$ **2**

Question 4 *(START A NEW PAGE)* **Marks**

(a) Find the exact value of $2\sin 15^\circ \cos 15^\circ$ **1**

(b) By firstly expanding $\cos(\theta+\alpha)$, find an expression for $\cos 2\theta$ **1**

(c) **(i)** If $0<\theta<90^\circ$, prove that

$$\sqrt{\frac{1-\cos 2\theta}{1+\cos 2\theta}}=\tan\theta$$ **2**

(ii) Hence, show that the exact value of $\tan 22\frac{1}{2}^\circ=\sqrt{2}-1$ **2**

(d) Find the values of k for which the equation $x^2-(k+3)x+4k=0$ has equal roots. **3**

(e) Find the equation of the normal to the curve $y=(2x^2-1)^2$ at the point $A(1,-1)$. **3**

Question 5 *(START A NEW PAGE)* **Marks**

(a) Prove by mathematical induction that for $n \geq 1$,

$$1^2 + 3^2 + 5^2 + ... + (2n-1)^2 = \frac{1}{3}n(2n-1)(2n+1)$$ **4**

(b) Consider the function $y = \dfrac{2x^2+1}{x^2-1}$.

(i) Write down the equations of the vertical asymptotes of this function. **1**

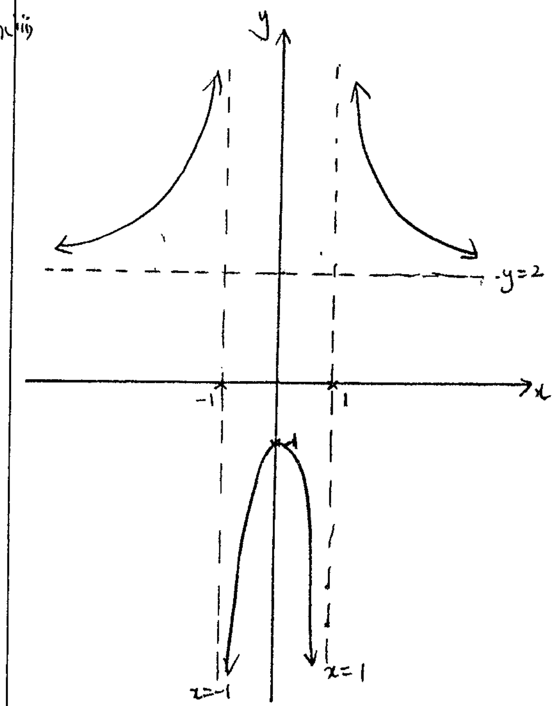
(ii) Find the horizontal asymptote of this function. **1**

(iii) Show that this is an even function and draw a neat sketch clearly showing all intercepts and asymptotes. **3**

(c) Find the value of the first term of the series $20 + 18 + 16 + ...$ that is less than 1. **3**

END OF EXAMINATON PAPER

Qn	Solutions	Marks	Comments+Criteria
5(b)(iii)	$y = \frac{2x^2+1}{x^2-1}$ For a function to be even $f(a) = f(-a)$ i.e. $f(a) = \frac{2a^2+1}{a^2-1}$ $f(-a) = \frac{2(-a)^2+1}{(-a)^2-1}$ $= \frac{2a^2+1}{a^2-1}$ $f(a) = f(-a) = \frac{2a^2+1}{a^2-1}$ $\therefore y = \frac{2x^2+1}{x^2-1}$ is an even function	1	even function
	y-int $\rightarrow$ let $x=0$ $y = \frac{2(0)^2+1}{0^2-1} = \frac{1}{-1} = -1$ x-int $\rightarrow$ let $y=0$ i.e. $0 = \frac{2x^2+1}{x^2-1}$ $2x^2 = -1$ $x^2 = -\frac{1}{2}$ no solutions $\therefore$ no x-int.	$\frac{1}{2}$	Intercepts

Qn	Solutions	Marks	Comm.
5(b)(iii)		$1\frac{1}{2}$	$-\frac{1}{2}$ for sloppy production of graph eg. graph must approach asymptote not sketch
(c)	$20 + 18 + 16 + \cdots$ $\boxed{a = 20}$ $d = 18 - 20$ $\boxed{d = -2}$ $n = ?$ $T_n < 1$ $a + (n-1)d < 1$ $20 + (n-1)\times -2 < 1$ $20 - 2n + 2 < 1$ $22 - 2n < 1$ $-2n < 1 - 22$ $-2n < -21$ $\therefore n > \frac{21}{2}$ $n > 10\frac{1}{2}$ $\therefore$ The 11th term is the first term to be less than 1 and it is $T_{11} = 20 + 10\times -2$ $= 20 + -20$ $= 0$	1 2	Correct a, d calculation $-\frac{1}{2}$ for not calculating value of 11th term.

Qn	Solutions	Marks	Comments+Criteria
	$= \frac{1}{3}(2k+1)\left[k(2k-1) + 3(2k+1)\right]$		
	$= \frac{1}{3}(2k+1)(2k^2 - k + 6k + 3)$		
	$= \frac{1}{3}(2k+1)(2k^2+5k+3)$		
	$= \frac{1}{3}(2k+1)(2k+3)(k+1)$		
	$=$ RHS		
	$\therefore$ The statement is true for $n=k+1$		
	Step 4: Stat. is true for $n=k+1$ if it is true for $n=k$ and since it is true for $n=1$, $\therefore$ it is true for $n=2,3,$ etc. $\therefore$ Statement is true for $n \geq 1$	½	Step 4
	OR step 1 + step 2 and step 4 are exactly the same. however step 3, can also be proven by expanding the LHS and expanding the RHS which proves LHS = RHS.		Students need to be aware that this is showing not proving! A Prove question must take one side of the statement and prove that it equals the other

Qn	Solutions	Marks	Comments+Criteria
5 (b)	$y = \frac{2x^2+1}{x^2-1}$		
	(i) $x^2 - 1 \neq 0$ $x^2 \neq 1$ $x \neq \pm\sqrt{1}$ $x \neq \pm 1$ $\therefore$ vertical asymptote equations are $x = 1$ and $x = -1$	1	many are writing $x^2-1=0$, $x^2=1$, $x=\pm 1$ accepted this time but they must realise $x^2-1 \neq 0$ Vertical asymptotes
	(ii) $\lim_{x \to \infty} \frac{2x^2+1}{x^2-1}$ $= \lim_{x \to \infty} \frac{\frac{2x^2}{x^2} + \frac{1}{x^2}}{\frac{x^2}{x^2} - \frac{1}{x^2}}$ $= \lim_{x \to \infty} \frac{2 + \frac{1}{x^2}}{1 - \frac{1}{x^2}}$ $= 2$ $\therefore$ Horizontal asymptote has equation $y = 2$	1	Horizontal asymptote

Qn	Solutions	Marks	Comments+Criteria
4(f)	$y+1=-\frac{1}{8}(x-1)$ $8y+8=-x+1$ $x+8y+7=0$ is the equation of the normal	1	equation of normal

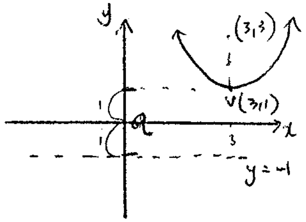
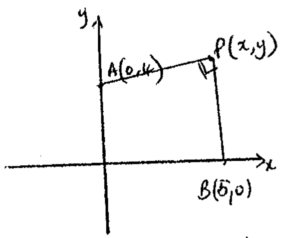
Qn	Solutions	Marks	Comments+Criteria
5(a)	Step 1: Prove that statement is true for $n=1$ $LHS=(2n-1)^2$ $=(2\times1-1)^2$ $=1^2$ $=1$ $RHS=\frac{1}{3}n(2n-1)(2n+1)$ $=\frac{1}{3}\times1(2\times1-1)(2\times1+1)$ $=\frac{1}{3}(1)(3)$ $=1$ $LHS=RHS=1$ $\therefore$ Statement is true for $n=1$	$\frac{1}{2}$	Step 1
	Step 2: Assume that statement is true for $n=k$ ie $1^2+3^2+5^2+\dots+(2k-1)^2=\frac{1}{3}k(2k-1)(2k+1)$	$\frac{1}{2}$	Step 2
	Step 3. Prove that statement is true for $n=k+1$ $1^2+3^2+5^2+\dots+(2k-1)^2+(2(k+1)-1)^2=\frac{1}{3}(k+1)(2(k+1)-1)(2(k+1)+1)$ $\underbrace{1^2+3^2+5^2+\dots+(2k-1)^2}+(2k+1)^2=\frac{(k+1)(2k+1)(2k+3)}{3}$ $\frac{1}{3}k(2k-1)(2k+1)+(2k+1)^2=\frac{(k+1)(2k+1)(2k+3)}{3}$ $LHS=\frac{1}{3}k(2k-1)(2k+1)+(2k+1)^2$ $=\frac{1}{3}k(2k-1)(2k+1)+\frac{1}{3}\times3(2k+1)^2$	$2\frac{1}{2}$	Step 3

Qn	Solutions	Marks	Comments+Criteria
4 c) iii)	$= \sqrt{\dfrac{\frac{\sqrt{2}-1}{\sqrt{2}}}{\frac{\sqrt{2}+1}{\sqrt{2}}}}$		1 to here
	$= \sqrt{\dfrac{\sqrt{2}-1}{\sqrt{2}} \times \dfrac{\sqrt{2}}{\sqrt{2}+1}}$		
	$= \sqrt{\dfrac{\sqrt{2}-1}{\sqrt{2}+1}}$		
	$= \sqrt{\dfrac{\sqrt{2}-1}{\sqrt{2}+1} \times \dfrac{\sqrt{2}-1}{\sqrt{2}-1}}$		$1\frac{1}{2}$ to here rationalising denominator
	$= \sqrt{\dfrac{(\sqrt{2}-1)^2}{2-1}}$		
	$= \sqrt{\dfrac{(\sqrt{2}-1)^2}{1}}$		
	$= \sqrt{(\sqrt{2}-1)^2}$		
	$= \sqrt{2}-1$		2 to here

Qn	Solutions	Marks	Comments+Criteria
4(d)	$x^2-(k+3)x+4k=0$		
	$a=1$ $b=-(k+3)$ $c=4k$		$\frac{1}{2}$ for correct a, b, c
	For an equation to have equal roots, $\Delta=0$		1 for $\Delta=0$
	$\Delta = b^2-4ac$		
	$= [-(k+3)]^2 - 4\times 1\times 4k$		
	$= (k+3)^2 - 16k$		$1\frac{1}{2}$ for solution
	$= k^2+6k+9-16k$		
	$= k^2-10k+9=0$		
	$(k-9)(k-1)=0$		
	$\therefore k=9, 1$		
4(e)	$y=(2x^2-1)^2$		
	$y' = 2(2x^2-1)\times 4x$ $= 8x(2x^2-1)$	1	Correct derivative
	at $x=1$ $y' = 8(2-1)$ $y' = 8$ (gradient of tangent)		
	$\therefore$ gradient of normal $= -\frac{1}{8}$	1	gradient of normal
	$y-y_1 = m(x-x_1)$		
	$y--1 = -\frac{1}{8}(x-1)$		

Qn	Solutions	Marks	Comments+Criteria
3(d)	$f(x) = \sqrt{2-\sqrt{x}}$ For $2-\sqrt{x}$: For $\sqrt{x}$: $2-\sqrt{x} \geq 0$ and $\boxed{x \geq 0}$ $-\sqrt{x} \geq -2$ $(\sqrt{x})^2 \leq (2)^2$ $\boxed{x \leq 4}$ $\therefore D: 0 \leq x \leq 4$	2	1 mark for solving $2-\sqrt{x} \geq 0$ and 1 mark for combining the two solutions to the final answer OR 1 mark for $0 \leq x$ 1 mark for $x \leq 4$

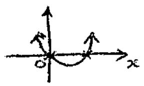
Qn	Solutions	Marks	Comments+Criteria
4(a)	$2\sin 15^\circ \cos 15^\circ$ $= \sin 2\times 15^\circ$ $= \sin 30^\circ$ $= \frac{1}{2}$	1	
(b)	$\cos(\theta+\alpha) = \cos\theta\cos\alpha - \sin\theta\sin\alpha$ $\therefore \cos 2\theta = \cos(\theta+\theta)$ $= \cos\theta\cos\theta - \sin\theta\sin\theta$ $= \cos^2\theta - \sin^2\theta$ or $= 1-2\sin^2\theta$ or $= 2\cos^2\theta - 1$	1	
(c) (i)	$LHS = \sqrt{\frac{1-\cos 2\theta}{1+\cos 2\theta}}$ $= \sqrt{\frac{1-(1-2\sin^2\theta)}{1+(2\cos^2\theta-1)}}$ $= \sqrt{\frac{\cancel{2}\sin^2\theta}{\cancel{2}\cos^2\theta}}$ $= \sqrt{\frac{\sin^2\theta}{\cos^2\theta}} = \frac{\sin\theta}{\cos\theta} = \tan\theta = RHS$	2	1 or $\cos 2\theta = \cos^2\theta -$ 1
(ii)	$\tan 22\frac{1}{2}^\circ = \sqrt{\frac{1-\cos 2\times 22\frac{1}{2}^\circ}{1+\cos 2\times 22\frac{1}{2}^\circ}}$ $= \sqrt{\frac{1-\cos 45^\circ}{1+\cos 45^\circ}}$ $= \sqrt{\frac{1-1/\sqrt{2}}{1+1/\sqrt{2}}}$	2	$\frac{1}{2}$ for $\theta = 22\frac{1}{2}$

Qn	Solutions	Marks	Comments+Criteria
3(a)			
(i)	The focal length 'a' = 2 units	1	
(ii)	Focus $(3,3)$	1	
(iii)	$(x-h)^2 = 4a(y-k)$ $(x-3)^2 = 4\times 2(y-1)$ $(x-3)^2 = 8(y-1)$ * $x^2-6x+9 = 8y-8$ $x^2-6x-8y+17=0$	1	1 mark to *
(b) i)	 $\angle APB$ is a right angle $\therefore m_{AP}\times m_{PB} = -1$ $\left(\frac{y-4}{x-0}\right)\times\left(\frac{y-0}{x-5}\right) = -1$ $\frac{y-4}{x}\times\frac{y}{x-5} = -1$ (*) $y(y-4) = -x(x-5)$ $y^2-4y = -x^2+5x$	2	1 mark for stating $m_1\times m_2=-1$ and showing (*) 1 mark for simplifying

Qn	Solutions	Marks	Comments+Criteria
3(b) (i)	$x^2+y^2-5x-4y=0$		
(ii)	$x^2-5x+\left(\frac{5}{2}\right)^2+y^2-4y+2^2 = 0+\left(\frac{5}{2}\right)^2+2^2$ $\left(x-\frac{5}{2}\right)^2+(y-2)^2 = \frac{25}{4}+4$ $= \frac{41}{4}$ $\therefore$ Centre $\left(\frac{5}{2}, 2\right)$ and radius $= \frac{\sqrt{41}}{2}$ units or 3.2 units	2	1 mark for completing the square correctly 1 mark for correct centre and radius
(c) (i)	$\alpha+\beta = 2$ and $\alpha\beta = -6$ $\alpha^2+\beta^2 = (\alpha+\beta)^2 - 2\alpha\beta$ * $= 2^2 - 2\times -6$ $= 4+12$ $= 16$	1	½ mark for correct line indicated (*) ½ mark for correct answer
(ii)	The quad. equ. has roots $x=\beta$ and $x=\alpha$ $\therefore (x-\beta)(x-\alpha)$ $= x^2-\alpha x-\beta x+\alpha\beta$ $= x^2-(\alpha+\beta)x+\alpha\beta$ * $= x^2-2x-6$ $\therefore$ possible values for $a=1$, $b=-2$ and $c=-6$	2	1 mark for * 1 mark for correct a, b & c values

Qn	Solutions	Marks	Comments+Criteria
2(a)(i)	$f(x) = 3x^2 - 2x + 5$ $f'(x) = 6x - 2$	1	
(ii)	$y = -\frac{2}{x^4}$ $y = -2x^{-4}$ $y' = 8x^{-5}$ $y' = \frac{8}{x^5}$	 ½ 1 ½	
(iii)	$f(x) = (x-2)(4x+3)^3$ let $u = x-2$, $v = (4x+3)^3$ $u' = 1$, $v' = 3(4x+3)^2 \times 4$ $= 12(4x+3)^2$ $f'(x) = u'v + v'u$ $= 1(4x+3)^3 + 12(4x+3)^2(x-2)$ $= (4x+3)^2\left[4x+3+12(x-2)\right]$ $= (4x+3)^2(4x+3+12x-24)$ $= (4x+3)^2(16x-21)$	3	1 mark for correct product rule and its use. 1 mark for correct differentiation. 1 mark for correct simplified answer
(iv)	$y = \frac{x+1}{\sqrt{x}}$ let $u = x+1$, $v = \sqrt{x} = x^{1/2}$ $u' = 1$, $v' = \frac{1}{2}x^{-1/2}$ $v' = \frac{1}{2x^{1/2}}$ $y' = \frac{u'v - v'u}{v^2}$ $= \frac{1 \cdot x^{1/2} - \frac{1}{2x^{1/2}} \cdot (x+1)}{(x^{1/2})^2}$	3	1 mark for correct quotient rule and its use

Qn	Solutions	Marks	Comments+Criteria
2(a)(iv)	$= \frac{x^{1/2} - \frac{x+1}{2x^{1/2}}}{x}$ $= \frac{\frac{2x-(x+1)}{2x^{1/2}}}{\frac{x}{1}}$ $= \frac{2x-x-1}{2x^{1/2}} \div \frac{x}{1}$ $= \frac{x-1}{2x^{1/2}} \times \frac{1}{x}$ $= \frac{x-1}{2x^{3/2}}$ $\therefore y' = \frac{x-1}{2\sqrt{x^3}}$		1 mark for correct differentiation 1 mark for correct simplified answer
b)	$r = \frac{m+2}{1} = m+2$ for a limiting sum to exist $\lvert r\rvert < 1$ or $-1 < r < 1$ ie. $\lvert m+2\rvert < 1$ case 1: $m+2 < 1$, $m < 1-2$, $m < -1$ case 2: $-(m+2) < 1$, $-m-2 < 1$, $-m < 3$, $m > -3$ $\therefore$ Possible values of m are $m < -1$ or $m \geq 3$ or $-3 < m < -1$ -3 -1	3	1 mark for stating the condition $\lvert r\rvert < 1$ 1 mark for each correct case (solution)

Qn	Solutions	Marks	Comments+Criteria
1(a)	$\frac{2x-3}{x} \times x^2 \geq -1 \times x^2$		
	$\frac{x^2(2x-3)}{x} \geq -x^2$		
	$x(2x-3) + x^2 \geq 0$	1	½ mark for multiplying by x^2 ½ mark for getting to *
	$2x^2 - 3x + x^2 \geq 0$		
	$3x^2 - 3x \geq 0$		
	$3x(x-1) \geq 0$ * 	½	½ mark for correct factorising and graph
	$x < 0$ or $x \geq 1$	1½	1½ mark for correct final solutions.
(b)	$m:n$ $3:2$ $x_1 = -4$ $y_1 = 3$ $x_2 = 8$ $y_2 = -5$		
	$P\left(\frac{mx_2+nx_1}{m+n}, \frac{my_2+ny_1}{m+n}\right)$	1	1 mark for correct formula and correct substitution of given x & y values.
	$P\left(\frac{3\times 8 + 2\times -4}{3+2}, \frac{3\times -5 + 2\times 3}{3+2}\right)$		
	$P\left(\frac{24-8}{5}, \frac{-15+6}{5}\right)$	1	1 mark for correct answer
	$P\left(\frac{16}{5}, \frac{-9}{5}\right)$		
(c)(i)	$y = \frac{4}{x}$		½ mark for correct derivative
	$y = 4x^{-1}$		
	$y' = -4x^{-2}$	1	½ mark for correct value of gradient of tangent
	at $x = 4$		
	$y' = -4(4)^{-2}$		
	$= \frac{-4}{16} = \frac{-1}{4}$		

Qn	Solutions	Marks	Comments+Criteria
1(c)(ii)	$y = \frac{\sqrt{x}}{2}$		
	$y = \frac{x^{1/2}}{2}$		½ mark for correct differentiation
	$y' = \frac{1}{2} \times \frac{1}{2} x^{-1/2}$	1	½ mark for correct value of gradient of tangent
	at $x = 4$		
	$y' = \frac{1}{4}(4)^{-1/2}$		
	$= \frac{1}{4} \cdot \frac{1}{4^{1/2}}$		
	$y' = \frac{1}{8}$		
(c)(iii)	$\tan\theta = \left\lvert \frac{m_1 - m_2}{1 + m_1 m_2} \right\rvert$		
	$= \left\lvert \frac{\frac{-1}{4} - \frac{1}{8}}{1 + \left(\frac{-1}{4} \times \frac{1}{8}\right)} \right\rvert$	1	½ mark for correct formula ½ mark for correct substitution
	$= \left\lvert \frac{-3/8}{31/32} \right\rvert$		
	$= \frac{12}{31}$		
	$\therefore \theta = 21^\circ\, 9'\, 40.54''$	1	½ mark for correct answer ½ mark for correct rounding to nearest degree
	$= 21^\circ$ (nearest degree)		
(d)	$\sum_{k=1}^{30} (4k+1) = (4\times 1 + 1) + (4\times 2 + 1) + (4\times 3 + 1) + \cdots + (4\times 30 + 1)$		1 mark for writing the series
	$= 5 + 9 + 13 + \cdots + 121$	1	
	AP with $d = 4$, $a = 5$, $l = 121$, $n = 30$	1	1 mark for correct formula
	$\therefore S_{30} = \frac{n}{2}(a + l)$		
	$= \frac{30}{2}(5 + 121) = 1890$	1	1 mark for correct substitution + answer