St Catherine's School

Year: 11
Subject: Mathematics Course
Time allowed: 1.5 hours
Extension 1
Date: September 2005

Student Number: 16361275

Directions to candidates:

- All questions are to be attempted.
- Marks may be deducted for careless or badly arranged work
- All necessary **working** must be shown in every question.
- The answers are to be written on the booklet provided
- Approved calculators and geometrical instruments are required.
- Hand in your work in **1 bundle**:
- Slip the question paper inside your exam booklet

Q.1	
Q.2	
Q.3	
Q.4	
Q.5	
Total	

Question 1

(a) Differentiate the following with respect to x:

(i) $y = 2x^3 + 5x^2 - x + 1$ (1m)

(ii) $y = (2x^2 - 1)^5$ (2m)

(iii) $y = \dfrac{2}{\sqrt{3x+1}}$ (2m)

(iv) $y = (x^2 + 1)^3(2x - 5)$ (3m)
(Leave the answer in factored form)

(b) Integrate the following:

(i) $\dfrac{dy}{dx} = 4x^3 - x + 1$ (1m)

(ii) $\dfrac{dy}{dx} = \dfrac{x^2 - 5x}{x}$ (1m)

(iii) $\dfrac{dy}{dx} = \dfrac{3}{(2x-1)^3}$ ✓ (2m)

Question 2

(a) Find the coordinates of the point that divides the interval AB, where A is (1,3) and B is (-4,6) externally in the ratio 3:2 (2m)

(b) Show using Mathematical Induction that

$$1^3 + 2^3 + 3^3 + \ldots\ldots + n^3 = \frac{n^2(n+1)^2}{4}$$ (4m)

(c) Find the common difference of the following arithmetic sequence

$$\log_e x, \log_e x^2, \log_e x^3, \log_e x^4, \ldots\ldots$$ (2m)

(d) How many terms of the series

3+5+7+9+.............

must be added to give a sum of 624? (4m)

Question 3.

(a) Find the equation of the tangent to the curve $y = x^3 - 12x + 1$ at x = 1 (3m)

(b) Find the values of x for which the function $y = x^3 - 12x^2 + 1$ is concave up? (3m)

(c) Consider the function $y = x^3 - 12x + 1$

(i) Find the coordinates of the stationary points and determine their nature. (3m)

(ii) Find any point(s) of inflexion (1.5m)

(iii) Sketch the function for $-3 \le x \le 3$, showing the above features (1.5m).

Question 4

(a) Find the acute angle to the nearest degree between the lines 2x-y+1=0 and $\sqrt{3}x + y - 2 = 0$ (3m)

(b)(i) On the same Number Plane sketch the graphs of $y = |x-1|$ and y=x (2m)

(ii) Hence or otherwise solve the inequality $|x-1| < x$ (2m)

(c) Susan borrows \$ 50000 from a bank. The interest is calculated monthly at the rate of 2% per month and is compounded monthly. She intends to pay back the loan with interest in equal monthly instalments. If \$M is the monthly instalment

(i) Explain why A_2 the amount she owes at the end of the second month is given by $A_2 = 50000(1.02)^2 - M(1+1.02)$ (1m)

(ii) Find the monthly instalment if the loan is to be repaid in 10 years' time. (4m)

Question 5.

(a) Explain using calculus why the point (1,0) is a horizontal point of inflexion for the curve $y = (x-1)^3$ (3m)

(b) Find $\lim_{x\to 2} \dfrac{x^2 - x - 2}{x-2}$

(3m)

(b) A rectangular prism has a square base of side length x cm and a height of y cm

(i) If the surface area of the prism is 216 cm^2, show that $y = \dfrac{54}{x} - \dfrac{x}{2}$ (2m)

(ii) If V is the volume of the prism, show that $V = 54x - \dfrac{x^3}{2}$ (1m)

(iii) Hence find the maximum volume

End of Paper

1

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1 (a) $y = 2x^3 + 5x^2 - x + 1$

$y' = 6x^2 + 10x - 1$ ✓ 1

(ii) $y = (2x^2 - 1)^5$

$y' = 5(2x^2-1)^4(4x)$ ✓

$= 20x(2x^2-1)^4$ ✓ 2

(iii) $y = \frac{2}{\sqrt{3x+1}}$

$y = 2(3x+1)^{-\frac{1}{2}}$ ✓

$y' = 2(-\frac{1}{2})(3x+1)^{-\frac{3}{2}}(3)$ ✓ 2

$= \frac{-3}{\sqrt{(3x+1)^3}}$

(iv) $y = (x^2+1)^3(2x-5)$

$y' = \underset{u}{(x^2+1)^3}\,\underset{v'}{(2)} + \underset{v}{(2x-5)}\,.\,\underset{u'}{3(x^2+1)^2(2x)}$ 2

$= 2(x^2+1)^2(x^2+1 + (2x-5).3.x)$ $\frac{1}{2}$

$= 2(x^2+1)^2(7x^2 - 15x + 1)$ $\frac{1}{2}$ 3

b) $y' = 4x^3 - x + 1$

$y = x^4 - \frac{x^2}{2} + x + c$ ✓

(ii) $y' = \frac{x^2 - 5x}{x}$

$y = \frac{x^2}{2} - 5x + c$

(iii) $y = 3(2x-1)^{-3}$ ✓

$= 3\frac{(2x-1)^{-2}}{-2 \times 2}$

$= \frac{-3}{4(2x-1)^2}$ ✓

2

Q.2 $A: (1,3)$; $B: (-4,6)$

$-3 : 2$.

$P: \left(\frac{12+2}{-1}, \frac{-18+6}{-1}\right)$ 2

$P: (-14, 12)$

b) Let $P(n): 1^3 + 2^3 + 3^3 + \ldots + n^3 = \frac{n^2(n+1)^2}{4}$ 4

Consider $P(1)$

LHS : 1^3.

RHS : $\frac{1^2(1+1)^2}{4} = 1$

$\therefore$ P(1) is true (1M)

Let P(k) be true

$1^3 + 2^3 + \ldots + k^3 = \frac{k^2(k+1)^2}{4}$. (A) $(\frac{1}{2})$

Consider $P(k+1)$:

$1^3 + 2^3 + \ldots + (k+1)^3 = \frac{(k+1)^2(k+2)^2}{4}$.

$1^3 + 2^3 + \ldots + (k+1)^3$

$= \underbrace{1^3 + 2^3 + \ldots + k^3} + (k+1)^3$

$= \frac{k^2(k+1)^2}{4} + (k+1)^3$. —— (by (A).)

$= (k+1)^2\left(\frac{k^2}{4} + (k+1)\right)$

$= \frac{(k+1)^2}{4}(k^2+4k+4)$

$= \frac{(k+1)^2 (k+2)^2}{4}$

$\therefore$ P(k+1) is true. if P(k) is true (2M)

Thus P(1) is true

& P(k+1) is true if P(k) is true.

$\therefore$ By the principle of Mathematical Induction P(n) is true for all $n \geq 1$. ($\frac{1}{2}$M)

c). $\log_e x, \log_e x^2, \log_e x^3, \ldots$

$\log_e x, 2\log_e x, 3\log_e x, \ldots$

2

arithmetic sequence with $\log_e x$ as the common difference.

d). $3+5+7+9+\ldots$

Arithmetic Series; $a = 3$

$d = 2$

$S_n = 624$

4

if n is the number of terms;

$\frac{n}{2}(6+(n-1).2) = 624$ ✓

$\frac{n}{2}(2n+4) = 624$

$n(n+2) = 624$

$n^2+2n-624 = 0$

$(n+26)(n-24) = 0$ 1.

$n \neq -26$ $n = 24$ $\frac{1}{2}$

$\therefore$ 24 terms are needed.

Q.3

a) $y = x^3 - 12x + 1$

$y' = 3x^2 - 12$

$y'_{at\ x=1} = 3-12$

$= -9$ ✓

at $x = 1$; $y = 1^3 - 12 + 1$

$= -10$ ✓

tgr. (1, −10)

grad. is −9.

$\therefore$ Equ of the tgr. is

$y + 10 = -9(x-1)$ ✓

or $9x + y + 1 = 0$.

b) $y = x^3 - 12x^2 + 1$

$y' = 3x^2 - 24x$

$y'' = \ldots$ ✓

Concave up when $y'' > 0$ ✓

$6x - 24 > 0$ ✓

$x > 4$ ✓

c) $y = x^3 - 12x + 1$

$y' = 3x^2 - 12$
$= 3(x^2 - 4)$
$= 3(x-2)(x+2)$

$y'' = 6x$

At stationary points; $y' = 0$
$(x-2)(x+2) = 0$

$x = -2$ or $x = 2$
$y = 17$ $\quad$ $y = -15$

$\therefore (-2, 17)$ and $(2, -15)$ are stationary pts

<u>nature</u> $\quad y'' = 6x$

$y''_{\text{at } x=-2} < 0 \quad \therefore$ <u>$(-2, 17)$ is a max turning pt.</u>

$y''_{\text{at } x=2} > 0 \quad \therefore$ <u>$(2, -15)$ is a minimum turning pt</u>

(11) At a pt. of inflexion
$y'' = 0$
$6x = 0$
$x = 0$
$y = 1$

$\therefore (0, 1)$ is a possible pt. of inflexion.

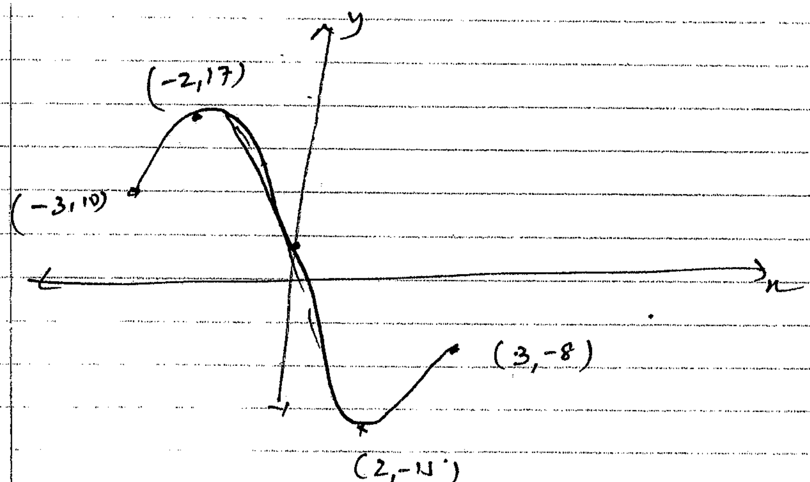
$y''_{\text{at } 0^-} < 0$ $\quad$ There is change in Concavity at $x = 0$

<u>$x = -3$</u>

$y = (-3)^3 - 12(-3) + 1$
$= -27 + 36 + 1$
$= 10$

<u>$x = 3$</u>

$y = (3)^3 - 12(3) + 1$
$= -8$.



Q.4

$2x - y + 1 = 0$ $\qquad$ $\sqrt{3}x + y - 2 = 0$

$m_1 = 2$ $\qquad$ $m_2 = -\sqrt{3}$

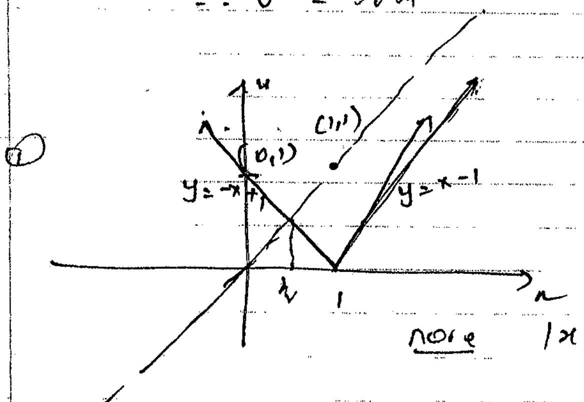
if θ is the acute angle between the given lines,

$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$$

$$= \left|\frac{2+\sqrt{3}}{1-2\sqrt{3}}\right|.$$

$\therefore \theta = 56^\circ 34'$

b)



note: labelling points is part of graphing curves.

note $|x-1| = x-1 \quad x \geq 1$

$= -x+1 \quad x < 1$

$y = x$ is parallel to $y = x-1$
and $y = x$ meets $y = -x+1$

$x = -x + 1$

$2x = 1$

$x = \frac{1}{2}, \; y = \frac{1}{2}.$

$(\frac{1}{2}, \frac{1}{2})$ is the point of intersection.

Thus from the figure; $|x-1| < x$, when

$x > \frac{1}{2}$

c) Let A_n be the amount owed at the end of n months.

(i) $A_1 = 50000(1.02) - M.$

$A_2 = A_1(1.02) - M$

$= 50000(1.02)^2 - M(1.02) - M.$

$= 50000(1.02)^2 - M(1 + 1.02)$

(ii) Loan is payed off in 10 years' time $\therefore A_{120} = 0$

$(\Rightarrow) A_{120} = 50000(1.02)^{120} - M(1 + 1.02 + \cdots + 1.02^{119})$

$\therefore M(1 + 1.02 + \cdots + 1.02^{119}) = 50000(1.02)^{120}$

g.s;

$M. \frac{1(1.02^{120} - 1)}{0.02} = 50000(1.02)^{120}$

$\therefore M = 50000(1.02)^{120} \times \frac{0.02}{(1.02^{120} - 1)}$

$= \$1102.40$

9.5 $\cdot\ y = (x-1)^3$

$y' = 3(x-1)^2$

$y'' = 6(x-1)$

at $x=1$; $y'=0$ and $y''=0$.

also at $x=1^-$; $y''<0$

and at $x=1^+$; $y''>0$

there is change in concavity at $x=1$

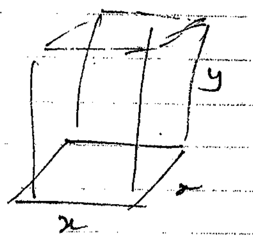
$\therefore$ (1,0) is a horizontal pt of inflexion

b) $\lim_{x\to 2} \frac{x^2-x-2}{x-2}$

$= \lim_{x\to 2} \frac{(x-2)(x+1)}{x-2}$

$= \lim_{x\to 2} (x+1) \qquad x\neq 2$

$= 3$.

b) 

$S.A = 2x^2+4xy$.

$\therefore 2x^2+4xy = 216$

$x^2+2xy = 108$

$2xy = 108-x^2$

$y = \frac{108}{2x} - \frac{x^2}{2x}$

$= \frac{54}{x} - \frac{x}{2}$.

(ii) $V = x^2y$

$= x^2\left(\frac{54}{x} - \frac{x}{2}\right)$

$= 54x - \frac{x^3}{2}$

For max vol; $\frac{dv}{dx} = 0$ and $\frac{d^2v}{dx^2} < 0$

$\frac{dv}{dx} = 54 - \frac{3x^2}{2}$

$\frac{dv}{dx} = 0$

$54 - \frac{3x^2}{2} = 0$

$3x^2 = 108$

$x^2 = 36$

$x = 6$.

($x\neq -6$; x is a length)

~~at x = 6;~~

$\frac{d^2v}{dx^2} = -3x$.

at $x=6$; $\frac{d^2v}{dx^2} < 0$.

Thus vol. is max. when $x=6$

$y = \frac{54}{6} - \frac{6}{2} = 9-3 = 6$.

$V = x^2y$

$= 6^2 \times 6$

$= 216$.

Thus max. vol. is 216cm^3.