check marks for.
2a, 7b(ii)

St Catherine's School
Year 12
Extension One Mathematics

Time allowed: Two hours
(Plus 5 minutes reading time)
Date: April 2003

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STUDENT NUMBER: 13322686

DIRECTIONS TO CANDIDATES

* ALL QUESTIONS are to be attempted.
* Board-approved calculators may be used.
  A table of standard integrals is provided.

Start a NEW PAGE for each question.
Write your student number on the cover sheet of each section.
All necessary working should be shown in every question.
Marks may be deducted for careless or badly arranged work.

HAND IN YOUR WORK IN 3 BUNDLES

Bundle One: Entire Question Paper attached to Questions 1 and 2	10 + 8.5	(24 marks)
Bundle Two: Questions 3 and 4	11 + 11.5	(24 marks)
Bundle Three: Questions 5, 6 and 7	8.5 + 12 + 9.5	(36 marks)

71/84 = 85%

10 + 8½

Total marks – 84
Attempt Questions 1-7
All questions are of equal value

Answer each question on a NEW PAGE. More paper is available.

Question 1 (12 marks) Start a NEW PAGE. Marks

(a) Solve the inequality $\frac{x^2-1}{x} \geq 0$ 4

(b) Find the coordinates of the point P which divides the interval AB externally in the ratio 1:4, where A = (-2,5) and B = (7,1). 3

(c) The acute angle between the line x – 2y + 3 = 0 and the line y = mx is 45°.

(i) Show that $\left|\frac{2m-1}{m+2}\right| = 1$ 3

(ii) Find the possible values of m. 2

Question 2 (12 marks) Start a NEW PAGE.

(a) Evaluate $\lim_{x \to 0} \frac{\sin 2x}{3x}$ 1

(b) Differentiate

(i) $x^2 \cos 3x$ 2

(ii) $\log_e \tan x$ 2

(iii) $e^{\sin x}$ 2

(c) (i) Sketch the curve $y = 2\cos(x + \frac{\pi}{2})$ for $0 \leq x \leq \frac{\pi}{2}$. 1

(ii) The area between this curve and the x-axis from $x = 0$ to $x = \frac{\pi}{2}$ is rotated about the x-axis. Find the volume of the solid formed. 4

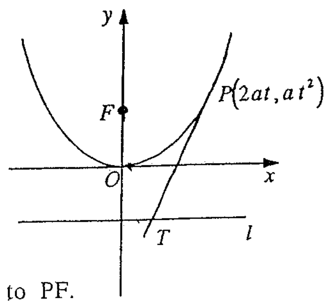
Question 3 (12 marks) Start a NEW PAGE.

(a) (i) Derive the equation of the tangent to the parabola $x = 2t$ and $y = t^2$ at the point P, where t=p. **2**

(ii) If Q is the point on the parabola where t = q, and OQ is parallel to the tangent at P (O is the origin), show that q = 2p **2**

(iii) M is the midpoint of PQ. If P and Q move along the parabola so that OQ always remains parallel to the tangent at P, show that the equation of the locus of M is $5x^2 = 18y$. **4**

(b) The tangent at the point P $(2at, at^2)$ on the parabola $x^2 = 4ay$ cuts the directrix l at T. F is the focus of the parabola.



(i) Find the coordinates of T **1**

(ii) Show that TF is perpendicular to PF. **3**

Question 4 (12 marks) Start a NEW PAGE.

(a) Find $\int_0^{\sqrt{3}} x\sqrt{1+x^2}\,dx$, using the substitution $u = 1 + x^2$. **3**

(b) Use the substitution u = cos x to find $\int \sin^3 x\,dx$ **3**

(c) By letting u = x - 1, find $\int x(x-1)^5\,dx$ **2**

(d) (i) Using the substitution $u = \tan x$, show that:

$$\int \tan^2 x \sec^2 x\,dx = \frac{\tan^3 x}{3} + c$$ **2**

(ii) Hence evaluate $\int_0^{\pi/4} \tan^2 x \sec^2 x\,dx$ **2**

Question 5 (12 marks) Start a NEW PAGE.

(a) Use the Principle of Mathematical Induction to prove that $7^n + 2$ is divisible by 3 for all integers $n \geq 1$. **5**

(b) (i) By considering the sum of the terms of an arithmetic series, show that: **2**

$$(1+2+\ldots+n)^2 = \frac{1}{4}n^2(n+1)^2$$

(ii) By using the Principle of Mathematical Induction, prove that: **5**

$$1^3 + 2^3 + \ldots + n^3 = (1+2+\ldots+n)^2 \text{ for all integers } n \geq 1$$

Question 6 (12 marks) Start a NEW PAGE.

(a) Evaluate $\int_0^{\pi/2} \cos^2 3x\,dx$ **3**

(b) (i) Show that $\sin x \cos x = \frac{1}{2}\sin 2x$ **1**

(ii) Hence or otherwise find the exact value of $\int_0^{\pi/8} \sin^2 x \cos^2 x\,dx$ **4**

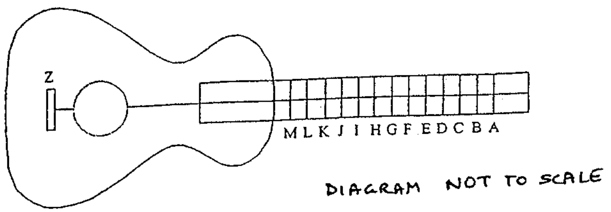
(c) Use the Table of Standard Integrals to find $\int \sec 4x \tan 4x\,dx$ **1**

(d) (i) On the same coordinate axes, draw the graphs of y = cos2x and y = sinx for $0 \leq x \leq 2\pi$. **2**

(ii) Find the number of solutions to the equation cos2x = sinx in the domain $0 \leq x \leq 2\pi$. **1**

Question 7 (12 marks) Start a NEW PAGE.

(a) On the keyboard of a guitar, the mark M is exactly halfway between A and Z. The 13 marks lettered A to M are such that their distances from Z form a Geometric Series. The length AZ is 52 cm.

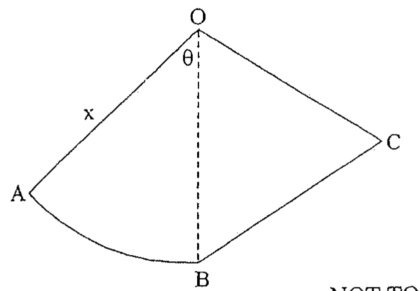


Find correct to 1 decimal place:

(i) the distance AB — 3

(ii) the distance FG. — 2

(b) The diagram shows a sector OAB of a circle, centre O and radius x metres. Arc AB subtends an angle θ radians at O. An equilateral triangle BCO adjoins the sector.



NOT TO SCALE

(i) For the figure bounded by ABCO show that the area is given by $\frac{1}{2}x^2(\theta+\frac{\sqrt{3}}{2})$ and the perimeter by $x(3+\theta)$ — 2

(ii) The perimeter of this figure is $(12-2\sqrt{3})$ metres. Show that its area in terms of θ is given by $A=\dfrac{(6-\sqrt{3})^2(2\theta+\sqrt{3})}{(\theta+3)^2}$. — 2

(iii) For what value of θ is the area A minimum? — 3

THE END!

a) $\frac{x^2-1}{x} \geq 0 \quad x \neq 0$

$x^2-1 \geq 0 \quad \cancel{A\ x>0}$

$(x+1)(x-1) \geq 0 \quad \cap\, x>0$ -1 1 Also $\boxed{-1 \leq x < 0}$

$\therefore x \leq -1, \boxed{x \geq 1}$ X $\frac{2}{4}$

b) $x = \frac{7\times1+4\times2}{3}$ $\quad y = \frac{1\times1-4\times5}{3}$

$= \frac{7+8}{3}$ $\quad = \frac{1-20}{3}$

$x = -5$ $\quad y = 6\frac{1}{3}$

$\therefore P(-5, 6\frac{1}{3})$ 3

c) $\tan\theta = \left|\frac{m_1-m_2}{1+m_1m_2}\right|$

(i) $2y = x+3$

$\theta = 45^\circ$ $\quad \therefore y = \frac{1}{2}x + 1\frac{1}{2}$

$\therefore \tan\theta = 1$

$m_1 = m$

$m_2 = \frac{1}{2}$

$\therefore \tan\theta = 1 = \left|\frac{m-\frac{1}{2}}{1+\frac{1}{2}m}\right| = \left|\frac{2m-1}{2+m}\right| \quad \therefore \left|\frac{2m-1}{m+2}\right| = 1$

3

(ii) $\frac{2m-1}{m+2} = 1$ or $-\left(\frac{2m-1}{m+2}\right) = 1$

$2m-1 = m+2$ $\quad -\frac{2m-1}{m+2} = -1$

$m = 3$ or $2m-1 = -m-2$

$3m = -1$

$\therefore m = -\frac{1}{3}$ ✓

10

$\therefore m = 3$ or $-\frac{1}{3}$

A4 — 7mm

-1-

a) $\lim_{x \to 0} \frac{\sin 2x}{3x} = \frac{2}{3} \lim_{x \to 0} \frac{\sin 2x}{2x} = \frac{2}{3} \times 1 = \frac{2}{3}$

$= \lim_{x\to 0} \cancel{\frac{2}{3} \cdot \frac{\sin x}{x}}$

$= \cancel{\frac{2}{3} \lim_{x \to 0} \frac{\sin x}{x}}$

$= \cancel{\frac{2}{3}(1)}$

$\cancel{= 2/3}$ X ?

b) i) $\frac{d}{dx} x^2 \cos 3x$ $\quad u' = 2x$, $v' = -3\sin 3x$

$= 2x\cos 3x - x^2 3\sin 3x$ $\quad$ 2

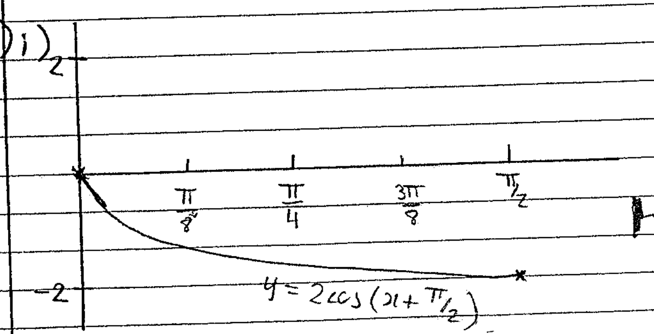
ii) $\frac{d}{dx} \log_e \tan x = \frac{f'(x)}{f(x)}$ $\quad$ 2

$= \frac{\sec^2 x}{\tan x}$

iii) $\frac{d}{dx} e^{\sin x}$ $\quad$ 2 ✓

$= \cos x\, e^{\sin x}$

c) i)



$2\cos 2x = 2\cos^2 x - 1$
$2\cos^2 x = \cos 2x +$
$\cos^2 x = \frac{1}{2}(\cos 2x +$

ii) $\pi \int_0^{\pi/2} 4\cos^2(x + \pi/2)\, dx$ ✓

$\cancel{\frac{4\pi}{2} \int_0^{\pi/2} \cos(2x + \pi)\, dx}$ X

$\cancel{= \frac{4\pi}{2} \left[\frac{1}{2}\sin(2x + \pi)\right]_0^{\pi/2}}$

$\cancel{= \frac{4\pi}{2}\left(\frac{1}{2}\sin 2\pi - \frac{1}{2}\sin \pi\right)} = \frac{4\pi}{2}(0)$

$2\cos 2x = 2\cos^2 x - 1$
$\cos^2 x = \frac{1}{2}(\cos 2x + 1)$
i.e. $\cos^2(x + \frac{\pi}{2}) = \frac{1}{2}[\cos(2x + \pi) + 1]$

$\therefore Vol = \pi \int_0^{\frac{\pi}{2}} \frac{1}{2} \cos(2x + \pi) + 1 \; dx$

$= \frac{\pi}{2}\left[\frac{\sin(2x+\pi)}{2} + x\right]_0^{\pi/2}$ $\quad$ $8\frac{1}{2}$

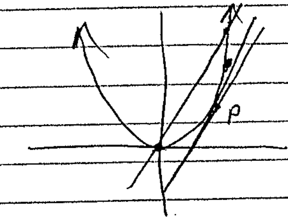
$= \frac{\pi}{2}\left[(0 + \frac{\pi}{2}) - 0\right] = \frac{\pi^2}{4}$ cu units.

A4 — 7mm

a) i) $x = 2t$, $y = t^2$ $\quad$ gradient at any point $= t = p$ ↖ show me

$y - y_1 = m(x - x_1)$
$y - p^2 = p(x - 2p)$
$y - p^2 = px - 2p^2$
$px - 2p^2 + p^2 - y = 0$

$\therefore px - y + p^2 = 0$ $\quad$ 1



ii) at q $\quad x = 2q,\ y = q^2$ $\quad$ and gradient is $\frac{y_2 - y_1}{x_2 - x_1}$

$= \frac{q^2 - 0}{2q - 0} = \frac{q^2}{2q}$

$= \frac{q}{2}$ ✓

OQ and the tangent at P are parallel

$\therefore$ gradient of both lines is equal

$\therefore \frac{q}{2} = p$

$\therefore q = 2p$ ✓ $\quad$ 2

iii) $M = \left(\frac{2p + 2q}{2}, \frac{p^2 + q^2}{2}\right) = \left(p + q, \frac{1}{2}(p^2 + q^2)\right)$ ✓

$x = p + q$ and $y = \frac{1}{2}(p^2 + q^2)$

$q = 2p$ as OQ always parallel to tangent at P.

$\therefore x = p + 2p$ and $y = \frac{1}{2}(p^2 + 4p^2)$ ✓

$= 3p$ $\quad y = \frac{1}{2}(5p^2)$ 2 ✓

$\therefore \frac{x}{3} = p$ 1 $\quad$ sub 1 into 2 $\quad y = \frac{1}{2}\left(5\left(\frac{x}{3}\right)^2\right)$

$y = \frac{5}{2}\left(\frac{x^2}{9}\right)$ $\quad$ 4

$y = \frac{5x^2}{18}$

$\therefore 5x^2 = 18y$ ✓

A4 — 7mm

b) i) $x^2 = 4ay$

$y = \frac{x^2}{4a}$

$y' = \text{gradient} = \frac{2x}{4a} = \frac{x}{2a}$

at P: gradient $= \frac{2at}{2a} = t$

$y - y_1 = m(x - x_1)$

$y - at^2 = t(x - 2at)$

$y - at^2 = tx - 2at^2$

$\therefore tx - y - at^2 = 0$

at T, $y = -a$

$\therefore tx + a - at^2 = 0$

$tx = at^2 - a$

$x = \frac{at^2 - a}{t} = \frac{at^{\cancel{2}}}{\cancel{t}} - \frac{a}{t}$

$= at - \frac{a}{t}$

$\therefore T\left(\left[at - \frac{a}{t}\right], -a\right)$ ✓ 1

ii) $m_{FT} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-a-a}{at - a/t - 0} = \frac{-2a}{at - a/t}$

$= \frac{-2at}{at^2 - a}$ ✓

3

$m_{FP} = \frac{at^2 - a}{2at - 0} = \frac{at^2 - a}{2at}$ ✓

$m_{FT} \times m_{FP} = \frac{-2at}{at^2 - a} \times \frac{at^2 - a}{2at} = -1$ ✓

$\therefore$ FT is pependicular to FP

A4 — 7mm

a) $\int_0^{\sqrt{3}} x\sqrt{1+x^2}\,dx$

$u = 1 + x^2$

$\frac{du}{dx} = 2x$

$du = 2x\,dx$

$\therefore dx = \frac{du}{2x}$

when $x = \sqrt{3}$, $u = 4$

when $x = 0$, $u = 1$ ✓

$= \int_1^4 \cancel{x}\sqrt{u}\,\frac{du}{2\cancel{x}}$

$= \frac{1}{2}\int_1^4 u^{1/2}\,du$ ✓

$= \frac{1}{2}\left[\frac{2u^{3/2}}{3}\right]_1^4 = \frac{1}{2}\left(5\frac{1}{3} - \frac{2}{3}\right)$

$= \frac{1}{2} \times 4\frac{2}{3}$ 3

$= 2\frac{1}{3}$ ✓

b) $\int \sin^3 x\,dx = \int \sin^2 x \sin x$

$u = \cos x$

$\frac{du}{dx} = -\sin x$

~~$du = -dx\sin x$~~

$dx = \frac{du}{-\sin x}$ ✓

$= \int (1 - \cos^2 x)\sin x\,dx$

$= \int (1 - u^2)\cancel{\sin x}\frac{du}{\cancel{\sin x}}$

$= -\int (1 - u^2)\,du = -\left(u - \frac{u^3}{3} + c\right)$

$= -u + \frac{u^3}{3} - c = -\cos x + \frac{\cos^3 x}{3} - c$ ✓ 3

c) $\int x(x-1)^5\,dx$

$u = x - 1$

$\frac{du}{dx} = 1$

$du = dx$

$x = u + 1$

$= \int (u+1)u^5\,du$ $1\frac{1}{2}$

$= \int u^6 + u^5\,du$

$= \frac{u^7}{7} + \frac{u^6}{6} + C$ ~~back in terms of x~~

$= \frac{(x-1)^7}{7} + \frac{(x-1)^6}{6} + C$

A4 — 7mm

d) i)

LHS $=\int \tan^2 x \sec^2 x \, dx$

$u = \tan x$

$\frac{du}{dx} = \sec^2 x$

$du = dx \sec^2 x$

$\therefore dx = \frac{du}{\sec^2 x}$

$= \int u^2 \sec^2 x \frac{du}{\sec^2 x}$

$= \int u^2 \, du$

$= \frac{u^3}{3} + C$

$= \frac{\tan^3 x}{3} + C = RHS$ 2

$\therefore \int \tan^2 x \sec^2 x \, dx = \frac{\tan^3 x}{3} + C$

ii) $\int_0^{\pi/4} \tan^2 x \sec^2 x \, dx$

$= \frac{\tan^3 \frac{\pi}{4}}{3} - 0 = \frac{1^3}{3} = \frac{1}{3}$ 2

A4 — 7mm

1. prove true for $n=1$

$7^1 + 2 = 7 + 2 = 9$ which is divisible by 3 ($9/3 = 3$)

$\therefore$ true for $n=1$

2. assume true for $n=k$

$7^k + 2 = 3M$ for some integer M.

3. prove true for $n=k+1$

$7^{k+1} + 2 = 7.7^k + 2$

$= 7.(3M-2)+2$ (from assumption)

$= 7 \times 3M - 14 + 2$

$= 7 \times 3M - 12$

$= 3(7M-4)$

$= 3N$ for some integer N

$\therefore$ true for $n=k+1$

This has been proven true for $n=1$, $n=k$, $n=k+1$ $\therefore$ for $n=2$, etc and thus for all positive integers n. needs work.

4½/5

b) i) $(1+2+\dots+n)^2$

$a=1$, $d=1$, terms $=n$ $\therefore l=n$

$S_n = \frac{n}{2}(a+l)$

$(S_n)^2 = \left(\frac{n}{2}(a+l)\right)^2$

$= \frac{n^2}{4}(a+l)^2$

2/2

$= \frac{1}{4}n^2(a+l)^2$

$= \frac{1}{4}n^2(1+n)^2$

ii) 1. prove true for $n=1$

LHS $= 1^3 = 1$, RHS $= 1^2 = 1$ $\therefore$ LHS = RHS $\therefore$ true for $n=1$

2. assume true for $n=k$

$1^3 + 2^3 + \dots + k^3 = (1+2+\dots+k)^2$

$\left[\frac{k+1}{2}(k+2)\right]^2 = \frac{(k+1)^2}{4}(k+2)^2$

4/5

3. prove true for $n=k+1$ i.e. prove $1^3+2^3+\dots+(k+1)^3 = (1+2+\dots+k+k+1)^2$

LHS $= S_k + T_{k+1} = (1+2+\dots+k)^2 + (k+1)^3$ (from assumption)

$= \left[\frac{k}{2}(1+k)\right]^2 + (k+1)^3 = (k+1)^2\left[\frac{k^2}{4} + k + 1\right]$

$= (k+1)^2\left[\frac{k^2+4k\ 4}{4}\right]$

$= \frac{(k+1)^2}{4}(k+2)^2 =$ R.H.S.

This has been proven true for $n=1$, $n=k$, $n=k+1$ $\therefore$ for $n=2$, $n=3$ etc and thus for all positive integers n.

a) $\int_0^{\pi/2} \cos^2 3x \, dx$

$\cos 2\theta = 2\cos^2\theta - 1$
$2\cos^2\theta = \cos 2\theta + 1$
$\cos^2\theta = \frac{1}{2}(\cos 2\theta + 1)$
$\therefore \cos^2 3x = \frac{1}{2}(\cos 6x + 1)$

$= \frac{1}{2}\int_0^{\pi/2} \cos 6x + 1 \, dx$

$= \frac{1}{2}\left[\frac{1}{6}\sin 6x + x\right]_0^{\pi/2}$

$= \frac{1}{2}\left[\frac{1}{6}\sin 3\pi + \pi/2 - 0\right]$

$= \frac{1}{2}\left(0 + \pi/2\right) = \frac{\pi}{4}$ ✓ $\frac{3}{3}$

b) i) RHS $= \frac{1}{2}\sin 2x = \frac{\sin 2x}{2}$

$= \frac{\sin(x+x)}{2} = \frac{\sin x \cos x + \sin x \cos x}{2}$

$= \frac{2\sin x \cos x}{2}$ ✓ $\frac{1}{1}$

$= \sin x \cos x = LHS \quad \therefore LHS = RHS$

$\therefore \sin x \cos x = \frac{1}{2}\sin 2x$

ii) $\int_0^{\pi/8} \sin^2 x \cos^2 x \, dx$

$\cos 2x = 1 - 2\sin^2 x$
$\sin^2 x = \frac{1}{2}(1 - \cos 2x)$

$= \int_0^{\pi/8} \frac{1}{4}\sin^2 2x \, dx$

$= \frac{1}{4}\int_0^{\pi/8} \frac{1}{2}(1 - \cos 4x) \, dx$

$= \frac{1}{8}\left[x - \frac{1}{4}\sin 4x\right]_0^{\pi/8}$

$= \frac{1}{8}\left(\frac{\pi}{8} - \frac{1}{4}\sin\frac{\pi}{2} - 0\right)$

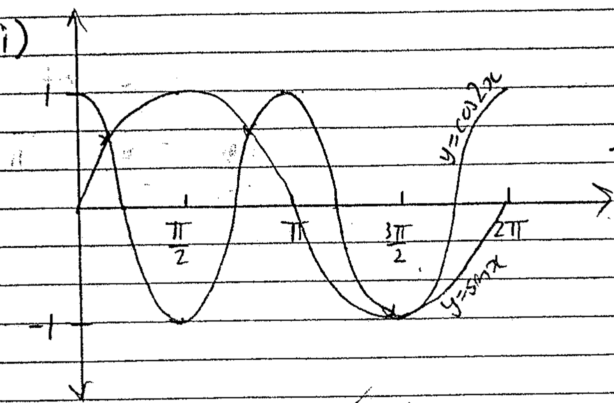
$= \frac{1}{8}\left(\frac{\pi}{8} - \frac{1}{4}\right)$

$= \frac{\pi}{64} - \frac{1}{32} = \frac{\pi - 2}{64}$ ✓ $\frac{4}{4}$

c) $\int \sec 4x \tan 4x \, dx$

$= \frac{1}{4}\sec 4x + C$ ✓ $\frac{1}{1}$

d) i)



✓ $\frac{2}{2}$

ii) 3 Solutions. ✓ $\frac{1}{1}$

/12 10362606

a) i) $AZ = 52\text{cm} = a = T_1$

$T_{13} = 26 = ar^{12} = 52 \times r^{12}$

$\therefore r^{12} = 0.5 \quad \therefore r = 0.9 \ (1dp)$ $\sqrt[12]{0.5}$

Excellent!

$AB = 52 - 52 \times 0.9 = 2.9\text{ cm} \ (1dp)$

5/5

ii) $F = 52 \times r^5$

$G = 52 \times r^6$

$FG = 52 \times r^5 - 52 \times r^6 = 52 \times 0.749... - 52 \times 0.707$

$= 2.2\text{ cm} \ (1dp)$

b) i) $AO = BO$ (radii of same circle) $= x$

$OC = BC = OB = x$ (equil Δ)

$AB = x\theta \quad (l = r\theta)$

$\therefore$ perimeter $= 3x + x\theta$

$= x(3 + \theta)$

area of $\Delta = \frac{1}{2} x . x . \sin 60$

$= \frac{1}{2} x^2 \sqrt{3}/2$

2/2

area of sector $= \frac{1}{2} x^2 \theta \quad (A = \frac{1}{2} r^2 \theta)$

$\therefore$ area $= \frac{1}{2} x^2 (\theta + \sqrt{3}/2)$

ii) $x(3 + \theta) = (12 - 2\sqrt{3})$

$\therefore x = \dfrac{12 - 2\sqrt{3}}{3 + \theta}$

$A = \frac{1}{2} x^2 (\theta + \sqrt{3}/2) = \frac{1}{2}\left(\dfrac{12 - 2\sqrt{3}}{3 + \theta}\right)^2 (\theta + \sqrt{3}/2)$

$= \dfrac{\frac{1}{2}(12 - 2\sqrt{3})^2 (\theta + \sqrt{3}/2)}{(\theta + \sqrt{3})^2}$ show ? $1\frac{1}{2}$/2

$= \dfrac{\frac{1}{2}(2(6 - \sqrt{3}))^2 \left(\frac{2\theta + \sqrt{3}}{2}\right)}{(\theta + \sqrt{3})^2} = \dfrac{(6 - \sqrt{3})^2 (2\theta + \sqrt{3})}{(\theta + 3)^2}$

$= \dfrac{\frac{1}{2} \times \frac{4}{2} (6 - \sqrt{3})^2 (2\theta + \sqrt{3})}{(\theta + 3)^2}$

~~$\frac{1}{2} \times 2 \times 3$~~

A4 — 7mm

(iii) $A = (6 - \sqrt{3})^2 \dfrac{(2\theta + \sqrt{3})}{(\theta + 3)^2}$

By the quotient rule

$A' = (6 - \sqrt{3})^2 \left[\dfrac{(\theta + 3)^2 (2) - (2\theta + \sqrt{3}).2(\theta + 3)}{(\theta + 3)^4} \right]$

Let $A' = 0$

$\therefore (\theta + 3)^2 - (2\theta + \sqrt{3})(\theta + 3) = 0$

i.e. $(\theta + 3)[\theta + 3 - 2\theta - \sqrt{3}] = 0$

$\therefore \theta = -3$ (Not valid) or $\theta = 3 - \sqrt{3}$

To test for min. A

θ^-	$\theta = 3 - \sqrt{3}$	θ^+
\	—	/

$\therefore$ Min.

$\therefore \theta = 3 - \sqrt{3}$ for A minimum.