Student Name:________________________

St. Catherine's School
Waverley

27th August 2009
PRELIMINARY ASSESSMENT TASK 3

Weighting 20%

Extension I Mathematics

Time allowed: *55 minutes*
Total marks: *46 marks*

INSTRUCTIONS

- Marks for each part of a question are indicated
- All questions should be attempted in the booklet provided
- All necessary working should be shown
- Start each question on a new page
- Approved scientific calculators and drawing templates may be used
- Marks may be deducted for badly arranged work.
- Diagrams should be drawn using PENCIL AND RULER

Question 1 **Marks**

Differentiate (leaving answers in simplest, factorised form):

(i) $y=\dfrac{3}{5x^2}$ 1

(ii) $y=\dfrac{5x^4-3x^2+2}{x}$ 2

(iii) $f(x)=\sqrt[3]{x^4}+3$ 2

(iv) $y=\dfrac{x^2}{1-x}$ 3

(v) $y=x\sqrt{3x-1}$ 3

Question 2 ***Start a new page***

(i) Find the equation of the tangent to the curve $y=2x^2-3x+1$ at the point (-1, 6). 2

(ii) Find the gradient of the normal to the curve $y=2\sqrt{x-2}$ at the point where $x=6$. 3

Question 3 ***Start a new page***

A can of coke is in the shape of a closed cylinder with height h cm and radius r cm.

(i) The volume of the can is 500 cm^3. Find an expression for h in terms of r. 2

(ii) Show that the surface area, S cm^2, of the can is given by

$$S=2\pi r^2+\frac{1000}{r}.$$ 2

(iii) If the area of metal used to make the can is to be minimized, find the radius of the can. 4

Question 4 *Start a new booklet* **Marks**

Use mathematical induction to prove that, for $n > 0$,

$$1 + 3 + 3^2 + \ldots + 3^{n-1} = \frac{1}{2}(3^n - 1).$$ 5

Question 5 *Start a new page*

A sales team sells 1200 calculators in its first month of operation. They plan to increase their sales by 150 calculators each month. How many calculators do they plan to sell:

(i) in the last month of the second year of operation; 2

(ii) over the entire two-year period? 2

Question 6 *Start a new page*

A ball is dropped from a height of 2 metres onto a hard floor and bounces. After each bounce, the maximum height reached by the ball is 75% of the previous maximum height. Thus, after it first hits the floor, it reaches a height of 1.5 metres before failing again, and after the second bounce, it reaches a height of 1.125 metres before falling again.

(i) What is the maximum height reached after the third bounce? 2

(ii) What kind of sequence is formed by the successive maximum heights? 1

(iii) What is the total distance travelled by the ball from the time it was first dropped until it eventually comes to rest on the floor? 3

Question 7 *Start a new page*

For the function $y = \dfrac{x^2 + 3}{x(x-3)}$,

(i) Find any stationary points and determine their nature. 3

(ii) Sketch the graph of this function showing any vertical and/horizontal asymptotes. 2

(iii) State the domain and range. 2

End of test

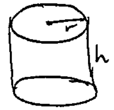
Year 11 [illegible]

Qn	Solutions	Marks	Comments: Criteria
1	(i) $y = \dfrac{3}{5x^2}$ $y = \dfrac{3}{5} \cdot \dfrac{1}{x^2}$ $y = \dfrac{3}{5} \cdot x^{-2}$ $\dfrac{dy}{dx} = \dfrac{-6}{5}x^{-3}$ $= \dfrac{-6}{5x^3}$	1	
	(ii) $y = \dfrac{5x^4 - 3x^2 + 2}{x}$ $y = \dfrac{5x^4}{x} - \dfrac{3x^2}{x} + \dfrac{2}{x}$ $y = 5x^3 - 3x + 2x^{-1}$ * $\dfrac{dy}{dx} = 15x^2 - 3 - 2x^{-2}$ $= 15x^2 - 3 - \dfrac{2}{x^2}$ ●	2	0.5 for * 1 for correct derivative 0.5 for correct last line ●
	(iii) $f(x) = \sqrt[3]{x^4} + 3$ $= x^{\frac{4}{3}} + 3$ * $f'(x) = \dfrac{4}{3}x^{\frac{1}{3}} + 0$ $= \dfrac{4\sqrt[3]{x}}{3}$	2	0.5 for * 1 correct derivative 0.5 for correct $\left(\dfrac{4\sqrt[3]{x}}{3}\right)$ $-\frac{1}{2}$ if not written in correct form ie. $\dfrac{4\sqrt[3]{x}}{3}$

Q1 continued

Qn	Solutions	Marks	Comments: Criteria
1	(iv) $y = \dfrac{x^2}{1-x}$ let $u = x^2$, $v = 1-x$ $u' = 2x$, $v' = -1$ $y' = \dfrac{u'v - v'u}{v^2}$ $= \dfrac{2x(1-x) - -1(x^2)}{(1-x)^2}$ $= \dfrac{2x - 2x^2 + x^2}{(1-x)^2}$ * $= \dfrac{2x - x^2}{(1-x)^2}$ ● $= \dfrac{x(2-x)}{(1-x)^2}$	3	− 0.5 if not simplified from * to ●
	(v) $y = x\sqrt{3x-1}$ let $u = x$, $v = \sqrt{3x-1}$ $u' = 1$, $v = (3x-1)^{\frac{1}{2}}$ $v' = \frac{1}{2}(3x-1)^{-\frac{1}{2}} \times 3$ $v' = \dfrac{3(3x-1)^{-\frac{1}{2}}}{2}$ $y' = u'v + v'u$ $= 1 \cdot (3x-1)^{\frac{1}{2}} + \dfrac{3(3x-1)^{-\frac{1}{2}}}{2} \cdot x$ $= (3x-1)^{-\frac{1}{2}}\left[(3x-1)^1 + \dfrac{3x}{2}\right]$ $= (3x-1)^{-\frac{1}{2}}\left[3x - 1 + \dfrac{3x}{2}\right]$ $= (3x-1)^{-\frac{1}{2}}\left(\dfrac{6x - 2 + 3x}{2}\right)$	3	

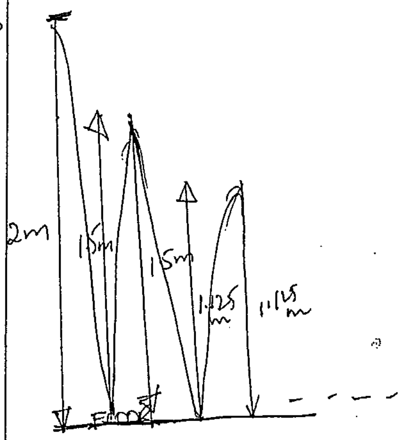
Qn	Solutions	Marks	Comments: Criteria
1	(iv) $=(3x-1)^{-1/2}\left(\frac{9x-2}{2}\right)$ $y' = \frac{(9x-2)}{2\sqrt{3x-1}}$		−0.5 if not simplified to
2	(i) $y = 2x^2-3x+1$ $y' = 4x-3$ at $x=-1$ $y' = -4-3$ $y' = -7$ (gradient of tangent $y-y_1 = m(x-x_1)$ $y-6 = -7(x--1)$ $y-6 = -7(x+1)$ $y-6 = -7x-7$ $7x+y+1=0$ is the equation of the tangent.	2	0.5 for derivative 0.5 for gradient 1 for correct equation
	(ii) $y = 2\sqrt{x-2}$ $y = 2(x-2)^{1/2}$ $y' = 2\times\frac{1}{2}(x-2)^{-1/2}$ $y' = (x-2)^{-1/2} = \frac{1}{\sqrt{x-2}}$ at $x=6$ $y' = \frac{1}{\sqrt{6-2}}$ $y' = \frac{1}{\sqrt{4}}$ $y' = \frac{1}{2}$ (gradient of tangent) ∴ gradient of normal $= -2$	3	1 mark for correct derivative 1 mark for gradient of tangent 1 mark for gradient of normal.

Qn	Solutions	Marks	Comments: Criteria
3	(i)  $V = \pi r^2 h = 500$ ✓ $h = \frac{500}{\pi r^2}$ ✓ — (1)	2	
	(ii) Surface area $= 2\pi r^2 + 2\pi r h$ ✗ $= 2\pi r^2 + 2\pi r\left(\frac{500}{\pi r^2}\right)$ from (1) $S = 2\pi r^2 + \frac{1000}{r}$ ✗ $S = 2\pi r^2 + 1000r^{-1}$	2	✓
	(iii) $\frac{dS}{dr} = 4\pi r - 1000r^{-2}$ $= 4\pi r - \frac{1000}{r^2}$ ✓ $\frac{dS}{dr} = 0$ ✗ i.e. $4\pi r - \frac{1000}{r^2} = 0$ $4\pi r^3 - 1000 = 0$ $4\pi r^3 = 1000$ $r^3 = \frac{1000}{4\pi}$ $r^3 = 79.577\ldots$ $r = 4.301270\ldots$ ✓ $= 4.3$ (to 1 d.p.) $\frac{d^2S}{dr^2} = 4\pi + 2000r^{-3}$ ✗ $= 4\pi + \frac{2000}{r^3}$ ✓ when $r = 6.8278\ldots$, $\frac{d^2S}{dr^2} > 0$ ∴ min.	4	✓ means 1 mark ✗ means 0.5 mark

Qn	Solutions	Marks	Comments: Criteria
Q4	Step 1: Prove that statement is true for $n=1$ $LHS = 3^{n-1} = 3^{1-1} = 3^0 = 1$ $RHS = \frac{1}{2}(3^n - 1) = \frac{1}{2}(3^1-1) = \frac{1}{2}.2 = 1$ ✓ $LHS = RHS$ $\therefore$ Statement is true for $n=1$ Step 2: Assume that statement is true for $n=k$ ie. $1+3+3^2+\dots+3^{k-1} = \frac{1}{2}(3^k-1)$ Step 3: Prove that statement is true for $n=k+1$ ie. $1+3+3^2+\dots+3^{k-1}+3^{k+1-1} = \frac{1}{2}(3^{k+1}-1)$ $1+3+3^2+\dots+3^{k-1}+3^k = \frac{1}{2}(3^{k+1}-1)$ $LHS = \frac{1}{2}(3^k-1)+3^k$ ✓ $= \frac{1}{2}.3^k - \frac{1}{2} + 3^k$ $= \frac{3}{2}.3^k - \frac{1}{2}$ ✓ $= \frac{3^1.3^k}{2} - \frac{1}{2}$ $= \frac{3^{k+1}}{2} - \frac{1}{2}$ ✓ $= \frac{1}{2}(3^{k+1}-1) = RHS$	4	✓ = 1 mark.

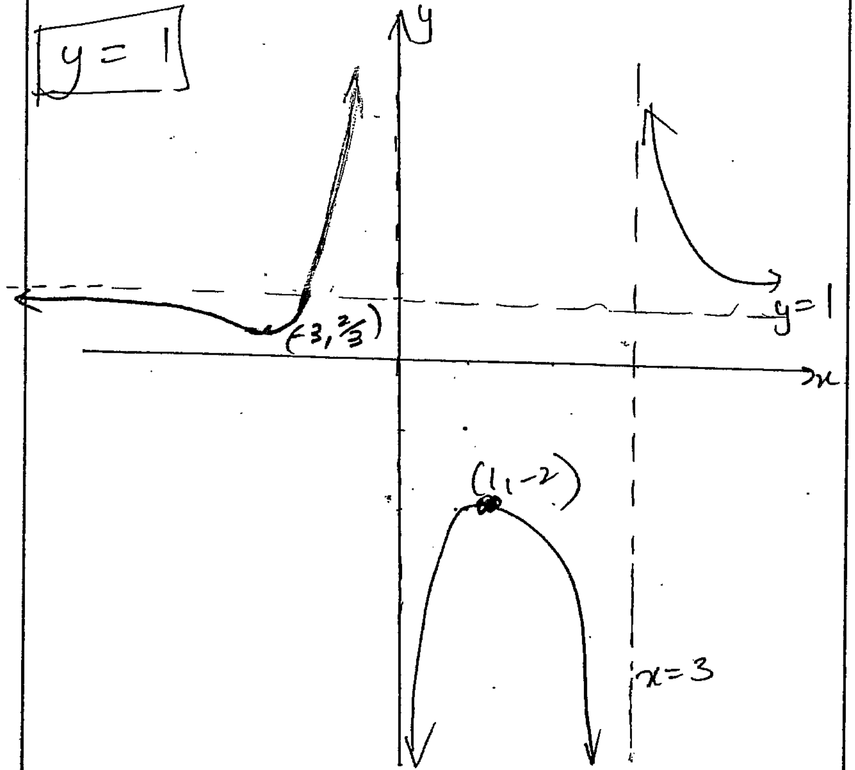
Q4 continued

Qn	Solutions	Marks	Comments: Criteria
	Step 4: If statement is true for $n=k$, then it is also true for $n=k+1$ i.e. if statement is true for $n=1$, then it is true for $n=1+1=2$ and for $n=2+1=3$ and so on --- $\therefore$ Statement is true for $n>0$		
Q5	$1200+1350+1500+1650+\dots$ (i) last month of the second year i.e. the 24^{th} month The series is an AP $\Rightarrow d = T_2 - T_1 = T_3 - T_2 = 150$ $T_n = a+(n-1)d$ ✓ $T_{24} = 1200+23\times 150 = 4650$ ✓ They plan to sell 4650 calculators in the 24^{th} month	2	✓ = 1 mark.
	(ii) $S_n = \frac{n}{2}[2a+(n-1)d]$ or $S_n = \frac{n}{2}(a+l)$ $S_{24} = \frac{24}{2}(1200+4650)$ ✓ $= 70200$ ✓ They plan to sell 70200 calculators over the entire 2-yr period.	2	$-\frac{1}{2}$ for small error in formula.

Qn	Solutions	Marks	Comments: Criteria
Q6			✓ = 1 mark.
	2m, 1.5m, 1.125m, ----		
(i)	maximum height reached after the 3rd bounce is 75% of 1.125 ✓	2	Alternate method of solution:
	$= \frac{75}{100} \times 1.125$		$T_n = ar^{n-1}$
	$= 0.84375$ ✓		$T_4 = 2(0.75)^{4-1}$
			$= 0.84375$
(ii)	The sequence formed is a Geometric progression (G.P.) ✓	1	
(iii)	There are 2 series formed:		Other methods of solution include:
	(1st) 2, 1.5, 1.125, 0.84375, --- (falling)		(i) $2\left(\frac{1.5}{1-0.75}\right) + 2$
	(2nd) 1.5, 1.125, 0.84375, ---- (rising)		$= 14\text{m}$
	So for 1st series $= \frac{a}{1-r}$	3	or
	$= \frac{2}{1-0.75}$ ✓		(ii) $2\left(\frac{2}{1-0.75}\right) - 2$
	$= 8$		$= 14\text{m}$
	So for 2nd series $= \frac{1.5}{1-0.75}$ ✓		
	$= 6$		
	∴ The total distance travelled is 14 metres	✓	

Qn	Solutions	Marks	Comments: Criteria
Q7(i)	$y = \frac{x^2+3}{x(x-3)}$		
	$y = \frac{x^2+3}{x^2-3x}$		
	let $u = x^2+3$, $u' = 2x$ | $v = x^2-3x$, $v' = 2x-3$		
	$y' = \frac{u'v - v'u}{v^2}$	2	Correct st pts
		1	Correct nature
	$= \frac{2x(x^2-3x) - (2x-3)(x^2+3)}{(x^2-3x)^2}$	1	correct assymptote
		1	correct shape graph
	$= \frac{2x^3-6x^2-2x^3-6x+3x^2+9}{(x^2-3x)^2}$	1	correct domain
		1	correct range.
	$y' = \frac{-3x^2-6x+9}{(x^2-3x)^2} = \frac{-3(x+3)(x-1)}{(x^2-3x)^2}$		
	Stationary points occur at $y'=0$,		
	i.e. $\frac{-3x^2-6x+9}{(x^2-3x)^2} = 0$		
	$\frac{-3x^2-6x+9}{-3} = 0$		
	$x^2+2x-3=0$		
	$(x+3)(x-1)=0$		
	$x=-3, \quad x=1$		
	Test for nature: $x=-3$		
	x: -4, -3, -2; y': <0, 0, >0 ∴ $\left(-3, \frac{2}{3}\right)$ is a minimum turning point		

Q7 continued

Qn	Solutions	Marks	Comments: Criteria
	$x = 1$		
	$\begin{array}{c\|c\|c\|c} x & 0.5 & 1 & 2 \\ \hline y & >0 & 0 & <0 \end{array}$ $(1, -2)$ is a maximum turning point.		
(ii)	vertical asymptotes $x(x-3)=0$, $\boxed{x=0}$, $x-3=0$, $\boxed{x=3}$		
	horizontal asymptotes $\lim_{x\to\infty} \frac{x^2+3}{x^2-3x}$ $= \lim_{x\to\infty} \frac{\frac{x^2}{x^2}+\frac{3}{x^2}}{\frac{x^2}{x^2}-\frac{3x}{x^2}}$ $= \lim_{x\to\infty} \frac{1-0}{1-0}$		
	$\boxed{y=1}$		
			
	D: For all real x, $x \neq 0, 3$		
	R: $y \geq \frac{2}{3}$, $y \leq -2$		