St Catherine's School

Year: 12
Subject: Extension II Mathematics
Time Allowed: 2 Hours
plus 5 min reading time
Date: April 2005

Exam number: 15227508

Directions to candidates:

- All questions are to be attempted.
- All questions are of equal value.
- All necessary **working** must be shown in every question.
- Full marks may not be awarded for careless or badly arranged work.
- Each question attempted should be started on a **new page.**
- Approved calculators and geometrical instruments are required.
- Hand up your questions in one e bundle and include the exam paper

TEACHER'S USE ONLY Total Marks	
Q1	15
Q2	16
Q3	16
Q4	$14\frac{1}{2}$
Q.5	15
TOTAL	$76\frac{1}{2}$/80

2nd: 65
$58\frac{1}{2}$

V good.

Question 1.

(a) If w is a complex root of $z^3 = 1$,

(i) show that w^2 is also a root (1m)

(ii) Show that $1 + w + w^2 = 0$ (1m)

(iii) Simplify $(1 - w^{-2})(1 - w^{-1})(1 - w^4)(1 - w^8)$ (2m)

(b) Consider the ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$

(i) Find the eccentricity. (1m)

(ii) Find the coordinates of the foci (1m)

(iii) State the equations of the directrices. (1m)

(iv) Sketch the ellipse showing the above features. (1m)

(v) Show clearly the position of the point R with coordinates

$(3\cos 60^\circ, 2\sin 60^\circ)$ $a\cos 60, b\sin\theta$ (1m)

(vi) Show that for any point P on the ellipse PS+PS'=6,
where S and S are the foci. (2m)

check (vii) Find the equation of the tangent to this ellipse at the point R. (2m)

(c) Find the angle between the asymptotes to a hyperbola, whose eccentricity is 2. (3m)

Question 2

(a) Let z = $\sqrt{3} + i$

(i) Write z in modulus Argument form. (2m)

(ii) Find the least positive integer value of n such that z^n is real

(2m)

(b) Solve for x: $\tan^{-1} x + \tan^{-1}(1-x) = \tan^{-1}\frac{9}{7}$ (3m)

(c) OAB is an equilateral triangle, where O is the origin.

Point A represents the complex number $2cis\frac{\pi}{4}$.

(i) Explain why B represents the Complex number $2cis\frac{7\pi}{12}$ (2m)

(ii) OD bisects the angle AOB meeting AB at D. Find D in Modulus Argument form.

(3m)

(Sketching the figure would help)

(d) α, β *and* γ are the roots of the polynomial equation $x^3 + 3x^2 + bx - 15 = 0$ and further $\alpha\beta = 3$

(i) Write the relations between the roots and coefficients. (1m)

(ii) Hence or otherwise, show that b=-37 (3m)

Question 3.

(a) $P(cp,\frac{c}{p})$ is a point on the rectangular Hyperbola $xy = c^2$

(i) Show that the equation of the ~~tangent~~ to the Hyperbola at the point P is $x + p^2y = 2cp$ (2m)

(ii) This tangent meets the x axis at M and the y axis at N. Find the coordinates of M and N. (2m)

(iii) Q is a point on the Number Plane so that OMQN is a rectangle. Find the locus of the point Q. (2m)

(b) Find the coordinates of the foci to the Hyperbola xy=1. (note that the eccentricity of the hyperbola is $\sqrt{2}$) (3m)

(c) (i) The line y=mx+c is a tangent to the Hyperbola H: $\frac{x^2}{16} - \frac{y^2}{9} = 1$

Show that $16m^2 - c^2 = 9$

(3m)

check

(ii) $y = m_1x + c_1$ *and* $y = m_2x + c_2$ are tangents drawn from the point $(1,\sqrt{6})$ to the Hyperbola, show that m_1 *and* m_2 are the roots of the equation $15m^2 + 2\sqrt{6}m - 15 = 0$

(3m)

(iii) Deduce that these tangents are at right angles

(1m)

$$m = \frac{-2\sqrt{6} \pm \sqrt{(2\sqrt{6})^2 - 4(15)(-15)}}{2(15)}$$

$=$

Question 4

(a) If $z = \cos\theta + i\sin\theta$,

(i) show that $z^n + \frac{1}{z^n} = 2\cos n\theta$, where n is an integer.

(1m)

(ii) Hence show that $\cos^5\theta = \frac{1}{16}(\cos 5\theta + 5\cos 3\theta + 10\cos\theta)$

(3m)

(b) Suppose that k is a double root of the polynomial P(x)=0.
Show that $P'(k)=0$

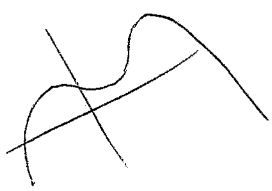
(1m)

(i) The polynomial $ax^5 + bx^3 + 6 = 0$ has a double root at x=1.
Find the values of a and b.

(2m)

(ii) Let $G(x)=1+x+\frac{x^2}{2}+\frac{x^3}{6}+\frac{x^4}{24}$. Show that G(x)=0 cannot have double roots.

(3m)



double root when $G'(x)=0$

(c) Find the eccentricity of the Hyperbola $\frac{x^2}{4}-\frac{y^2}{9}=1$

(2m)

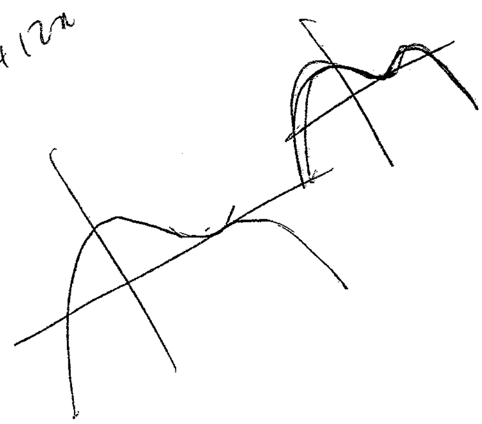
(d) Write $\frac{x^3+1}{(x-1)(x+2)}$ as sum of partial Fractions

(4m)

check

$24+24x+12x^2 \times 4x^3+x^4$

ans



Question 5

(a) (i) On an Argand Diagram, sketch clearly the locus of z :
$|z-(4+3i)|=5$ (1m)

(iv) P represents the point on the locus, which has the maximum modulus.
Find the value of this modulus and also find the Argument of the complex number represented by the point P. (3m)

(iii) If Q represents the point on the locus, which has an argument of $\tan^{-1}2$, find the modulus of Q. (2m)

(b) (i) Use De Moivre's theorem and the expansion of $(\cos\theta+i\sin\theta)^5$ to express $\cos 5\theta$ *and* $\sin 5\theta$ in powers of $\cos\theta$ *and* $\sin\theta$
(3m)

(ii) Hence show that $\tan 5\theta=\dfrac{5\tan\theta-10\tan^3\theta+\tan^5\theta}{1-10\tan^2\theta+5\tan^4\theta}$ (2m)

(iii) Show that $\tan\frac{\pi}{5}$, $\tan\frac{2\pi}{5}$, $\tan\frac{3\pi}{5}$ *and* $\tan\frac{4\pi}{5}$ are the roots of the equation $x^4-10x^2+5=0$ (3m)

√ (c) Show that $\tan(\sin^{-1}x)=\dfrac{x}{\sqrt{1-x^2}}$ (2m)

END OF PAPER

Question 1. 15227508

a. i) $z^3 = \text{cis}\,0$

$z = (\text{cis}\,0)^{\frac{1}{3}}$

$= (\text{cis}\,2k\pi)^{\frac{1}{3}}$ where k is an integer.

$= \text{cis}\,\frac{2k\pi}{3}$, using De Moivre's

~~where k=~~

$z_0 = \text{cis}\,0$

$z_1 = \text{cis}\,\frac{2\pi}{3} = w$

$z_2 = \text{cis}\,\frac{4\pi}{3}$

$w = \text{cis}\,\frac{2\pi}{3}$

$w^2 = \left(\text{cis}\,\frac{2\pi}{3}\right)^2$

$= \text{cis}\,\frac{4\pi}{3}$, using De Moivre's

$= z_2$ ✓

$\therefore w^2$ is also a root.

ii) roots: $1, w, w^2$

sum of roots:

$1 + w + w^2 = -\frac{b}{a}$ ✓

$1 + w + w^2 = 0$

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iii) $\left(1 - \frac{1}{w^2}\right)\left(1 - \frac{1}{w}\right)(1-w)(1-w^2)$

$\left(\frac{w^2-1}{w^2}\right)\left(\frac{w-1}{w}\right)(1-w)(1-w^2)$

$\left(\frac{w^3 - w^2 - w + 1}{w^3}\right)\left(1 - w^2 - w + w^3\right)$

now $w^3 = 1$

$0 = 1 + w + w^2$

$\therefore \left(\frac{1 - w^2 - w + 1}{1}\right)\left(1 - w^2 - w + 1\right)$

$= \left(3 - (1 + w + w^2)\right)\left(3 - (1 + w + w^2)\right)$

$= 3 \times 3$

$= \underline{9}$ ✓✓

4/

b. i) $\frac{x^2}{9} + \frac{y^2}{4} = 1$

$a^2 = 9 \quad ; a = \pm 3$

$b^2 = 4 \quad ; b = \pm 2$

$b^2 = a^2(1 - e^2)$

$4 = 9(1 - e^2)$

$\frac{4}{9} = 1 - e^2$

$e^2 = \frac{5}{9}$

$e = \frac{\sqrt{5}}{3} \quad (e > 0)$ ✓

ii) foci $(\pm ae, 0)$

$= (\pm \cancel{3} \frac{\sqrt{5}}{\cancel{3}}, 0)$ ✓

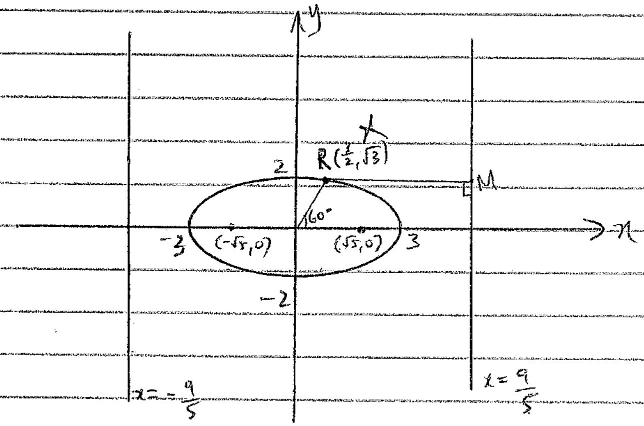
$= (\pm \sqrt{5}, 0)$

iii) directrices: $x = \pm \frac{a}{e}$

$= \pm 3 \div \frac{\sqrt{5}}{3}$

$= \pm \frac{9}{5}$ ✓

iv)



✓

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v) $(3\cos 60, 2\sin 60)$

$= \left(3\left(\frac{1}{2}\right), \frac{\cancel{2}\sqrt{3}}{\cancel{2}}\right)$

$= \left(\frac{3}{2}, \sqrt{3}\right)$

vi) By def'n:

$PS = e\,PM$

$PS = e\left(\frac{a}{e} - a\cos\theta\right)$

$= a - ae\cos\theta \quad — ①$

$PS' = e\,PM'$

$= e\left(\frac{a}{e} + a\cos\theta\right)$

$= a + ae\cos\theta \quad — ②$

① + ②: $PS + PS' = 2a$ ✓✓

$= 2(3)$

$= 6$

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vii) $\frac{2x}{9} + \frac{2y\frac{dy}{dx}}{4} = 0$

$\therefore \frac{dy}{dx} = -\frac{\cancel{2}x}{9} \times \frac{4}{\cancel{2}y}$

$= \frac{-4x}{9y}$

At $R(3\cos 60^\circ, 2\sin 60^\circ)$

$\frac{dy}{dx} = \frac{-\cancel{2}4(\cancel{3}\cos 60)}{\cancel{3}9(\cancel{2}\sin 60)}$

$= \frac{-2\cos 60}{3\sin 60}$

$= -2\left(\frac{1}{2}\right) \div 3\left(\frac{\sqrt{3}}{2}\right)$

$= -1 \times \frac{2}{3\sqrt{3}}$

$= -\frac{2}{3\sqrt{3}}$

$\therefore$ tgt: $y - y_1 = m(x - x_1)$

$y - 2\sin 60 = \frac{-2}{3\sqrt{3}}(x - 3\cos 60)$ ✓

$3\sqrt{3}y - 6\sqrt{3}\sin 60 = -2x + 6\cos 60$

$3\sqrt{3}y - 9 = -2x + 3$ $\quad$ 8/9

tgt: $0 = 2x + 3\sqrt{3}y - 12$

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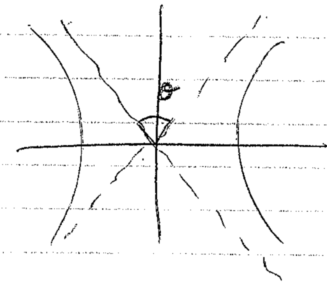
c. asymptotes: $y = \pm \frac{b}{a}x$



$b^2 = a^2(e^2 - 1)$

$= a^2(4-1)$

$= 3a^2$

~~$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$~~

$\therefore b = \pm\sqrt{3}a$ ✓

$\therefore$ asymptotes:

$y = \pm \frac{\sqrt{3}\cancel{a}}{\cancel{a}} x$

$= \pm\sqrt{3}x$

$\therefore m_1 = \sqrt{3}$

$m_2 = -\sqrt{3}$

$\tan\theta = \left|\frac{m_2 - m_1}{1 + m_1 m_2}\right|$

$= \left|\frac{-\sqrt{3} - \sqrt{3}}{1 + (\sqrt{3})(-\sqrt{3})}\right|$

$= \left|\frac{-2\sqrt{3}}{1-3}\right|$

$= \left|\frac{-\cancel{2}\sqrt{3}}{-\cancel{2}}\right|$ ✓ $\quad$ 3/

$= \sqrt{3}$

$\therefore \theta = 60^\circ$

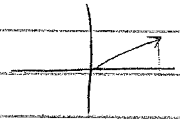
$\therefore \angle$ between asymptotes is 60° $\quad$ 15/16

Question 2. 15227508

a. i) $z = \sqrt{3} + i$

$|z| = \sqrt{(\sqrt{3})^2 + 1^2}$

$= 2$



$\arg z = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$

$= \frac{\pi}{6}$

2/

$\therefore z = 2 \operatorname{cis} \frac{\pi}{6}$ ✓

ii) $z^n = \left(2 \operatorname{cis} \frac{\pi}{6}\right)^n$

$= 2^n \operatorname{cis} \frac{n\pi}{6}$, ~~where n ε~~ using De Moivre's

$z^1 = 2 \operatorname{cis} \frac{\pi}{6}$

$z^2 = \cancel{4 \operatorname{cis} \frac{\pi}{6}}\ 4 \operatorname{cis} \frac{2\pi}{6}$

$z^3 = \cancel{8 \operatorname{cis} \frac{\pi}{3}}\ 8 \operatorname{cis} \frac{3\pi}{6}$

$z^4 = 2^4 \operatorname{cis} \frac{4\pi}{6}$

2/

$z^5 = 2^5 \operatorname{cis} \frac{5\pi}{6}$ ✓

$z^6 = 2^6 \operatorname{cis} \frac{6\pi}{6} = 64 \operatorname{cis} \pi$

$= -64$ which is real

$\therefore$ least +ve integer $n = 6$ for z^n to be real

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b. • let $\theta = \tan^{-1} x$

$\therefore \tan\theta = x$

• let $\alpha = \tan^{-1}(1-x)$

$\therefore \tan\alpha = 1-x$

• $\therefore \theta + \alpha = \tan^{-1}\frac{9}{7}$

$\therefore \tan(\theta+\alpha) = \frac{9}{7}$

$\frac{\tan\theta + \tan\alpha}{1 - \tan\theta\tan\alpha} = \frac{9}{7}$

$\frac{x + 1 - x}{1 - (x)(1-x)} = \frac{9}{7}$

$\frac{1}{1-(x-x^2)} = \frac{9}{7}$

3

$\frac{9}{7} = \frac{1}{1-x+x^2}$

Well done!

$9(x^2 - x + 1) = 7$

$9x^2 - 9x + 9 - 7 = 0$

$9x^2 - 9x + 2 = 0$

$x = \frac{9 \pm \sqrt{9^2 - 4(9)(2)}}{2(9)}$

$= \frac{9 \pm 3}{18} = \frac{2}{3}, \frac{1}{3}$

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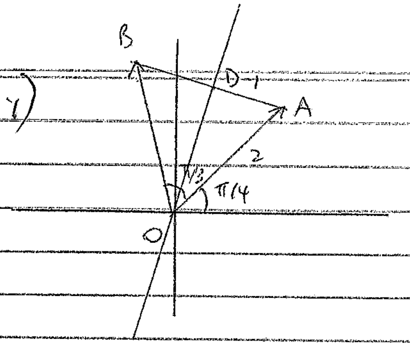
c. i)



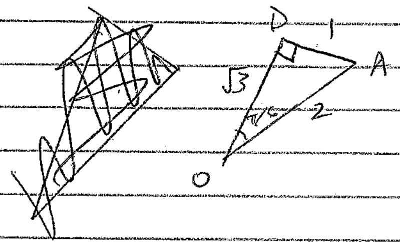
~~BA~~ To rotate A anticlockwise by $\frac{\pi}{3}$ to ~~find B:~~

$B = \text{cis}\,\frac{\pi}{3} \times A$

$= \left(\text{cis}\,\frac{\pi}{3}\right)\left(2\,\text{cis}\,\frac{\pi}{4}\right)$

$= 2\,\text{cis}\,\frac{7\pi}{12}$ ✓

ii)



$\tan\frac{\pi}{6} = \frac{1}{OD}$

$\frac{1}{\sqrt{3}} = \frac{1}{OD}$

$\therefore OD = \sqrt{3}$ ✓

$\therefore D = \sqrt{3}\,\text{cis}\left(\frac{\pi}{4} + \frac{\pi}{6}\right)$

$= \sqrt{3}\,\text{cis}\,\frac{5\pi}{12}$

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d. i) Let $P(x) = x^3 + 3x^2 + 6x - 15 = 0$

roots: α, β, γ

$\alpha\beta = 3$ — (4)

~~product of roots:~~

~~$\alpha\beta\gamma = 15$~~

~~$\therefore \gamma = 5$~~

~~$\alpha + \beta + \gamma = -3$~~

~~$\alpha\beta + \beta\gamma + \alpha\gamma = 6$~~

~~$3 + \beta\gamma + \alpha\gamma = 6$~~

$\alpha + \beta + \gamma = -3$ — (1)

$\alpha\beta + \beta\gamma + \alpha\gamma = 6$ — (2)

$\alpha\beta\gamma = 15$ — (3)

ii) From (3) and (4)

$3\gamma = 15$

$\therefore \gamma = 5$ — (5) ✓

Sub (5) into (1)

$\alpha + \beta + 5 = -3$

$\alpha + \beta = -8$ — (6)

Sub (4) and (5) into (2)

$3 + \gamma(\alpha + \beta) = 6$

$3 + 5(\alpha + \beta) = 6$

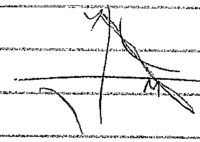
Sub (6) in

$3 + 5(-8) = 6$

16/16

Question 3. 15227508

a. i) $xy = c^2$



Diff:

$x\frac{dy}{dx} + y = 0$

$\therefore \frac{dy}{dx} = \frac{-y}{x}$

At P: $\frac{dy}{dx} = -\frac{\cancel{c}}{p} \times \frac{1}{\cancel{c}p}$

$= -\frac{1}{p^2}$

tgt: $y - y_1 = m(x - x_1)$

$y - \frac{c}{p} = \frac{-1}{p^2}(x - cp)$

$p^2 y - cp = -x + cp$ ✓

$0 = x + p^2 y - 2cp$

$2cp = x + p^2 y$ as req'd.

ii) M when $y = 0$

$2cp = x$

$\therefore M(2cp, 0)$

N when $x = 0$

$2c\cancel{p} = p^{\cancel{2}} y$

$y = \frac{2c}{p}$ $\therefore N(0, \frac{2c}{p})$

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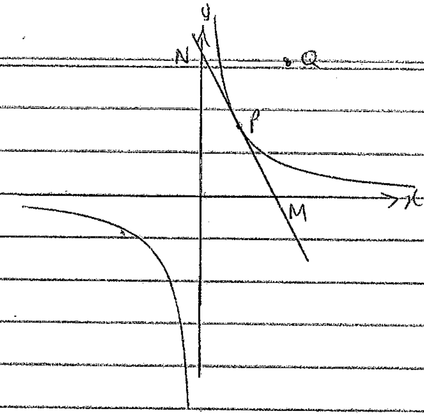
iii)



$Q(2cp, \frac{2c}{p})$

$\therefore x = 2cp$ — ①

$y = \frac{2c}{p}$ — ②

~~From ①~~

From ②

$p = \frac{2c}{y}$ — ③

Sub ③ into ①

$x = 2c\left(\frac{2c}{y}\right)$ $x^{2^{2^4}}$!!

$= \frac{\cancel{2}^{4}c^2}{y}$

$\therefore xy = \cancel{2}^{4}c^2$

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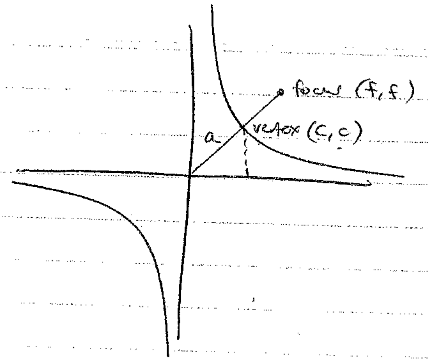
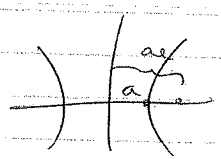
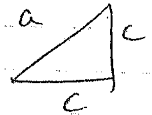
b.



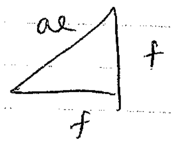
$\boxed{xy=1}$ $\quad \therefore c^2=1$

$c=1$





$$c^2+c^2=a^2$$
$$1+1=a^2$$
$$2=a^2$$
$$a=\pm\sqrt{2}$$



$$f^2+f^2=(ae)^2$$
$$=a^2e^2$$
$$2f^2=2\times 2$$
$$=4$$
$$f^2=2$$
$$\therefore f=\pm\sqrt{2}$$

$\therefore$ focus: $(\sqrt{2},\sqrt{2})$ and $(-\sqrt{2},-\sqrt{2})$

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c. $y=mx+c$ — ①

$\frac{x^2}{16}-\frac{y^2}{9}=1$ — ②

Solve simult:

$$\frac{x^2}{16}-\frac{(mx+c)^2}{9}=1$$

$$\cancel{\frac{x^2}{16}-\frac{(m^2x^2+2mxc+c^2)}{9}=1}$$

$$\frac{x^2}{16}-\frac{m^2x^2+2mxc+c^2}{9}=1$$

$$9x^2-16m^2x^2-32mxc-16c^2=144$$

$\therefore$ ~~16m²x²+9~~

$$16m^2x^2-9x^2+32mxc+16c^2+144=0$$

if tgt, $\therefore \Delta=0$

$$\Delta=b^2-4ac=0$$
$$0=(32mc)^2-4(16m^2-9)(16c^2+144)$$
$$=1024m^2c^2-(64m^2-36)(16c^2+144)$$
$$=1024m^2c^2-(1024m^2c^2+9216m^2-576c^2-5184)$$
$$=-9216m^2+576c^2+5184$$

(÷ by 576) $\quad 0=-16m^2+c^2+9$

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ii) when line is tgt to H:

$$16m^2 - c^2 = 9 \quad \text{—— ①}$$

$l_1 : y = m_1 x + c_1$

$l_2 : y = m_2 x + c_1$

With l_1:

$$y = m_1 x + c_1$$

Sub pt in: $\sqrt{6} = m_1 + c_1$

$\therefore c_1 = \sqrt{6} - m_1$ — ②

Because l_1 is a tgt, sub ② into ①

From ① $c_1^2 = 16m_1^2 - 9$

$\therefore (\sqrt{6} - m_1)^2 = 16m_1^2 - 9$

$6 - 2\sqrt{6}m_1 + m_1^2 = 16m_1^2 - 9$

$$0 = 15m_1^2 + 2\sqrt{6}m_1 - 15$$

Similarly,
With l_2:

$\therefore 0 = 15m_2^2 + 2\sqrt{6}m_2 - 15$

$\therefore m_1$ and m_2 are roots of:

$$0 = 15m^2 + 2\sqrt{6}m - 15$$

$\therefore m_1 m_2 =$ product of roots $= -1$

From quadratic:

iii) $m = \dfrac{-2\sqrt{6} \pm \sqrt{(2\sqrt{6})^2 - 4(15)(-15)}}{2(15)}$

$= \dfrac{-2\sqrt{6} \pm \sqrt{924}}{30}$

$= \dfrac{-2\sqrt{6} \pm 2\sqrt{231}}{30}$

$= \dfrac{-\sqrt{6} \pm \sqrt{231}}{15}$

$\therefore m_1 = \dfrac{-\sqrt{6} + \sqrt{231}}{15}$, $m_2 = \dfrac{-6 - \sqrt{231}}{15}$

$m_1 \times m_2$

$= \left(\dfrac{-\sqrt{6} + \sqrt{231}}{15}\right)\left(\dfrac{-\sqrt{6} - \sqrt{231}}{15}\right)$

$= \dfrac{6 - 231}{225}$

$= \dfrac{-225}{225}$

$= -1$

$m_1 \times m_2 = -1$

$\therefore m_1 \perp m_2$

16/16.

Question 4. 15227508

a. i) $z^n + \frac{1}{z^n}$

$= (\text{cis}\,\theta)^n + \frac{1}{(\text{cis}\,\theta)^n}$

$= \text{cis}\,n\theta + \frac{1}{\text{cis}\,n\theta}$ (using De Moivre's)

$= \text{cis}\,n\theta + \text{cis}(-n\theta)$

$= \cos n\theta + i\sin n\theta + \cos(-n\theta) + i\sin(-n\theta)$

$= \cos n\theta + \cancel{i\sin n\theta} + \cos n\theta - i\sin n\theta$

$= 2\cos n\theta$

$= LHS$

$\therefore z^n + \frac{1}{z^n} = 2\cos n\theta$

ii) $z + \frac{1}{z} = 2\cos\theta$

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1

$\left(z+\frac{1}{z}\right)^5 = z^5 + 5z^4\frac{1}{z} + 10z^3\frac{1}{z^2} + 10z^2\frac{1}{z^3} + 5z\frac{1}{z^4} + \frac{1}{z^5}$

$(2\cos\theta)^5 = \left(z^5 + \frac{1}{z^5}\right) + (5)\left(z^3 + \frac{1}{z^3}\right) + 10\left(z + \frac{1}{z}\right)$

$32\cos^5\theta = 2\cos 5\theta + 5(2\cos 3\theta) + 10(2\cos\theta)$

$= 2\cos 5\theta + 10\cos 3\theta + 20\cos\theta$

$\therefore \cos^5\theta = \frac{1}{16}(\cos 5\theta + 5\cos 3\theta + 10\cos\theta)$

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b. $P(x) = 0$

$P(x) = (x-k)^2 Q(x)$

$P'(x) = uv' + vu'$

$= (x-k)^2 Q'(x) + Q(x)\, 2(x-k)$

$= (x-k)\left((x-k)Q'(x) + 2Q(x)\right)$

$P'(k) = (k-k)\left((x-k)Q'(x) + 2Q(x)\right)$

$= 0$

i) $P(x) = ax^5 + bx^3 + 6$

$P'(x) = 5ax^4 + 3bx^2$

double root at 1:

$P(1) = 0$

$\therefore 0 = a + b + 6$ — ①

$P'(1) = 0$

$\therefore 0 = 5a + 3b$ — ②

cont'd

From ①

$\cancel{-6 = a+b}$

$a = -6 - b$ — ③

sub ③ into ②

$0 = 5(-6-b) + 3b$

$= -30 - 5b + 3b$

$30 = -2b$

$\therefore b = -15$ — ④

sub ④ into ③

$a = -6 + 15$

$= 9$

2

$\therefore \boxed{\begin{array}{l} a = 9 \\ b = -15 \end{array}}$

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ii) $G(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24}$

$\cancel{G'(x) = 1 + 2x}$

$G'(x) = 1 + \frac{2x}{2} + \frac{3x^2}{6} + \frac{4x^3}{24}$

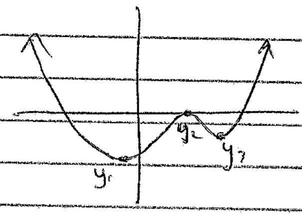
$= 1 + x + \frac{x^2}{2} + \frac{x^3}{6}$

If $G(x)=0$ has double roots,

$\therefore G'(x) = 0$

$0 = 1 + x + \frac{x^2}{2} + \frac{x^3}{6}$

$= 6 + 6x + 3x^2 + x^3$



If $G(x)$ has double roots,

$y_1 \times y_2 \times y_3 = 0$

However, $G'(x) \neq 0$

$\therefore G(x)$ doesn't have any double roots.

(next page working out)

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b. ii)

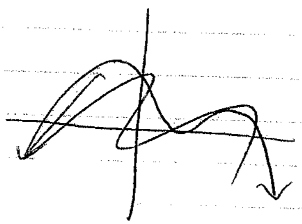
$$G'(x) = \cancel{4x + x + \frac{x^2}{2} + \frac{x^3}{6}}$$

$$\cancel{= 6x + 12}$$

$$= 1 + \frac{2x}{2} + \frac{3x^2}{6} + \frac{4x^3}{24}$$

$$= 1 + x + \frac{x^2}{2} + \frac{x^3}{6}$$

$$= 6 + 6x + 3x^2 + x^3$$

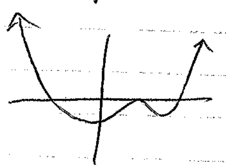


to have double root,

~~st. pts~~

y values st. pts multiplied together $= 0$

?



$$G'(x) = x^3 + 3x^2 + 6x + 6$$

solve: test $P(1) \neq 0$

$P(-1) \neq 0$

$P(2) \neq 0$

$P(-2) \neq 0$

$P(3) \neq 0$

$P(-3) \neq 0$

$P(6) \neq 0$

$P(-6) \neq 0$

$\therefore \; G'(x) \neq 0$

1/2 /3

c. $\frac{x^2}{4} - \frac{y^2}{9} = 1$

$a^2 = 4$

$b^2 = 9$

$$a^2 = b^2(e^2 - 1)$$

$$4 = 9(e^2 - 1)$$

$$\frac{4}{9} = e^2 - 1$$

$$e^2 = \frac{13}{9}$$

$$e = \frac{\sqrt{13}}{3} \quad (e > 0)$$

d. $\frac{x^3+1}{(x-1)(x+2)} = \frac{x^3+1}{(x^2+x-2)}$

$$\begin{array}{r|l} & x - 1 \\ x^2+x-2 & x^3 + 0x^2 + 0x + 1 \\ & x^3 + x^2 - 2x \\ \hline & -x^2 + 2x + 1 \\ & -x^2 - x + 2 \\ \hline & 3x - 1 \end{array}$$

$$\therefore \frac{x^3+1}{(x-1)(x+2)} = (x-1) + \frac{3x-1}{(x-1)(x+2)}$$

cont'd

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$\cancel{x^3+1 \equiv (x-1)(x-1)(x}$

$$\frac{x^3+1}{(x-1)(x+2)} = (x-1) + \frac{A}{x-1} + \frac{B}{x+2}$$

$$x^3+1 = (x-1)^2(x+2) + A(x+2) + B(x-1)$$

let $x=1$

$$2 = 3A$$

$$\boxed{\therefore A = \frac{2}{3}}$$

let $x=-2$

$$-7 = B(-3)$$

$$\boxed{\frac{7}{3} = B}$$

✓ 4

$$\therefore \frac{x^3+1}{(x-1)(x+2)} = (x-1) + \frac{2}{3(x-1)} + \frac{7}{3(x+2)}$$

$\frac{14\frac{1}{2}}{16}$

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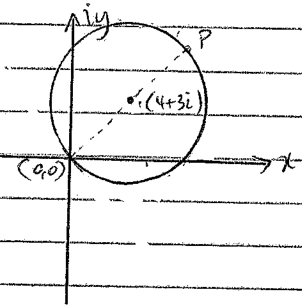
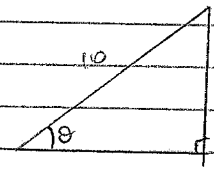
Question 5.

a. i) $|z-(4+3i)| = 5$



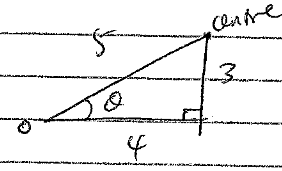
✓ 1

iv) From diagram, P has max modulus



✓

$\therefore \max |z| = 10$



$\tan\theta = \frac{3}{4}$

3 ✓

$\therefore \theta = 36°52'11.63''$

$= 36°52'$ (nearest min)

$\therefore$ arg of P : $36°52'$

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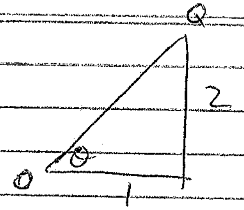
iii)



$\tan\theta = 2$

$\therefore \theta = 63°26'5.82''$

$\sin\theta = \frac{2}{OQ}$

$\therefore OQ = 2.236..$

$= 2.2$ (2dp)

$\therefore |Q| = 2.2$

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1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 1

b. i) $(\cos\theta + i\sin\theta)^5 = (c + is)^5$

$= \cos 5\theta + i\sin 5\theta$ (using de Moivre's

$\therefore \cos 5\theta + i\sin 5\theta = c^5 + 5c^4(is) + 10c^3(is)^2 + 10c^2(is)^3 + 5c(is)^4 + (is)^5$

Equate real:

$\cos 5\theta = \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta$

Equate imaginary:

$\sin 5\theta = 5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta$ 3

ii) $\tan 5\theta = \frac{\sin 5\theta}{\cos 5\theta}$

$= \frac{5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta}{\cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta}$

(÷ by $\cos^5\theta$)

$\tan^5\theta = \frac{5\tan\theta - 10\tan^3\theta + \tan^5\theta}{1 - 10\tan^2\theta + 5\tan^4\theta}$ 2

as req'd

iv) let $\tan 5\theta = 0$

$$\therefore 0 = \frac{5\tan\theta - 10\tan^3\theta + \tan^5\theta}{1 - 10\tan^2\theta + 5\tan^4\theta}$$

$$= \tan\theta(5 - 10\tan^2\theta + \tan^4\theta) \quad \text{—— ①}$$

Now,

$\tan 5\theta = 0$

$5\theta = 0, \pi, 2\pi, 3\pi, 4\pi, 5\pi, 6\pi, 7\pi, 8\pi, 9\pi$

$\theta = 0, \frac{\pi}{5}, \frac{2\pi}{5}, \frac{3\pi}{5}, \frac{4\pi}{5}, \pi, \frac{6\pi}{5}, \frac{7\pi}{5}, \frac{8\pi}{5}, \frac{9\pi}{5}$

~~From ① $\tan\theta$ has roots $0, \pi$~~

~~$\therefore (5 - \tan^2\theta + \tan^4\theta)$~~

From ①:

let $x = \tan\theta$

$$0 = \underset{\substack{\downarrow \\ \text{roots} \\ \tan 0,\ \tan\pi}}{x}\underbrace{(5 - 10x^2 + x^4)}_{\text{rest of roots}}$$

$\therefore$ In eqn $0 = x^4 + 10x^2 - 5$,

roots are $\tan\frac{\pi}{5}, \tan\frac{2\pi}{5}, \tan\frac{3\pi}{5}, \tan\frac{4\pi}{5}$,

$\tan\frac{6\pi}{5}, \tan\frac{7\pi}{5}, \tan\frac{8\pi}{5}, \tan\frac{9\pi}{5}$

(cont'd)

now $\tan\frac{\pi}{5} = \tan\frac{6\pi}{5}$

$\tan\frac{2\pi}{5} = \tan\frac{7\pi}{5}$

$\tan\frac{3\pi}{5} = \tan\frac{8\pi}{5}$

$\tan\frac{4\pi}{5} = \tan\frac{9\pi}{5}$

$\therefore \tan\frac{\pi}{5}, \tan\frac{2\pi}{5}, \tan\frac{3\pi}{5}, \tan\frac{4\pi}{5}$

are roots of $x^4 - 10x^2 + 5 = 0$

3

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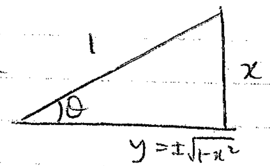
c. let $\sin^{-1} x = \theta$

$\therefore \sin\theta = x$



$$x^2 + y^2 = 1$$

$$y^2 = 1 - x^2$$

$$y = \pm\sqrt{1-x^2}$$

$\therefore \tan\theta = \dfrac{x}{\sqrt{1-x^2}}$, when $\tan\theta$ is positive

$\therefore \tan(\sin^{-1} x) = \dfrac{x}{\sqrt{1-x^2}}$ ✓ 2

$\frac{15}{16}$