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Year: 12
Subject: Extension II Mathematics
Time Allowed: 2 hours plus 5 mins reading time
Date: April 2002

Exam number: ____________

Directions to candidates:

- All questions are to be attempted.
- All questions are of equal value.
- All necessary **working** must be shown in every question.
- Full marks may not be awarded for careless or badly arranged work.
- Each question attempted should be started on a **new page.**
- Approved calculators and geometrical instruments are required.
- This page is a cover sheet for **Section A**. Write a cover page for **Section B** and include your number.
- Hand in your work in **2 bundles:**
  Section A Questions 1, 2 and 3.
  Section B Questions. 4 and 5

TEACHER'S USE ONLY Total Marks
A
B
TOTAL

Question 1 (15 marks)

a) If $A = 3+4i$ and $B = 5-13i$ write in the form of $a+ib$ (5)

i) $A+B$

ii) AB

iii) $\frac{A}{B}$

iv) $\sqrt{A}$

b) The point z is rotated anti-clockwise about the origin through $\frac{\pi}{2}$ radians to the point z'. Prove that $z' = iz$. (3)

c) Let $w = \frac{1+2i}{1-i} + \frac{1}{i}$. Express w in mod-arg form. (3)

d) Find the Cartesian equation of the locus in the Argand Plane defined by $|z-1| = \mathrm{Re}(z)$. Describe in geometrical terms this locus. (2)

e) Z_1, Z_2, Z_3 represent three different complex numbers where $z_1 z_3 = z_2^2$ and O is the origin. Show geometrically that OZ_2 bisects the angle $Z_1 O Z_3$ (Hint: Look at arguments) (2)

Question 2 (15 marks)

a) $\frac{x^2}{9} - \frac{y^2}{7} = 1$ is a hyperbola

i) Find its eccentricity, the foci and sketch the hyperbola (3)

ii) Let PQ be the latus rectum with P in the first quadrant. Show that the gradient of the tangent at P is equal to the eccentricity. (3)

b) $P(ct, \frac{c}{t})$ lies on the rectangular hyperbola $xy = c^2$

i) Find the equation of the normal and tangent at P. (2)

ii) Show that the normal at P cuts the hyperbola again at $Q(-\frac{c}{t^3}, -ct^3)$ (3)

iii) The normal at P meets the x-axis at A and the tangent at P meets the y-axis at B. M is the mid-point of AB. Find the co-ordinates of A and B and find the locus of M as P moves on the hyperbola. (4)

Question 3 (15 marks)

a) De Moivre's Theorem states $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$ (7)

i) Use the method of mathematical induction to prove De Moivre's Theorem for positive, non zero integers.

ii) By making the substitution $n = -m$, where m is positive, show that the theorem also holds for negative, non-zero integers.

iii) Does the theorem hold for $n = 0$? Show.

b) If $\frac{2x+31}{(x-1)^3(x+2)} = \frac{a}{(x-1)} + \frac{b}{(x-1)^2} + \frac{c}{(x-1)^3} + \frac{d}{(x+2)}$ (5)

find the values of a, b, c, d.

c) If $ax^4 + bx^3 + dx + e = 0$ has a triple root show that $4a^2d + b^3 = 0$ (3)

a, b, d, e all non-zero

Question 4 (15 marks)

a) α, β, δ are the roots of the equation $x^3 - 7x^2 + 18x - 7 = 0$

i) Note that to evaluate $\alpha^2 + \beta^2$ by using the sum and product of roots the following result can be used $\alpha^2 + \beta^2 = (\alpha+\beta)^2 - 2\alpha\beta$,

Find similar results for the expressions

A) $\alpha^2 + \beta^2 + \delta^2$ B) $\alpha^2\beta^2 + \beta^2\delta^2 + \delta^2\alpha^2$ (3)

ii) Hence or otherwise evaluate $(1$ [illegible] $+ \beta^2)(1 + \delta^2)$ (2)

check in my own set of questions

~~b)~~ Show tha $7^n + 15^n$ is divisible by 11 for all odd $n \geq 1$ (4)

c) i) Find real numbers a and [illegible] that

$x^4 + x^3 + x^2 + x + 1 \equiv (x^2 + ax + 1)(x^2 + bx + 1)$ (3)

ii) Given that $x = \cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}$ is a solution of $x^4 + x^3 + x^2 + x + 1 = 0$

Find the exact value of $\cos\frac{2\pi}{5}$. (3)

4

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Question 1

a) i) $A+B = 8-9i$

ii) $AB = (3+4i)(5-13i)$

$= 15-39i+20i+52$

$= \cancel{67}-19i$

iii) $\dfrac{A}{B} = \dfrac{3+4i}{5-13i} \times \dfrac{5+13i}{5+13i}$

$= \dfrac{15+39i+20i-52}{25+169}$

$= -\dfrac{37}{194} + \dfrac{59}{194}i$

iv) $\sqrt{A} = \sqrt{3+4i}$

let $\sqrt{3+4i} = a+ib$

then $3+4i = a^2+2ab\,i-b^2$

$3=a^2-b^2$ $\quad$ $4=2ab$

$2=ab$

$\dfrac{2}{a}=b$

subbing in $\quad 3=a^2-\dfrac{4}{a^2}$

$3a^2=a^4-4$

$a^4-3a^2-4=0$

$(a^2-4)(a^2+1)=0$

$\downarrow$

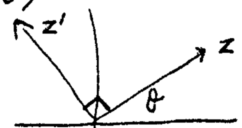
no real solns

$a^2=4$

$a=\pm 2$ and $b=\pm\dfrac{1}{2}$

$a+ib = \pm(2+\dfrac{i}{2})$

b) $z=r(\cos\theta+i\sin\theta)$



$z'=r(\cos(\theta+\pi/2)+i\sin(\theta+\pi/2))$

now $\cos(\theta+\pi/2) = \cos\theta\cos\pi/2 - \sin\theta\sin\pi/2$

$= -\sin\theta$

and $\sin(\theta+\pi/2) = \sin\theta\cos\pi/2 + \sin\pi/2\cos\theta$

$= \cos\theta$

so $z' = r(-\sin\theta + i\cos\theta)$

$= r(i^2\sin\theta + i\cos\theta)$

$= i.r(\cos\theta+i\sin\theta)$

$= iz$

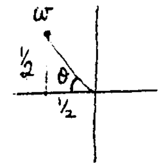
c) $w = \dfrac{1+2i}{1-i} + \dfrac{1}{i}$

$= \dfrac{i-2+(1-i)}{i+1}$

$= \dfrac{-1}{i+1} \times \dfrac{i-1}{i-1}$

$= \dfrac{1-i}{-1-1}$

$= \dfrac{i-1}{2}$



$|w| = \sqrt{(1/2)^2+(1/2)^2}$

$= \dfrac{1}{\sqrt{2}}$

$\theta=\pi/4$

$\text{Arg}(w) = \pi - \pi/4 = 3\pi/4$

$w = \dfrac{1}{\sqrt{2}}\,\text{cis}\,3\pi/4$

d) $|z-1| = \text{Re}(z)$

$(x-1)^2+y^2=x^2$

$x^2-2x+1+y^2=x^2$

$y^2=2x-1$

This is a horizontal parabola

e) $z_1 z_3 = z_2 . z_2$

taking arguments

$\arg(z_1 z_3) = \arg(z_2 z_2)$

$\arg z_1 + \arg z_3 = \arg z_2 + \arg z_2$

$\arg z_3 - \arg z_2 = \arg z_2 - \arg z_1$

ie. $\angle Z_3 O Z_2 = \angle Z_2 O Z_1$

and hence Z_2 bisects $\angle Z_3 O Z_1$

Question 2.

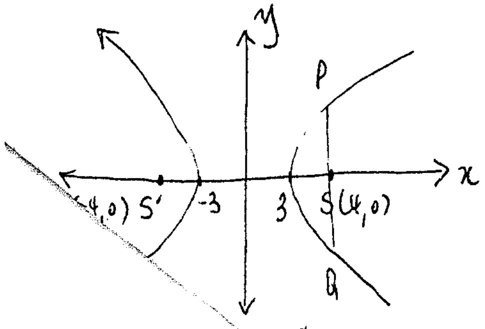
a) $\frac{x^2}{9} - \frac{y^2}{7} = 1$

i) $b^2 = a^2(e^2 - 1)$

$7 = 9(e^2 - 1)$

$e = 4/3 \quad (e > 0)$ 1

$S(\pm ae, 0) = S(\pm 4, 0)$ 1



1

P has x co-ordinate 4 so ½

$\frac{16}{9} - \frac{y^2}{7} = 1$

$\frac{y^2}{7} = \frac{7}{9}$

$y^2 = 49/9$

$y = \pm 7/3$

P in the 1st quadrant so $P(4, 7/3)$ 1

$\frac{x^2}{9} - \frac{y^2}{7} = 1$

$\frac{2x}{9} - \frac{2y}{7} \cdot \frac{dy}{dx} = 0$

$\frac{dy}{dx} = \frac{7x}{9y}$ 1

at P $\frac{dy}{dx} = \frac{28}{9 \cdot 7/3}$

$= 4/3$ ½

$= e$

b) i) $xy = c^2$

$y = c^2 x^{-1}$

$\frac{dy}{dx} = -\frac{c^2}{x^2}$

at P $y' = -c^2$ ... 1

eqn of tangent at P

$y - \frac{c}{t} = -\frac{1}{t^2}(x - ct)$

$yt^2 - ct = -x + ct$

$x + yt^2 - 2ct = 0$

gradient of normal at P

$m = t^2$

eqn of normal

$y - \frac{c}{t} = t^2(x - ct)$

$yt - c = t^3 x - ct^4$

$t^3 x - yt + c - ct^4 = 0$

ii) Solve simultaneously

$t^3 x - yt + c - ct^4 = 0$ — ①

$xy = c^2$ — ②

$y = \frac{c^2}{x}$ — ③

sub ③ into ①

$t^3 x - \frac{c^2 t}{x} + c - ct^4 = 0$

$t^3 x^2 - c^2 t + (c - ct^4)x = 0$

using quadratic formula

$x = \frac{-(c - ct^4) \pm \sqrt{(c - ct^4)^2 + 4t^3 \cdot ct}}{2t^3}$

$= \frac{ct^4 - c \pm \sqrt{c^2 - 2c^2t^4 + c^2t^8 + 4c^2t}}{}$

$t^3x^2 + (c - ct^4)x - c^2t = 0$

$t^3x^2 - ct^4x + cx - c^2t = 0$

$xt^3(x - ct) + c(x - ct) = 0$

$(xt^3 + c)(x - ct) = 0$

$x = -\frac{c}{t^3}$ or $x = ct$.

x

$= \frac{2ct^4}{2t^3}$ and $-\frac{2c}{2t^3}$

$= ct$ and $-\frac{c}{t^3}$

this is P

this is Q

Q has x co-ordinate $-\frac{c}{t^3}$

sub into $xy = c^2$

$-\frac{c}{t^3}y = c^2$

$y = -ct^3$

$Q\left(-\frac{c}{t^3}, -ct^3\right)$

iii) normal at P $t^3x - yt + c - ct^4 = 0$

at A, $y = 0$

$t^3x + c - ct^4 = 0$

$x = \frac{ct^4 - c}{t^3}$

$A\left(\frac{ct^4 - c}{t^3}, 0\right)$

tangent at P $x + yt^2 - 2ct = 0$

at B, $x = 0$

$yt^2 - 2ct = 0$

$y = \frac{2ct}{t^2} = \frac{2c}{t}$

$B\left(0, \frac{2c}{t}\right)$

M is midpoint of AB

$M = \left(\frac{ct^4 - c}{2t^3}, \frac{c}{t}\right)$ need to eliminate t

$y = \frac{c}{t} \Rightarrow t = \frac{c}{y}$

$$x = \frac{c\left(\frac{c^4}{y^4}\right) - c}{2\left(\frac{c^3}{y^3}\right)}$$

$$x = \frac{c^5 - cy^4}{2yc^3}$$

$= c^4 - y^4$.

$2c^2xy = c^4 - y^4$.

Question 3.

a) i Prove for $n \geq 1$

Step 1. Show for $n = 1$

$(\cos\theta + i\sin\theta)^1 = \cos\theta + i\sin\theta$

$= \cos 1\theta + i\sin 1\theta$

Step 2. Assume for $n = k$

$(\cos\theta + i\sin\theta)^k = \cos k\theta + i\sin k\theta$

and show for $n = k + 1$

$(\cos\theta + i\sin\theta)^{k+1}$

$= (\cos\theta + i\sin\theta)^k(\cos\theta + i\sin\theta)$

$= (\cos k\theta + i\sin k\theta)(\cos\theta + i\sin$

from assumption

$= \cos k\theta\cos\theta + i\cos k\theta\sin\theta$

$+ i\sin k\theta\cos\theta - \sin k\theta\sin$

$= (\cos k\theta\cos\theta - \sin k\theta\sin\theta)$

$+ i(\cos k\theta\sin\theta + \sin k($

$= \cos(k\theta + \theta) + i\sin(k\theta + \theta)$

$= \cos((k+1)\theta) + i\sin((k+1)\theta)$

Step 3. Thus if result is true for n

ii) now let $n = -m$

$(\cos\theta + i\sin\theta)^n$

$= (\cos\theta + i\sin\theta)^{-m}$

$= \dfrac{1}{(\cos\theta + i\sin\theta)^m}$

$= \dfrac{1}{\cos m\theta + i\sin m\theta} \times \dfrac{\cos m\theta - i\sin m\theta}{\cos m\theta - i\sin m\theta}$

$= \dfrac{\cos m\theta - i\sin m\theta}{\cos^2 m\theta + \sin^2 m\theta}$

$= \cos m\theta - i\sin m\theta$

$= \cos(-m\theta) + i\sin(-m\theta)$

$= \cos n\theta + i\sin n\theta$

$\therefore$ result is true for negative integers

i) if $n = 0$

$(\cos\theta + i\sin\theta)^0$

$= \cos(0) + i\sin 0$

$1 + 0$

for $n = 1$

b) $2x + 31$

$= a(x-1)^2(x+2) + b(x-1)(x+2) + c(x+2) + d(x-1)^3$

let $x = 1$

$33 = 3c$

$\underline{c = 11}$

let $x = -2$

$27 = -27d$

$\underline{d = -1}$

let $x = 0$

$31 = 2a - 2b + 2c - d$

$= 2a - 2b + 23$ —①

let $x = -1$

$29 = 4a - 2b + c - 8d$

$= 4a - 2b + 19$ —②

①−②

$2 = -2a + 4$

$-2 = -2a$

$\underline{a = 1}$

$29 = 4 - 2b + 19$

$\underline{b = -3}$

c) $P(x) = ax^4 + bx^3 + dx + e$

$P'(x) = 4ax^3 + 3bx^2 + d$

$P''(x) = 12ax^2 + 6bx$

if $P(x)$ has triple root, then it is a single root of $P''(x)$

$12ax^2 + 6bx = 0$

$6x(2ax + b) = 0$

$x = 0$ or $x = -\dfrac{b}{2a}$

not a triple root of $P(x)$

so $x = -b/2a$ is the triple r

$\therefore P'\left(\dfrac{-b}{2a}\right) = 0$

$0 = 4a\left(\dfrac{-b^3}{8a^3}\right) + 3b.\dfrac{b^2}{4a^2} + d$

$0 = -\dfrac{b^3}{2a^2} + \dfrac{3b^3}{4a^2} + d$

$0 = \dfrac{-2b^3 + 3b^3 + 4a^2d}{4a^2}$

$0 = 4a^2d + b^3$

4

1) A)

$(\alpha+\beta+\gamma)^2$

$= \alpha^2+\alpha\beta+\alpha\gamma+\beta\alpha+\beta^2+\beta\gamma+\alpha\gamma+\beta\gamma+\gamma^2$

$= \alpha^2+\beta^2+\gamma^2+2(\alpha\beta+\beta\gamma+\alpha\gamma)$

$\therefore \alpha^2+\beta^2+\gamma^2 = (\alpha+\beta+\gamma)^2-2(\alpha\beta+\beta\gamma+\alpha\gamma)$

B) $(\alpha\beta+\beta\gamma+\alpha\gamma)^2$

$= \alpha^2\beta^2+\alpha\beta^2\gamma+\alpha^2\beta\gamma+\alpha\beta^2\gamma+\beta^2\gamma^2+\alpha\beta\gamma^2$

$+\alpha^2\beta\gamma+\alpha\beta\gamma^2+\alpha^2\gamma^2$

$\therefore \alpha^2\beta^2+\beta^2\gamma^2+\alpha^2\gamma^2 =$

$= (\alpha\beta+\beta\gamma+\alpha\gamma)^2-2\alpha\beta\gamma(\alpha+\beta+\gamma)$

ii) $(1+\alpha^2)(1+\beta^2)(1+\gamma^2)$

$= (1+\alpha^2+\beta^2+\alpha^2\beta^2)(1+\gamma^2)$

$= 1+\alpha^2+\beta^2+\alpha^2\beta^2+\gamma^2+\alpha^2\gamma^2$

$+\gamma^2\beta^2+\alpha^2\beta^2\gamma^2$

$= 1+(\alpha^2+\beta^2+\gamma^2)+(\alpha^2\beta^2+\beta^2\gamma^2+\alpha^2\gamma^2)$

$+\alpha^2\beta^2\gamma^2$

from the polynomial.

$(\alpha+\beta+\gamma) = 7$

$(\alpha\beta+\alpha\gamma+\beta\gamma) = 18$

$\alpha\beta\gamma = 7$

now $\alpha^2+\beta^2+\gamma^2 = 7^2-2.18$

$= 13$

and $\alpha^2\beta^2+\alpha^2\gamma^2+\beta^2\gamma^2 = 18^2-2.7.7$

$= 226$

so $(1+\alpha^2)(1+\beta^2)(1+\gamma^2)$

$= 1+13+226+49$

$= 289$

b) Step 1 $n=1$

$7^1+15^1 = 22$

$= 2\times 11$ $\therefore$ true for $n=1$

Step 2 Assume for $n=k$

$7^k+15^k = 11Q$ for some integer Q

and show for $7^{k+2}-15^{k+2}$

$7^{k+2}-15^{k+2}$

$= 7^k.7^2-15^k.15^2$

$= 7^2(7^k-15^k)+176.15^k$

$= 7^2.11Q + 11.16\ 15^k$

from assumption

$= 11[7^2Q+16.15^k]$

$= 11P$ for some integer P

Step 3: Yadda Yadda (You kn what goes here!)

$x^4+x^3+x^2+x+1$

c) $\equiv (x^2+ax+1)(x^2+bx+1)$

$\equiv x^4+bx^3+x^2+ax^3+abx^2+a$

$+x^2+bx+1$

$\equiv x^4+x^3(b+a)+x^2(2+ab)$

$+x(b+a)+1$

equating co-efficents

$a+b=1$

$2+ab=1$

Solving simultaneously

$\cancel{a=b-1}$ $1-b$

$2+b(\cancel{b}-b)=1$

$2+b^2-b=1$ $\quad 2+b^1-b^2=$

$b^2-b+1=0$ $\quad b^2-b-$

$b = \frac{1\pm\sqrt{5}}{2}$ $\therefore a = 1-(\frac{1\pm}{2}$

$a = \frac{1 \mp \sqrt{5}}{2}$

Hence without loss of generality

$a = \frac{1+\sqrt{5}}{2}$, $b = \frac{1-\sqrt{5}}{2}$

ii) if $x = \cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}$

is a soln of $x^4+x^3+x^2+x+1=0$

then it is a soln of either

$(x^2+ax+1)=0$ or $(x^2+bx+1)=0$

$\therefore \cos 2\pi/5 + i\sin 2\pi/5 =$

$\frac{-a \pm \sqrt{a^2-4}}{2}$ or $\frac{-b \pm \sqrt{b^2-4}}{2}$

now $b^2 = \frac{6-2\sqrt{5}}{4}$ and $a^2 = \frac{6+2\sqrt{5}}{4}$

so $a^2-4<0$ and $b^2-4<0$

$\therefore \cos 2\pi/5 = -\frac{a}{2}$ or $-\frac{b}{2}$

(by equating real parts)

$= \frac{-1-\sqrt{5}}{2}$ or $\frac{\sqrt{5}-1}{2}$

now as $\cos 2\pi/5 > 0$

$\cos 2\pi/5 = \frac{\sqrt{5}-1}{2}$.

Question 5

a) i) $\frac{x^2}{25} + \frac{y^2}{9} = 1$

$\frac{2x}{25} + \frac{2y}{9}\frac{dy}{dx} = 0$

$\frac{dy}{dx} = \frac{-9x}{25y}$

at $P(x_1, y_1)$ $y' = \frac{-9x_1}{25y_1}$

eqn of tangent

$y - y_1 = \frac{-9x_1}{25y_1}(x - x_1)$

$25yy_1 - 25y_1^2 = -9x_1x + 9x_1^2$

$9x_1x + 25y_1y = 9x_1^2 + 25y_1^2$

as P lies on ellipse then

$\frac{x_1^2}{25} + \frac{y_1^2}{9} = 1$

$9x_1^2 + 25y_1^2 = 225$

$\therefore$ eqn of tangent

$9x_1x + 25y_1y = 225$

ii) gradient of PS = gradient SQ

$m_{PS} = \frac{y_1 - 0}{x_1 - 4}$

$m_{QS} = \frac{0 - y_2}{4 - x_2}$

equating

$\frac{y_1}{x_1-4} = \frac{y_2}{x_2-4}$

$y_1x_2 - 4y_1 = y_2x_1 - 4y_2$

$4(y_1 - y_2) = x_1y_2 - x_2y_1$

iii) tangent @ P $9x_1x + 25y_1y = 225$ — 1

tangent @ Q $9x_2x + 25y_2y = 225$ — 2

Solving simultaneously

① × y_2 and ② × y_1

$9x_1xy_2 + 25y_1y_2y = 225y_2$ — 3

$9x_2xy_1 + 25y_1y_2y = 225y_1$ — 4

③ − ④

$9x(x_1y_2 - x_2y_1) = 225(y_2 - y_1)$

$x = \frac{225(y_2 - y_1)}{9(x_1y_2 - x_2y_1)}$

$\quad = 25 \cdot \frac{1}{4}$ (from part ii)

$= 25/4$

$\therefore$ Tangent at P and Q meet on the line $x = 25/4$

eqn of directrix corresponding to $S(4,0)$

$x = a/e$

$b^2 = a^2(1-e^2)$

$9 = 25(1-e^2)$

$e^2 = 16/25$

$e = 4/5$

directrix $x = \frac{5}{4/5} = 25/4$

$\therefore$ tangents meet on the directrix

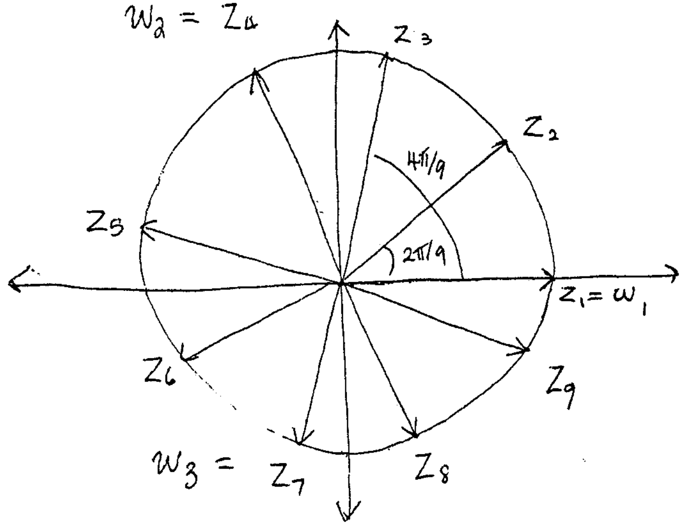
b) i) $\frac{z^9-1}{z^3-1} = \frac{(z^3)^3-1}{z^3-1}$

$= \frac{(z^3-1)(z^6+z^3+1)}{(z^3-1)}$

$= z^6+z^3+1$

ii) $z^9-1=0$ has 9 solutions equally spaced around the unit circle starting at $z=1$ and $2\pi/9$ radians apart

$w^3-1=0$ has 3 solutions " " " " and $2\pi/3$ radians apart.



iii) $6\pi/9 = 2\pi/3 \quad \therefore \quad w_2 = z_4$

$12\pi/9 = 4\pi/3 \quad \therefore \quad w_3 = z_7$

$z_1 = w_1$

and $\overline{z_2} = z_9$, $\overline{z_3} = z_8$, $\overline{z_5} = \overline{z_6}$

so z^6+z^3+1

$= (z-z_2)(z-\overline{z_2})(z-z_3)(z-\overline{z_3})$

$(z-z_5)(z-\overline{z_5})$

note $(z-w)(z-\overline{w})$

$= (z^2 - 2z\mathrm{Re}(w) + 1)$

so z^6+z^3+1

$= (z^2 - 2\cos(\frac{2\pi}{9})z + 1) \times$

$(z^2 - 2\cos(4\pi/9)z + 1) \times$

$(z^2 - 2\cos(8\pi/9)z + 1)$

iv) equating co-efficients of z^2

$0 = (1+1+1+2\cos 2\pi/9 \cdot 2\cos 4\pi/9 + 2\cos 4\pi/9 \cdot 2\cos 8\pi/9 + 2\cos 8\pi/9 \cdot 2\cos 2\pi/9)$

$0 = 3 + 4(\cos 2\pi/9 \cos 4\pi/9 + \cdots$

$-\frac{3}{4} = \cos 2\pi/9 \cos 4\pi/9 + \cos 4\pi/9 \cos 8\pi/9 + \cos 8\pi/9 \cos 2\pi/9$