St Catherine's School

Year: 12
Subject: Extension I Mathematics
Time allowed: 2 hours
(plus 5 mins reading time)
Date: August 2002

Stephanie Sim

Exam number: 12520204

Directions to candidates:

- All questions are to be attempted.
- All questions are of equal value.
- All necessary **working** must be shown in every question.
- Full marks may not be awarded for careless or badly arranged work.
- Approved calculators and geometrical instruments are required.

- Each **section** should be started on a **new booklet**.
- Hand in your work in **3 bundles**:
  - Section A Questions 1 and 2.
  - Section B Questions. 3 and 4
  - Section C Questions. 5, 6 and 7.

TEACHER'S USE ONLY Total Marks
A
B
C
TOTAL

Section A

Question 1

a) Use the table of standard integrals to find the exact value of $\int_0^1 \frac{dx}{\sqrt{4-x^2}}$ 2

b) Find $\frac{d}{dx}\sin^{-1}\sqrt{1-x}$ 2

c) Evaluate $\sum_{n=3}^{7}(3n-1)$ 1

d) Let A be the point $\overset{x_1\ y_1}{(-6,2)}$ and let B be the point $\overset{x_2\ y_2}{(4,7)}$. Find the coordinates of the point P, which divides the interval AB externally in the ratio $\underset{k\ l}{3:2}$. 2

e) Is $x-2$ a factor of $x^3+3x-14$? Give reason for your answer. 2

f) Use the substitution $u=x^2-1$ to evaluate $3\int_1^2 x\sqrt{x^2-1}\ dx$ 3

Question 2

a) Let $f(x)=2x^2+x$. Use the definition $f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$ to find the derivative of $y=f(x)$. 2

b) i) Find $\int \frac{e^{2x}}{3+e^{2x}}\,dx$ 1

ii) Evaluate $\int_0^{\frac{\pi}{2}} \sin^2(\tfrac{1}{2}x)\ dx$ 3

c) i) Write down the expansion of tan(A+B) 1

ii) Hence find the value of tan 105° in simplest surd form. 2

d) Solve for x; $\frac{4}{5-x}\geq 1$ 3

Section B (Start a new booklet)

Question 3

2

a) Write the expansion of $(2x-y)^5$

b) Find the term independent of x in the binomial expansion $(x^2+\frac{2}{x})^6$ 3

c) The function $f(x)=x-2\sin x$ has a zero near x = 1.7. Use one application of Newton's method to find a second approximation to the zero. Write your answer correct to three significant figures. 3

d) A smooth piece of ice is projected up a smooth inclined (sloping) surface. Its distance x in metres up the surface is at time t seconds is $x=6t-t^2$. 4

i) Find velocity, v, and acceleration, $\ddot{x}$.

ii) In which direction is the ice moving and in which direction is acceleration when t = 2

iii) Use your answer from (ii) to explain whether the piece of ice is increasing in speed or decreasing in speed when t=2.

iv) Find when and where the piece of ice is stationary.

Question 4

1

a) i) Given that $x^2+4x+13 \equiv (x+a)^2+b^2$, find the values of a and b.

ii) Use the result of (i) to find $\int_{-2}^{1} \frac{1}{x^2+4x+13}\,dx$ 2

b) The volume, V, of a sphere of radius r mm is increasing at a constant rate of $200\,\text{mm}^3$ per second. 4

i) Find $\frac{dr}{dt}$ when r = 50.

ii) Determine the rate of increase of the surface area, S of the sphere when the radius is 50 mm.

$$\left(V=\frac{4}{3}\pi r^3 \qquad S=4\pi r^2\right)$$

c) A particle, whose displacement is x, moves in simple harmonic motion. 5

Find x as a function t if:

$$\ddot{x}=-16x \text{ and } x=\sqrt{3} \text{ and } \dot{x}=12 \text{ when } t=0.$$

Section C (Start a new booklet)

Question 5

a) $P(2p, p^2)$ and $Q(2q, q^2)$ are two points on the parabola $x^2 = 4y$. The variable chord PQ is such that it is always parallel to the line $y = x$.

i) Find the gradient of PQ and hence show that $p + q = 2$ 1

ii) Given that the equation of the *normal* at P is $x + py = 2p + p^3$. Write down the equation of the *normal* at Q, and hence find the coordinates of the point of intersection R, of these normals. 3

iii) Prove that the locus of R is the straight line $x - 2y + 12 = 0$. 2

b) A garden sprinkler is positioned at the centre of a large, flat lawn. Water droplets are projected from the sprinkler at a fixed speed of 20ms and at an angle, ϑ, above the horizontal. The acceleration due to gravity is $10\,ms^{-2}$.

i) Use integration to show that the horizontal displacement x metres and the vertical displacement y metres of the water droplets is after time t seconds are given by 2

$$x = 20t\cos\vartheta \text{ and } y = 20t\sin\vartheta - 5t^2$$

ii) Show that the horizontal range, R, of the water droplets is given by $R = 40\sin 2\vartheta$. 2

iii) The garden sprinkler rotates in a circle to water the lawn. If the angle of projection varies between 15° and 45° above the horizontal, find the exact area of that part of the lawn that can be watered in this way. 2

Question 6

a) The acceleration of an object is given by $a = 12e^{2x}$ m/s/s. If the object leaves the origin with a velocity of $2\sqrt{3}$ m/s and its velocity is always positive, find the displacement when the velocity is 5m/s, correct to two decimal places. 4

b) i) Find the general solution to the equation $\sin 2x = 2\sin^2 x$. 3

ii) Show that if $0 < x < \frac{\pi}{4}$, then $\sin 2x > 2\sin^2 x$ 2

iii) Find the area enclosed between the curves $y = \sin 2x$ and $y = 2\sin^2 x$ for $0 \le x \le \frac{\pi}{4}$ 3

Please turn over for question 7

Question 7

a) In a flock of 1000 chickens, the number P, infected with a disease at time t years is given by $P = \dfrac{1000}{1+ke^{-1000t}}$ where k is a constant.

(i) Show that, eventually, all the chickens will be infected. 1

ii) Suppose that when time $t = 0$, exactly one chicken was infected. After how many days will 500 chickens be infected? 3

b) A particle is moving in a straight line. After time, t seconds it has displacement x metre from a fixed point O on the line, velocity, v ms^{-1} given by $v = \dfrac{1-x^2}{2}$ and acceleration, a ms^{-2}. Initially the particle is at O.

i) Find an expression for acceleration in terms of x. 1

ii) Show that $\dfrac{2}{1-x^2} = \dfrac{1}{1+x} + \dfrac{1}{1-x}$ and hence find an expression for x in terms of t. 3

(iii) Describe the initial conditions for displacement, velocity and acceleration and hence describe the motion of the particle, explaining whether it moves to the left or the right of O and whether it slows down or speeds up. 3

(iv) What is the limiting position of the particle ? 1

End of examination

2002

Stephanie Sim.

Question 1

a) $\int_0^1 \frac{dx}{\sqrt{4-x^2}} = \left[\sin^{-1}\left(\frac{x}{2}\right)\right]_0^1$

$= \sin^{-1}\frac{1}{2} - \sin^{-1}0$

$= \frac{\pi}{6} - 0$

$= \frac{\pi}{6}$ ②

b) $\frac{d}{dx}\sin^{-1}\sqrt{1-x} =$

let $u = \sqrt{1-x} = (1-x)^{1/2}$

$\frac{du}{dx} = -\frac{1}{2}(1-x)^{-1/2}$

let $y = \sin^{-1}u$

$dy/du = \frac{1}{\sqrt{1-u^2}}$

$dy/dx = \frac{du}{dx} \times dy/du$

$= \frac{-1}{2\sqrt{1-x}} \times \frac{1}{\sqrt{1-(1-x)}}$

$= \frac{-1}{2\sqrt{1-x}} \times \frac{1}{\sqrt{x}}$ ②

$= \cancel{\frac{-1}{2\sqrt{x(1-x)}}}$

c) $\sum_{n=3}^{7}(3n-1) = 8 + 11 + 14 + 17 + 20$

$= 70$ ①

d) A $(6,2)$ B$(4,7)$ $m:n = -3:2$ $3:-2$

$x = \frac{mx_2+nx_1}{m+n}$ $y = \frac{my_2+ny_1}{m+n}$

$= \frac{3(4)-2(6)}{3-2}$ $= \frac{3(7)-2(2)}{3-1}$ ②

$= 24$ $= 17$

ie. ~~(24, 17)~~

e) If $(x-2)$ is a factor of $x^3+3x-14$

then when $x=2$ $x^3+3x-14=0$

$P(2) = (2)^3 + 3(2) - 14$

$= 8 + 6 - 14$

$= 0$ ②

$\therefore$ $(x-2)$ is a factor by the factor theorem

f) $3\int_1^2 x\sqrt{x^2-1}\,dx$ $u = x^2-1$

$\frac{du}{dx} = 2x$

$du = 2x\,dx$

$\frac{1}{2}du = x\,dx$

at $x=2$ $u=3$

$x=1$ $u=0$

$= \frac{3}{2}\int_0^3 u^{1/2}\,du$

$= \frac{3}{2}\left[\frac{2}{3}u^{3/2}\right]_0^3$

$= \left[u^{3/2}\right]_0^3$

$= (3)^{3/2}$

$= \sqrt{27}$ ③

$= 3\sqrt{3}$

Question Two

(a) $f(x) = 2x^2 + x$

$f(x+h) = 2(x+h)^2 + (x+h)$

$= 2(x^2 + 2xh + h^2) + (x+h)$

$= 2x^2 + 4xh + 2h^2 + x + h$

$f'(x) = \lim_{h\to 0}\frac{f(x+h) - f(x)}{h}$

$= \lim_{h\to 0}\frac{2x^2+4xh+2h^2+x+h-(2x^2+x)}{h}$

$= \lim_{h\to 0}\frac{4xh + 2h^2 + h}{h}$

$= \lim_{h\to 0}\frac{h(4x+2h+1)}{h}$

$= \lim_{h\to 0} 4x+2h+1$

② $= 4x+1$

(b) i) $\int\frac{e^{2x}}{3+e^{2x}}\,dx$

① $= \frac{1}{2}\ln(3+e^{2x}) + C$

ii) $\int_0^{\pi/2}\sin^2\left(\frac{1}{2}x\right)dx$

$\frac{1}{2}\int 1 - \cos x\,dx$

$= \frac{1}{2}\left[x - \sin x\right]_0^{\pi/2}$

$= \frac{1}{2}\left[\frac{\pi}{2} - 1\right]$

c) i) $\tan(A+B) = \frac{\tan A + \tan B}{1-\tan A\tan B}$

ii) $\tan(105) = \tan(60+45°)$

$= \frac{\tan 60 + \tan 45}{1 - \tan 60 \cdot \tan 45}$

$= \frac{\sqrt{3}+1}{1-\sqrt{3}.1} \times \frac{1+\sqrt{3}}{1+\sqrt{3}} = \frac{4+2\sqrt{3}}{-2}$

$= -2-\sqrt{3}$

d) $\frac{4}{5-x} \geq 1$

METHOD ONE: squaring Both Sides

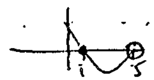
$\frac{4}{5-x}(5-x)^2 \geq (5-x)^2$, $x \neq 5$

$4(5-x) \geq 25 - 10x + x^2$

$20 - 4x \geq 25 - 10x + x^2$

$0 \geq x^2 - 6x + 5$

$0 \geq (x-5)(x-1)$



$1 \leq x < 5$ ③

METHOD TWO Critical Points Method.

$\frac{4}{5-x} \geq 1$

$x \neq 5$

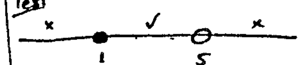
Solve equality

$\frac{4}{5-x} = 1$

$4 = 5-x$

$x = 1$

Test



$\boxed{5 \leq x < 5}$

Question 3

a) $(2x-y)^5 = {}^5C_0(-y)^5(2x)^0 + {}^5C_1(-y)^4(2x)^1 + {}^5C_2(-y)^3(2x)^2 + {}^5C_3(-y)^2(2x)^3 + {}^5C_4(-y)^1(2x)^4 + {}^5C_5(-y)^0(2x)^5$

$= -1y^5 + 5y^4 \times 2x + 10(-y^3)(4x^2) + 10y^2 . 8x^3 + 5(-y)(16x^4) + 1(32 \times 5)$

(2) $= -y^5 + 10xy^4 - 40x^2y^3 + 80x^3y^2 - 80x^4y + 32x^5$

b) $T_{k+1} = {}^nC_k \left(\frac{2}{x}\right)^k (x^2)^{n-k}$ $n=6$

$= {}^6C_k \left(\frac{2}{x}\right)^k (x^2)^{6-k}$ 1

$= {}^6C_k (2^k)(x^{-k}) (x)^{12-2k}$

$= {}^6C_k \, 2^k \, x^{12-3k}$

now $x^{12-3k} = x^0$

$\therefore k=4$

$\therefore T_5$ is the independent term. $\frac{1}{2}$

$T_5 = {}^6C_4 \left(\frac{2}{x}\right)^4 . (x^2)^2$ (1)

$= 15 \, \frac{16}{x^4} . x^4$ 1

(3) $= 15 \times 16$

$= \underline{240}$

$\frac{1}{2}$ formula' c) $f(x) = x - 2\sin x$ has a root near 1.7

$f'(x) = 1 - 2\cos x$ ($\frac{1}{2}$)

$\frac{1}{2}$

$\frac{1}{2}$ (3) $x = 1.7 - \frac{f(1.7)}{f'(1.7)}$ ($\frac{1}{2}$)

$= 1.7 + \left.\frac{1.64\ldots}{0.99\ldots}\right\}$ if in degrees!

$= \cancel{3.34}$ 1.93 ($\frac{1}{2}$)

$= \boxed{1.92527.}$ $= 1.93$ (3 sig figs)

1.7 is in radians

($\frac{1}{2}$) $f(1.7) = -0.28332962$

($\frac{1}{2}$) $f'(1.7) = 1.257688989$

($\frac{1}{2}$) for rads.

3d) $x = 6t - t^2$

i) $\dot{x} = 6 - 2t$

(1) $\ddot{x} = -2$

ii) when $t=2$ $\dot{x} = 6 - 2(2)$

$= 2$

$\therefore$ it is moving to the right at 2 m/s

(1) when $t=2$ $\ddot{x} = -2$

acceleration is in the neg direction (constantly)

iii) The ice is decreasing in speed since $\dot{x}$ and $\ddot{x}$ are in opposition

(1)

iv) stat when $\dot{x}=0$

$0 = 6 - 2t$

$6 = 2t$

(1) $t = 3$

$x = 6(3) - (3)^2$

$= 18 - 9$

$\therefore$ Ice is stationary when $t=3$ at 9m up the slope.

Question 4

a i) $x^2 + 4x + 13 \equiv (x+a)^2 + b^2$

$(x^2+4x+4) + 9 = (x+a)^2 + b^2$

$(x+2)^2 + 9 = (x+a)^2 + b^2$ (1)

ii) $\int_{-2}^{1} \frac{dx}{(x+2)^2+9}$

$= \left[\frac{1}{3}\tan^{-1}\frac{x+2}{3}\right]_{-2}^{1}$

$= \frac{1}{3}\left[\tan^{-1}(1) - \tan^{-1}(0)\right]$

$= \frac{1}{3}\left(\frac{\pi}{4}\right)$ (2)

$= \frac{\pi}{12}$

b) i) $V = \frac{4}{3}\pi r^3$

$\frac{dV}{dr} = 4\pi r^2$

$\frac{dV}{dt} = 200 \text{ mm}^3/s$

$\frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dt}$

(2) $200 = 4\pi r^2 \times \frac{dr}{dt}$ (2)

when $r = 50$

$200 = 4\pi(50)^2 \times \frac{dr}{dt}$

$\frac{200}{4\pi(2500)} = \frac{dr}{dt}$

$\frac{50}{2500\pi}$

$\boxed{\frac{1}{50\pi} = \frac{dr}{dt}}$

b) ii) $S = 4\pi r^2$

$\frac{ds}{dr} = 8\pi r$

$\frac{ds}{dt} = \frac{ds}{dR} \times \frac{dr}{dt}$ (2)

$= 8\pi r \times \frac{1}{50\pi}$

$= \frac{8 \times 50}{50}$

$= 8 \text{ mm}^2/\text{sec}$ (2)

c) $\ddot{x} = -16x$ $\boxed{\text{at } t=0 \quad x=\sqrt{3} \quad \dot{x}=12}$

$$\boxed{\begin{aligned} x &= a\cos(nt+\alpha) \\ \dot{x} &= -na\sin(nt+\alpha) \\ \ddot{x} &= -n^2 a\cos(nt+\alpha) \end{aligned}}$$ (1)

at $t=0$ $x=\sqrt{3}$ $\dot{x}=12$

① $\sqrt{3} = a\cos\alpha$ (1/2)

② $12 = -na\sin\alpha$ (1/2)

but $\boxed{n=4}$ $\frac{1}{2}$ since $\ddot{x} = -n^2 x$

③ $12 = -4a\sin\alpha$

solving ① + 3 simult

$\frac{\sqrt{3}}{3} = \frac{a\cos\alpha}{-a\sin\alpha}$ (5)

$-\frac{3}{\sqrt{3}} = \frac{a\sin\alpha}{a\cos\alpha}$

$\boxed{-\frac{3}{\sqrt{3}} = \tan\alpha}$

ie $-\sqrt{3} = \tan\alpha$

$\therefore \alpha = -\frac{\pi}{3}$ (1) or $\frac{2\pi}{3}$ $\frac{5\pi}{3}$

$a = 2\sqrt{3}$ (1)

$\therefore x = 2\sqrt{3}\cos\left(4t - \frac{\pi}{3}\right)$ (1/2)

also $x = 2\sqrt{3}\sin\left(4t + \frac{\pi}{6}\right)$

Question 5

a) i) $M_{PQ} = 1$ since $PQ \parallel y = x$

$M_{PQ} = \frac{p^2-q^2}{2p-2q}$

$= \frac{(p-q)(p+q)}{2(p-q)}$

$= \frac{p+q}{2}$

$1 = \frac{p+q}{2}$

(1) $\therefore p+q = 2$

Finding gradient (1/2)

equating to 1 (1/2)

b) i) normal at p

$x + py = 2p + p^3$ ①

normal at q

$x + qy = 2q + q^3$ ②

① − ②

$py - qy = 2p - 2q + p^3 - q^3$

$y(p-q) = 2(p-q) + (p-q)(p^2+pq+q^2)$

y coord of R → $$\boxed{\begin{aligned} y &= p^2+pq+q^2+2 \\ y &= (p+q)^2 - pq + 2 = 6 - pq \end{aligned}}$$

subst y coordinate ①

$x + p(p^2+pq+q^2+2) = 2p + p^3$

$x + p^3 + p^2q + pq^2 + 2p = 2p + p^3$

$x + p^2q + pq^2 = 0$

$$\boxed{\begin{aligned} x &= -pq(p+q) \quad \text{but } p+q=2 \\ x &= -2pq \end{aligned}}$$

$\therefore R(-2pq, 6-pq)$

$x: -pq(p+q)$

$y: p^2+pq+q^2+2$

$y: -2pq$

$x:$

manipulation to get y is 1 mark.

writing $x =$

$y =$

(2) Using $p+q=2$ (17)

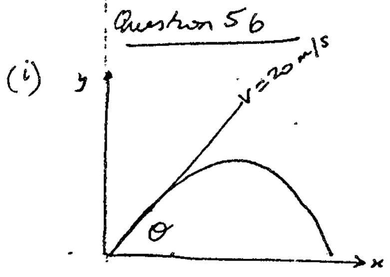
ii) $x = -2pq$ $\quad y = 6 - pq$

$p = \frac{x}{-2q}$ $\quad y = 6 - q\left(\frac{x}{-2q}\right)$

$y = 6 + \frac{x}{2} \longrightarrow x - 2y + 12 = 0$

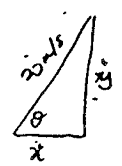
Question 5b

(i)



now initially

(2)



$\sin\theta = \frac{\dot{y}}{20}$

$\therefore \dot{y} = 20\sin\theta$

$\cos\theta = \frac{\dot{x}}{20}$

$\dot{x} = 20\cos\theta$

horizontal motion

$\ddot{x} = 0$

$\dot{x} = c_1$

at $t=0$ $\dot{x} = 20\cos\theta$

$\therefore \dot{x} = 20\cos\theta$

$x = 20\cos\theta\, t + c_2$

at $t=0$ $x=0$

$0 = [20\cos\theta](0) + c_2$

$c_2 = \cancel{20\cos\theta\ldots}$

$\boxed{\therefore x = 20\cos\theta\, t \quad \therefore y = -5t^2 + 20\sin\theta\, t}$

vertical motion

$\ddot{y} = -10$

$\dot{y} = -10t + k_1$

now at $t=0$ $\dot{y} = 20\sin\theta$

$\dot{y} = -10t + 20\sin\theta$

$y = -5t^2 + 20\sin\theta t + k_2$

now at $t=0$ $y=0$

$0 = k_2$

(ii) Range of x

ie find x when $y=0$

or find t when $y=0$

then sub into x

(2)

at $y=0$

$0 = -5t^2 + 20t\sin\theta$

$0 = t(20\sin\theta - 5t)$

$0 = 5t(4\sin\theta - t)$

$\therefore t = 0$ or $t = 4\sin\theta$

at $t = 4\sin\theta$

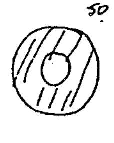
$x = 20\cos\theta \times 4\sin\theta$

$x = 80\sin\theta.\cos\theta$

$= 40(2\sin\theta.\cos\theta)$

$= 40\sin 2\theta$

iii)



so $A = \pi(R^2 - r^2)$

$= \pi(40^2 - 20)^2$

$= \pi(1600 - 400)$

$= 1200\pi$ sq m

at $\theta = 45°$ $x = 40\sin 90°$

$= 40$ ie $R = 40$

at $\theta = 15$ $x = 40\sin 30$

$= 20$ ie $r = 20$

pag

(6a) $a = \frac{d}{dx}(\frac{1}{2}v^2)$

$\frac{d}{dx}(\frac{1}{2}v^2) = 12e^{2x}$

$\frac{1}{2}v^2 = 6e^{2x} + c$ (1)

$v^2 = 12e^{2x} + c$

when $x=0$ $v = 2\sqrt{3}$

$(2\sqrt{3})^2 = 12e^{2(0)} + c$

$12 = 12e^{2(0)} + c$

$12 = 12 + c$

$c = 0$

$v^2 = 12e^{2x}$ (1)

$v = \pm\sqrt{12e^{2x}}$

but $v > 0$ (particle left $x \geq 0$, $v>0$)

$\therefore v = 2\sqrt{3e^{2x}}$

$= 2\sqrt{3}e^{x}$ (exp) (1)

$\frac{25}{12} = e^{2x}$

$2x = \ln\frac{25}{12}$

$x = \frac{1}{2}\ln\frac{25}{12}$

when $v = 5$ find x

$5 = 2\sqrt{3e^{2x}}$

$5 = 2\sqrt{3}e^{2x}$

$\frac{5}{2\sqrt{3}} = e^{2x}$

$\log_e(\frac{5}{2\sqrt{3}}) = 2x$

$x = \frac{1}{2}\log_e(\frac{5}{2\sqrt{3}})$ or $\frac{1}{2}\log_e\frac{5\sqrt{3}}{6}$

$= 0.18$ (2dp)

$x = \frac{5}{2\sqrt{3}}$ (1)

$x = \ln\frac{5}{2\sqrt{3}}$

6b(i) $\sin 2x = 2\sin^2 x$ for $0 \le x < \pi$

$2\sin x\cos x = 2\sin^2 x$

$2\sin x\cos x - 2\sin^2 x = 0$

$2\sin x(\cos x - \sin x) = 0$

$\sin x = 0$ or $\cos x - \sin x = 0$

$\cos x = \sin x$

2 $\quad 1 = \tan x$

∴ The general solution is

$x = \pi n + (-1)^n(0)$, $n\pi + \frac{\pi}{4}$

$= n\pi$, $n\pi$, $\frac{\pi}{4}$

ii) If $0 < x < \frac{\pi}{4}$, then

$\sin 2x > 2\sin^2 x$

Consider $\sin 2x - 2\sin^2 x > 0$

$= 2\sin x\cos x - 2\sin^2 x > 0$

$= 2\sin x(\cos x - \sin x)$

since $\sin x > 0$ for $0 < x < \frac{\pi}{4}$

and $\cos x > \sin x$ for $0 < x < \frac{\pi}{4}$

then $2\sin x(\cos x - \sin x) > 0$

for $0 < x < \frac{\pi}{4}$

There are other possible methods

iii) Area $= \int_0^{\pi/4} y_{\text{upper}} - y_{\text{lower}}\, dx$

$= \int_0^{\pi/4} \sin 2x - 2\sin^2 x\, dx$

3 $\quad = \int_0^{\frac{\pi}{4}} \sin 2x - (1 - \cos 2x)\, dx$

$= \left[-\frac{1}{2}\cos 2x - x + \frac{1}{2}\sin 2x\right]_0^{\pi/4}$

$= \left[-\frac{\pi}{4} + \frac{1}{2}\right] - \left[-\frac{1}{2}\right] = 1 - \frac{\pi}{4}$ sq u

Question 7

a) $P = \frac{1000}{1 + ke^{-1000t}}$ $\cancel{\frac{1000}{k} + e^{1000t}}$

i) $\lim_{t \to \infty} \frac{1000}{1 + ke^{-1000t}}$

~~since~~ $\lim_{t \to \infty} e^{-1000t} = 0$

then $\lim_{t \to \infty} \frac{1000}{1 + ke^{-1000t}}$

$= \frac{1000}{1 + 0}$

1 $\quad = 1000$

ii) ~~at~~ $t = 0$; $P = 1$

$P = \frac{1000}{1 + ke^0}$

$1 = \frac{1000}{1 + k}$

∴ $k = 999$

∴ $P = \frac{1000}{1 + 999e^{-1000t}}$

when $P = 500$

$500 = \frac{1000}{1 + 999e^{-1000t}}$

$2 = 1 + 999e^{-1000t}$

∴ $999e^{-1000t} = 1$

$e^{-1000t} = \frac{1}{999}$

2 $\quad -1000t = \ln\left(\frac{1}{999}\right)$

$t \doteq 2.5$ days

7 (b) (i)

$v = \frac{1}{2}(1 - x^2)$

$\frac{dv}{dx} = -x \quad \therefore a = v\frac{dv}{dx} = \frac{x^3 - x}{2}$

(b) (ii)

$\frac{1}{1+x} + \frac{1}{1-x} = \frac{(1-x)+(1+x)}{(1+x)(1-x)} = \frac{2}{1-x^2}$

$\frac{dx}{dt} = \frac{1-x^2}{2} \Rightarrow \frac{dt}{dx} = \frac{2}{1-x^2}$

$\frac{dt}{dx} = \frac{1}{1+x} + \frac{1}{1-x}$

$t = \ln(1+x) - \ln(1-x) + c$

when $t = 0$, $x = 0$ ∴ $c = 0$

∴ $t = \ln\frac{1+x}{1-x}$ 1

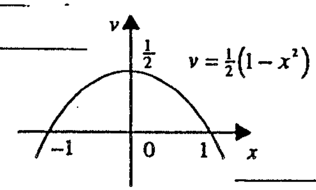
$\frac{1+x}{1-x} = e^t$

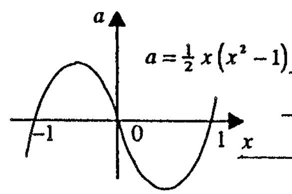
$1 + x = e^t - xe^t$

$x(e^t + 1) = e^t - 1$

∴ $x = \frac{e^t - 1}{e^t + 1} = \frac{1 - e^{-t}}{1 + e^{-t}}$ 1

(b) (iii)





Initially the particle is at O, moving right at speed of $0 \cdot 5\,\text{ms}^{-1}$ and slowing down (since v and a have opposite signs for $0 < x < 1$).
The particle continues to move right while slowing down for $x < 1$. As $t \to \infty$, $x \to \frac{1-0}{1+0} = 1$.
Its limiting position is 1 m to the right of O.

(10)