St Catherine's School

Year: 12
Subject: Extension I Mathematics
Time Allowed: 1.5 Hours
plus 5 min reading time
Date: April 2005

Exam number: 15227508

Directions to candidates:

- All questions are to be attempted.
- All questions are of equal value.
- All necessary **working** must be shown in every question.
- Full marks may not be awarded for careless or badly arranged work.
- Each question attempted should be started on a **new page**.
- Approved calculators and geometrical instruments are required.
- Hand up your questions in two bundle
  - Bundle 1- Q.1,2 and 3.Plus the question paper
  - Bundle 2 Q.4 and 5.

TEACHER'S USE ONLY Total Marks	
Q1	12
Q2	12
Q3	12
Q4	12
Q.5	$9\frac{1}{2}$
TOTAL	$57\frac{1}{2} + 2\frac{1}{2}$

$59\frac{1}{2}$

Question 1

(a) Solve for x: $\dfrac{3}{x-2} \leq 1$ (3m)

(b) (i) Find the acute angle between the lines y=2x-1 and $y = \dfrac{1}{3}x + 1$ (2m)

(ii) Find the acute angle between the lines x=3 and $y = \sqrt{3}x$ (2m)

(c) A(-2,7) and B(8,-2) are two points on the Number Plane. Find the coordinates of the point P, which divides AB externally in the ratio 3 : 2. (2m)

(d) Find: $\sum_{r=4}^{20}(2r + 2^r)$ (3m)

Question 2 (Start a new page)

(a) Use Mathematical Induction to show that

$$\frac{1}{2\times3}+\frac{1}{3\times4}+\frac{1}{4\times5}+\dots\dots+\frac{1}{(n+1)(n+2)}=\frac{n}{2(n+2)}$$ for $n \geq 1$ (4m)

(b) $P\ (2ap, ap^2)$ and $Q\ (2aq, aq^2)$ are points on the parabola $x^2 = 4ay$ and PQ is a focal chord.

(i) Find the equation of PQ and hence show that pq= -1 (2m)

(ii) Show that the equation of the tangent at P is $px - y = ap^2$ (2m)

(iii) Prove that the tangents at P and Q meet on the directrix of the parabola. (2m)

(c) Find $\sin(2\cos^{-1}\frac{1}{3})$ (2m)

Question 3. (Start a new page)

(a) Sketch the following functions clearly stating the Domain and the Range.

(i) $y = \tan^{-1}\frac{x}{2}$ (2m)

(ii) $y = 3\cos^{-1}\frac{x}{5}$ (3m)

(iii) $y=\sin^{-1}\sqrt{x}$ (3m)

(b) (i) Show that $\frac{d}{dx}(x\ \cos^{-1}x - \sqrt{1-x^2}) = \cos^{-1}x$ (2m)

(ii) Hence evaluate $\int_{-1}^{1}\cos^{-1}x\ dx$ (2m)

Question 4 (Start a new page)

(a) Differentiate:

(i) $y = \sin^{-1}x^2$ (2m)

check (ii) $y = x\tan^{-1}\sqrt{x}$ (3m)

(b) Integrate:

(i) $\int\frac{dx}{1+4x^2}$ (2m)

check (ii) $\int\frac{dx}{\sqrt{16-9x^2}}$ (3m)

(c) Consider y = sec x $\left(0 \le x < \frac{\pi}{2}\right)$, Show that its inverse function is $y = \cos^{-1}\frac{1}{x}$ (2m)

Question 5. (Start a new page)

(a) (i) Differentiate $y = \cos^{-1} x + \cos^{-1}(-x)$ (1m)

(ii) Hence or otherwise show that $\cos^{-1} x + \cos^{-1}(-x) = \pi$ 2 ~~(1m)~~

(b) (i) Show that the equation of the normal to the parabola $x^2 = 4ay$ at a point P $(2ap, ap^2)$ is $py + x = 2ap + ap^3$

(2m)

(ii) This normal meets the axis of the parabola at M. Find the coordinates of the point M (1m)

(iii) PM is produced to a point N, so that PM=MN. Show that the coordinates of the point N is $(-2ap, 4a + ap^2)$

(2m)

(iv) Find the Cartesian equation of the locus of N.

(1m)

$\cos x = \theta$

(c) (i) If y= $\sin^{-1}\overbrace{(\cos x)}^{\theta}$, $\cancel{-\pi} < x \leq \pi$

Show that $\dfrac{dy}{dx}$ is 1 or -1 $\sin y = \theta$ (2.5m)

(ii) Find the values of x for which $\dfrac{dy}{dx}$ is 1 (1.5m)

END OF PAPER

Question 1. 15227508

a. ~~lot~~

$$0 \le 1 - \frac{3}{x-2} \qquad \boxed{x \neq 2}$$

$$0 \le \frac{x-2-3}{x-2}$$

$$0 \le \frac{x-5}{x-2}$$

For this to be true:

$x - 5 \ge 0$, $x - 2 \ge 0$ or $x - 5 \le 0$, $x - 2 \le 0$

$x \ge 5$ $\quad$ $x > 2$ $\quad$ $x \le 5$ $\quad$ $x < 2$

2 5 $\qquad$ 2 5

$\therefore x < 2, \; x \ge 5$

3

b. i) $y = 2x - 1$

$y = \frac{1}{3}x + 1$

$\therefore m_1 = 2$

$m_2 = \frac{1}{3}$

$$\tan\theta = \left|\frac{m_2 - m_1}{1 + m_1 m_2}\right|$$

$$= \left|\frac{\frac{1}{3} - 2}{1 + (2)(\frac{1}{3})}\right|$$

$$= |-1|$$

$$= 1$$

$\therefore \theta = 45°$

2

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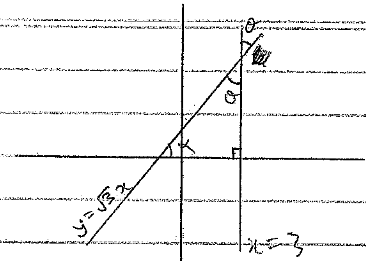
ii)



$x = 3$

$y = \sqrt{3}x$

$\therefore m_1 = \sqrt{3}$

Find α:

$\tan\alpha = m_1$

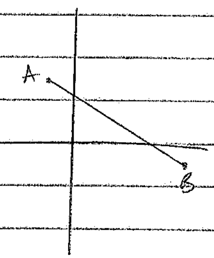
$= \sqrt{3}$

$\therefore \alpha = 60°$

2

$\therefore \theta = 30°$ ($\angle$ sum of a $\triangle = 180°$)

c. ratio = 3 : −2



$$x = \frac{mx_2 + nx_1}{m+n}$$

$$= \frac{3(8) + (-2)(2)}{3-2}$$

$= 28$

$$y = \frac{my_2 + ny_1}{m+n}$$

$$= \frac{3(-2) + (-2)(7)}{3-2}$$

$= -20$

2

$\therefore P(28, -20)$

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d. $\sum_{r=4}^{20} (2r + 2^r)$

~~= 24 + 42 + 76 + ... + 1048616~~

$= \sum_{r=4}^{20} 2r + \sum_{r=4}^{20} 2^r$

$= \underbrace{(8+10+12+\cdots+40)}_{} + \underbrace{(2^4+2^5+\ldots 2^{20})}_{}$

$a = 8$, $d = 2$, $n = 17$, $l = 40$

$a = 16$, $r = 2$, $n = 17$

$= \frac{n}{2}(a+l) + \frac{a(r^n-1)}{r-1}$ ✓

$= \frac{17}{2}(8+40) + \frac{16(2^{17}-1)}{2-1}$

$= 408 + 2097136$

$= 2097544$ ✓

3

Question 2.

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a. let $P(n) = \frac{1}{2\times3} + \frac{1}{3\times4} + \frac{1}{4\times5} + \ldots + \frac{1}{(n+1)(n+2)} = \frac{n}{2(n+2)}$

1. consider $P(1)$

$LHS = \frac{1}{2\times3} = \frac{1}{6}$ ✓

$RHS = \frac{1}{2(3)} = \frac{1}{6}$

$= LHS$ ✓

$\therefore P(1)$ is true ✓

2. Assume $P(k)$ is true

$\frac{1}{2\times3} + \frac{1}{3\times4} + \ldots + \frac{1}{(k+1)(k+2)} = \frac{k}{2(k+2)}$ ✓

Consider $P(k+1)$

$\frac{1}{2\times3} + \frac{1}{3\times4} + \ldots + \frac{1}{(k+1)(k+2)} + \frac{1}{(k+2)(k+3)} = \frac{k+1}{2(k+3)}$

$LHS = \frac{k}{2(k+2)} + \frac{1}{(k+2)(k+3)}$

$= \frac{k(k+3) + 2}{2(k+2)(k+3)}$ ✓

$= \frac{k^2+3k+2}{2(k+2)(k+3)}$

$= \frac{\cancel{(k+2)}(k+1)}{2\cancel{(k+2)}(k+3)}$ ✓

$= \frac{k+1}{2(k+3)} = RHS$

cont'd

$\therefore P(k+1)$ is true if $P(k)$ is true

$\therefore$ It has been proven that $P(1)$ is true; and that $P(k+1)$ is true if $P(k)$ is true ✓

$\therefore$ through mathematical induction, $P(n)$ is true; $n \geq 1$ ✓ 4

b. i) $P(2ap, ap^2)$

$Q(2aq, aq^2)$

$m_{PQ}: \dfrac{y_2 - y_1}{x_2 - x_1}$

$= \dfrac{aq^2 - ap^2}{2aq - 2ap}$

$= \dfrac{\cancel{a}\cancel{(q-p)}(q+p)}{2\cancel{a}\cancel{(q-p)}}$

$= \dfrac{q+p}{2}$ ✓

$\therefore PQ: \quad y - y_1 = m(x - x_1)$

$y - ap^2 = \dfrac{q+p}{2}(x - 2ap)$

$2y - \cancel{2ap^2} = (q+p)x - 2apq - \cancel{2ap^2}$ 2

- Eqn of PQ: $\therefore 2y = (q+p)x - 2apq$ ✓
- Focal chord: ie. runs thru $(0, a)$, so sub $(0,a)$ in:

$2a = -2apq$ ✓

ii) $y = \dfrac{x^2}{4a}$

$y' = \dfrac{\cancel{2a}}{\cancel{4a}} \cdot \dfrac{2x}{2\cancel{4}a} = \dfrac{x}{2a}$

At P, $y' = \dfrac{\cancel{2}ap}{2a}$

$= p$ ✓

$\therefore$ tgt: $y - y_1 = m(x - x_1)$

$y - ap^2 = p(x - 2ap)$

$= px - 2ap^2$ ✓ 2

$\therefore px - y = ap^2$

iii) tgt at P: $px - y = ap^2$ —— ①

Similarly:

tgt at Q: $qx - y = aq^2$ —— ②

① × q : $pqx - qy = ap^2q$ —— ③

② × p : $pqx - py = aq^2p$ —— ④

③ − ④ $\quad -qy + py = ap^2q - aq^2p$

$y\cancel{(p-q)} = apq\cancel{(p-q)}$

$y = apq$ ✓

Now $pq = -1$ (proven in (i)) 2

$\therefore y = -a$ ✓

which lies on the directrix ($y = -a$)

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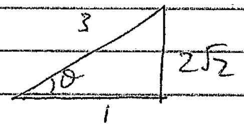
c. $\sin\left(2\cos^{-1}\frac{1}{3}\right)$

~~Let...~~

Let $\cos^{-1}\frac{1}{3} = \theta$

$\therefore \cos\theta = \frac{1}{3}$



$\therefore \sin\theta = \frac{2\sqrt{2}}{3}$

now, $\sin 2\theta = 2\sin\theta\cos\theta$

$= 2\left(\frac{2\sqrt{2}}{3}\right)\left(\frac{1}{3}\right)$

$\therefore \sin\left(2\cos^{-1}\frac{1}{3}\right) = \frac{4\sqrt{2}}{9}$

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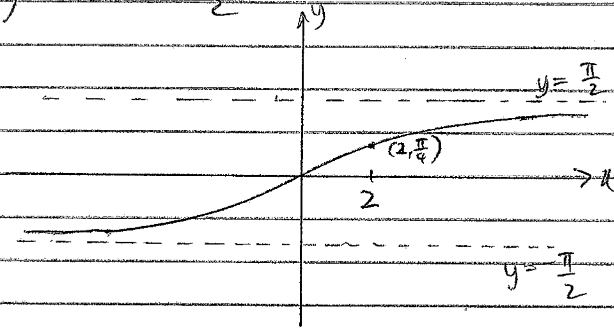
Question 3.

a. i) $y = \tan^{-1}\frac{x}{2}$



2

D; all x

R; $-\frac{\pi}{2} < y < \frac{\pi}{2}$

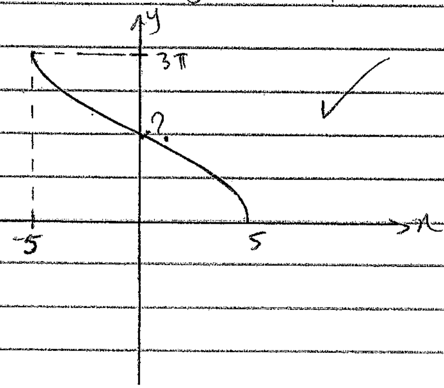
ii) $y = 3\cos^{-1}\frac{x}{5}$

D: $-1 \le \frac{x}{5} \le 1$

$\therefore -5 \le x \le 5$

R; $0 \le \cos^{-1}\frac{x}{5} \le \pi$

$\therefore 0 \le 3\cos^{-1}\frac{x}{5} \le 3\pi$



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iii) $y = \sin^{-1}\sqrt{x}$

D: ~~[illegible]~~

~~[illegible]~~

• $-1 \le \sqrt{x} \le 1$

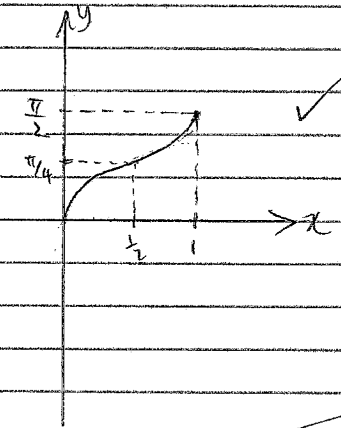
• $x \geqslant 0$

$\therefore$ Domain:

$0 \le x \le 1$ ✓

R: $0 \le \sin^{-1}\sqrt{x} \le \frac{\pi}{2}$ ✓

$\therefore y = \sin^{-1}\sqrt{x}$:

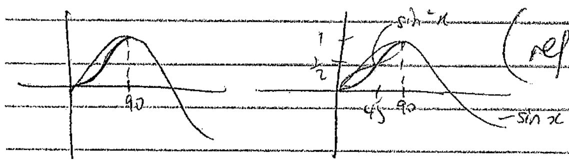


✓ 3

$\sin y = \sqrt{x}$

$\sin^2 y = x$

inverse: $\sin^2 x = y$



(reflected on $y = x$)

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6. i) let $y = x\cos^{-1}x - \sqrt{1-x^2}$

~~$\frac{dy}{dx}$ [illegible]~~

$$\frac{dy}{dx} = x\left(\frac{-1}{\sqrt{1-x^2}}\right) + \cos^{-1}x - \left(\frac{1}{2}(1-x^2)^{-\frac{1}{2}} \cdot -2x\right)$$

$$= \frac{-x}{\sqrt{1-x^2}} - \cos^{-1}x + \frac{x}{\sqrt{1-x^2}}$$

$= \cos^{-1}x$ as req'd ✓ 2

ii) $\int_{-1}^{1} \cos^{-1}x \, dx$

$$= \left[x\cos^{-1}x - \sqrt{1-x^2}\right]_{-1}^{1}$$

$$= \left(\cos^{-1}1 - \sqrt{1-1^2}\right) - \left(-\cos^{-1}(-1) - \sqrt{1-(-1)^2}\right)$$

$= 0 - (-\pi)$

$= \pi$ ✓ 2

Question 4. $12 + 9\frac{1}{2}$ 15227508

a. i) $y = \sin^{-1} x^2$

$$\frac{dy}{dx} = \frac{2x}{\sqrt{1-x^4}}$$ ✓

ii) $y = x\tan^{-1}\sqrt{x}$

$$\frac{dy}{dx} = uv' + vu'$$

$$= x\left(\frac{\frac{1}{2}x^{-\frac{1}{2}}}{1+x}\right) + \tan^{-1}\sqrt{x}$$

$$= \frac{x}{2\sqrt{x}(1+x)} + \tan^{-1}\sqrt{x}$$ ✓

b. i) $\int \frac{1}{1+4x^2}\,dx$

$$= \int \frac{1}{4(\frac{1}{4}+x^2)}\,dx$$

$$= \frac{1}{4}\int \frac{1}{\frac{1}{4}+x^2}\,dx$$

$$= \frac{1}{4}\left(2\tan^{-1}(2x) + c\right)$$ ✓

$$= \frac{1}{2}\tan^{-1} 2x + C$$

ii) $\int \frac{1}{\sqrt{16-9x^2}}\,dx$

$$= \int \frac{1}{3\sqrt{(\frac{16}{9}-x^2)}}\,dx$$ ✓

$$= \frac{1}{3}\sin^{-1}\left(\frac{3x}{4}\right) + C$$

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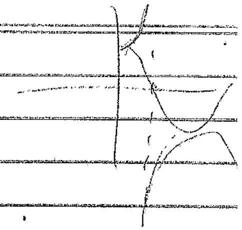

c. $y = \sec x$

$$= \frac{1}{\cos x}$$

Inverse function: x and y swap:

$$x = \frac{1}{\cos y} \qquad \boxed{\cos y \neq 0 \;\therefore y \neq \frac{\pi}{2}}$$ ✓

$$\cos y = \frac{1}{x}$$

$$\therefore y = \cos^{-1}\left(\frac{1}{x}\right) \qquad \left(0 \le x < \frac{\pi}{2}\right)$$

$\frac{1}{2}$/1

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Question 5.

a. i) $y = \cos^{-1}x + \cos^{-1}(-x)$

$\frac{dy}{dx} = \frac{-1}{\sqrt{1-x^2}} + \frac{-\,-1}{\sqrt{1-x^2}}$

$= 0$

ii) $\therefore \int 0\,dx = k$ (constant)

$\therefore \cos^{-1}x + \cos^{-1}(-x) = k$

sub in a pt $(0,0)$?

$\cos^{-1}0 + \cos^{-1}0 = k$?

$\therefore k = \pi$

$\therefore \cos^{-1}x + \cos^{-1}(-x) = \pi$

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b. i) $y = \frac{x^2}{4a}$

$y' = \frac{2x}{4a}$

$= \frac{x}{2a}$

At P, $y' = \frac{2ap}{2a}$

$= p$

$\therefore$ m of normal: $-\frac{1}{p}$

$\therefore$ eqn of normal:

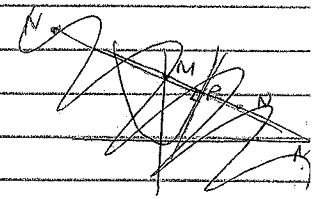
$y - y_1 = m(x - x_1)$

2

$y - ap^2 = -\frac{1}{p}(x - 2ap)$

$py - ap^3 = -x + 2ap$

$py + x = 2ap + ap^3$



ii) M when $x = 0$

$py = 2ap + ap^3$

$y = 2a + ap^2$

$\therefore M(0,\ 2a + ap^2)$

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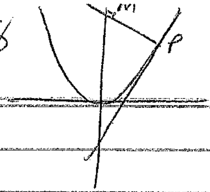

iii) If ~~[illegible]~~ $PM = MN$

$\therefore M$ is midpoint of NP

$P(2ap, ap^2)$
$M(0, 2a+ap^2)$
$N(x, y)$

$\cancel{2ap = \frac{0+x}{2}}$

$\cancel{4ap = x}$

$0 = \frac{2ap + x}{2}$

$0 = 2ap + x$

$\therefore x = -2ap$

2

$2a + ap^2 = \frac{ap^2 + y}{2}$

$4a + 2ap^2 = ap^2 + y$

$4a + ap^2 = y$

$\therefore N(-2ap, 4a+ap^2)$

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iv) $N(-2ap, 4a+ap^2)$

$\therefore X = -2ap$ — ①

$Y = 4a + ap^2$ — ②

From ①

$\frac{X}{-2a} = p$ — ③

Sub ③ into ②

$Y = 4a + a\left(\frac{X}{-2a}\right)^2$

1

$= 4a + \frac{\cancel{a}X^2}{4a^{\cancel{2}}}$

$= 4a + \frac{x^2}{4a}$

$4ay = 16a^2 + x^2$

$\therefore x^2 = 4ay - 16a^2$

$= 4a(y - 4a)$

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c. i) $y = \sin^{-1}(\cos x)$ $\quad x>0$

$$\frac{dy}{dx} = \frac{-\sin x}{\sqrt{1-\cos^2 x}}$$

$$= \frac{-\sin x}{\sin x}$$ $\quad \sqrt{\sin^2 x} = |\sin x|$

$$= -1$$

however,

$y = \sin^{-1}(\cos(-x)) = y = \sin^{-1}(\cos x)$ $\quad x>0$

$\quad -x<0$

$$\therefore \frac{dy}{dx} = \frac{-\sin(-x)}{\sqrt{1-(\cos(-x))^2}}$$

$$= \frac{-\sin x}{\sqrt{1-\cos^2 x}}$$

$$= \frac{\sin x}{\sin x}$$

$$= 1$$

2 + 2

/4.

ii) $\frac{dy}{dx} = 1$ when ~~x<0~~

x is negative, as this makes the differential positive

$\therefore -\pi < x < 0$

$9\frac{1}{2}$ + 2

/12