St Catherine's School

Year: 12
Subject: Mathematics Extension 1
Time allowed: 2 hours
Date: August 2005

Student Number: 15992581

Directions to candidates:

- All questions are to be attempted.
- Marks may be deducted for careless or badly arranged work
- All necessary **working** must be shown in every question.
- Answers to questions 1-4 are to be written in Booklet 1
- Answers to questions 5-7 are to be written in Booklet 2
- Additional booklets available if required
- Start a new page for each question

TEACHERS' USE ONLY

Question 1	a)	b)	c)	d)	
Question 2	a)	b)	c)		
Question 3	a)	b)	c)	d)	
Question 4	a)	b)	c)	d)	
Question 5	a)	b)	c)	d)	
Question 6	a)	b)			
Question 7	a)	b)	c)		
TOTAL					

Questions 1-4 are to be done in your first booklet.
Questions 5-7 are to be done in your second booklet.
Extra booklets are available if needed.

Question 1 (12 marks)

a) Solve for x and graph on the number line: *(3)*

$$\frac{5}{x-3} \le 3$$

b) P(4,-6) Q (9,4). Find the co-ordinates of the point R which divides PQ externally in the ratio 8:3 *(2)*

c) Find the general solution to *(3)*

$$\tan(\theta + \frac{\pi}{4}) = \sqrt{3}$$

d) Prove using Mathematical Induction: *(4)*

$$\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4} \qquad n \ge 1$$

Question 2 (12 marks)

Start a new page

a) Find the term with the largest co-efficient in the expansion of *(4)*

$$(3x+5)^{10}$$

b) Find the constant term in the expansion of *(4)*

$$((2x - \frac{1}{x^2})^9$$

c) Using the expansion of $(1+x)^n$ and differentiating both sides, prove that *(4)*

$$\sum_{k=1}^{n} (-1)^{k+1} k.{}^nC_k = 0$$

Question 3 (12 marks)

Start a new page

a) i) Show that (2)

$$\frac{d}{dx}(\sin^{-1} x + \cos^{-1} x) = 0$$

ii) Hence or otherwise graph $f(x) = \sin^{-1} x + \cos^{-1} x$ (2)

b) Show that for $x \neq -1$ (3)

$$\tan^{-1} x + \tan^{-1}\frac{1-x}{1+x} = \frac{\pi}{4}$$

c) Sketch the graph of $y = 2\sin^{-1} 4x$ stating domain and range (3)

d) Simplify $\sin 2\theta(\tan\theta + \cot\theta)$ (2)

Question 4 (12 marks)

Start a new page

a) Evaluate $\int_0^3 \frac{3x+1}{\sqrt{x+1}}dx$ by substitution, using $u = x+1$ (3)

b) Find $\int_0^{\frac{1}{2}} \frac{dx}{1+4x^2}$ (3)

c) A committee of 5 people is selected at random from a group of 8 men and 4 women.

i) How many different committees can be chosen? (2)

ii) What is the probability that a particular man, Peter, and a particular woman, Sally, are selected for the committee. (2)

d) Express $\frac{{}^nC_7}{{}^nC_4}$ in simplest form. (2)

Question 5 (12 marks)

Start a new booklet

a) $P(x) = x^3 + 3x^2 + 6x - 5$ has a root α between 0 and 1.

Using $\alpha = 0.5$ as a first approximation, use one application of Newton's method to find a better approximation for α (to 1 decimal place). (2)

b) i) Express $\sqrt{3}\sin x + \cos x$ in the form $R\sin(x+\alpha)$ (2)

ii) Hence or otherwise find the general solution to (2)

$$\sqrt{3}\sin x + \cos x = 1$$

c) Given that $\frac{dx}{dt} = 5x$ and $x = 5$ when $t = 0$, find x in terms of t (3)

d i) Sketch the function $y = (x-1)(x-3), x\varepsilon R$, (1)

ii) Explain why y= $f(x)$ does not have an inverse function. (1)

iii) Suggest a restriction on the domain of $f(x)$ so that an inverse function $f^{-1}(x)$ exists (1)

Question 6 (12 marks)

Start a new page

a) $P(2ap, ap^2)$ and $Q(2aq, aq^2)$ are two points on the parabola $x^2 = 4ay$

i) Show that the equation of the tangent at P is $px - y = ap^2$ *(1)*

ii) R is the point of intersection of the tangents at P and Q. Show that R has co-ordinates $(a(p+q), apq)$ *(2)*

iii) If the chord PQ passes through the focus, show that $pq = -1$ *(2)*

iii) Find the locus of point R *(1)*

b) A particle moves in Simple Harmonic Motion with velocity v when it is x units to the right of the origin, given by

$$v = \sqrt{-12 + 8x - x^2}$$

i) Find the two positions where the particle comes to rest. *(2)*

ii) What is the amplitude of the motion? *(1)*

iii) Find the acceleration as a function of x *(2)*

iv) What is the period of the motion? *(1)*

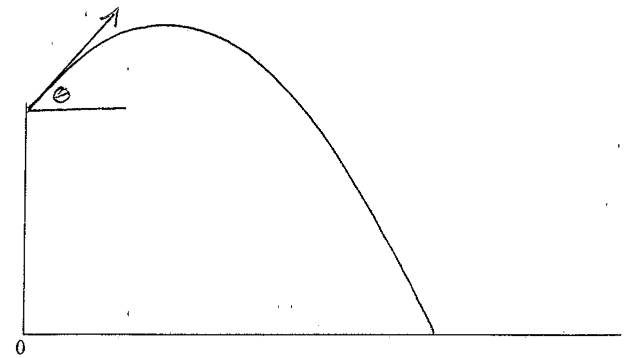
Question 7 (12 marks)

Start a new page.

a) A spherical balloon is being inflated at a rate of 100 cm^3 per second. Find the rate of increase of surface area when the radius of the balloon is 10cm. *(4)*

NB For a sphere, volume $V = \frac{4}{3}\pi r^3$ and surface area $A = 4\pi r^2$

b) A particle is projected from the top of a tower 50 m high with a velocity of 40 m/s at an angle of θ to the horizontal.



Place the origin at the base of the tower as shown. Use $g = -10 m/s^2$

i) Show that the position of the particle at time t is given by

$$x = 40t\cos\theta \text{ and } y = -5t^2 + 40t\sin\theta + 50 \quad (2)$$

ii) The range of the projectile is 200m. Show that the two possible values of θ are given by

$$5\tan^2\theta - 8\tan\theta + 3 = 0 \quad (2)$$

iii) Hence find the two possible values of θ *(2)*

c) In Sydney, 40% of homes have only one occupant. If 4 homes are selected at random, find the probability that at least 3 homes have only one occupant. *(2)*

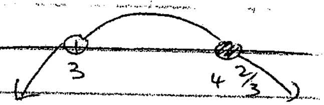
Ext 1 Maths Aug 2005 SOLUTIONS

Q1. a) $\frac{5}{x-3} \le 3 \qquad x \ne 3$

$5(x-3) \le 3(x-3)^2$

$(x-3)(5-3(x-3)) \le 0$

$(x-3)(14-3x) \le 0$



$\therefore \; x < 3, \quad x \ge 4\frac{2}{3}$

b) $P(4,-6) \quad Q(9,4) \quad l:m = 8:-3$

$R\left(\frac{-3\times4+8\times9}{5}, \frac{-3\times-6+8\times4}{5}\right)$

$= (12, \quad 10)$

c) $\tan\left(\theta+\frac{\pi}{4}\right) = \sqrt{3}$

$\theta+\frac{\pi}{4} = n\pi+\frac{\pi}{3} \qquad (n \in I)$

$\theta = n\pi + \frac{\pi}{12}$

d) $P(1)$ is $\quad 1^3 = \frac{1\times2^2}{4}$ True for 1

$P(k)$ is $\quad 1^3+2^3+\dots k^3 = \frac{k^2(k+1)^2}{4}$

Assume true for k.

$P(k+1)$ is $\quad 1+2^3+\dots k^3+(k+1)^3 = \frac{(k+1)^2(k+2)^2}{4}$

$LHS = \frac{k^2(k+1)^2}{4}+(k+1)^3$ using $P(k)$

$= (k+1)^2\left(\frac{k^2}{4}+k+1\right)$

$= (k+1)^2\left(\frac{k^2+4k+4}{4}\right)$

$= \frac{(k+1)^2(k+2)^2}{4} \qquad = RHS$

$\therefore$ True for $n=k+1$ if true for $n=k$

True for $n=1$ $\therefore$ true for $n=2$ $\therefore$ true for all integer $n \ge 1$

Q2

a) $(3x+5)^{10} \Rightarrow {}^{10}C_r(3x)^{10-r}(5)^r$ ← ge[neral] te[rm]

for some r, $T_r > T_{r+1}$ — find largest

${}^{10}C_r\, 3^{10-r}\, 5^r > {}^{10}C_{r+1}\, 3^{9-r}\, 5^{r+1}$

${}^{10}C_r \times 3 > {}^{10}C_{r+1} \times 5$

$\frac{10!}{(10-r)!\,(r)!}\times 3 > \frac{10!}{(9-r)!\,(r+1)!}\times 5$

$\frac{3}{10-r} > \frac{5}{r+1}$ $\qquad 0<r<10$ so denom both +ve

$3r+3 > 50-5r$

$8r > 47$

$r > 5\frac{7}{8} \qquad \therefore r=6$

$\therefore$ Largest term $\; {}^{10}C_6(3x)^4 5^6 = 265781250\, x$[^4]

b) $\left(2x-\frac{1}{x^2}\right)^9 \Rightarrow {}^9C_r(2x)^{9-r}(-x^{-2})^r$

$= {}^9C_r\, 2^{9-r}(-1)^r x^{9-3r}$

$\therefore x^0$ when $r=3$

Constant term is $\; {}^9C_3\, 2^6(-1)^3 = -5$[376]

c) $(1+x)^n = {}^nC_0 + {}^nC_1x + {}^nC_2x^2 + {}^nC_3x^3 + \dots {}^nC_rx^r + \dots$

$\frac{d}{dx} \quad n(1+x)^{n-1} = {}^nC_1 + 2\,{}^nC_2x + 3\,{}^nC_3x^2 + \dots r\,{}^nC_rx^{r-1} + n$[…]

set $x=-1 \quad 0 = {}^nC_1 - 2\,{}^nC_2 + 3\,{}^nC_3 \dots (-1)^{r-1}r\,{}^nC_r + \dots$

$= \sum_{k=1}^{n}(-1)^{k+1}\cdot k\cdot {}^nC_k$

Q3

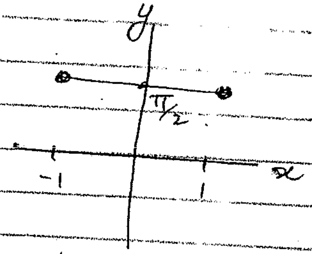
a) $\frac{d}{dx}(\sin^{-1}x + \cos^{-1}x) = \frac{1}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-x^2}}$

$= 0$

$\therefore \sin^{-1}x + \cos^{-1}x = \text{constant}.$

Set $x = 0$ $\quad \sin^{-1}0 + \cos^{-1}0 = \frac{\pi}{2}$

natural domain $-1 \le x \le 1$.



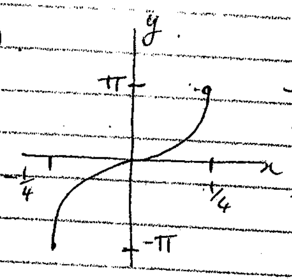
b) $\tan\left(\tan^{-1}x + \tan^{-1}\frac{1-x}{1+x}\right) = \dfrac{x + \frac{1-x}{1+x}}{1 - x\left(\frac{1-x}{1+x}\right)}$

$= \dfrac{x + x^2 + 1 - x}{1 + x - x + x^2}$

$= 1 = \tan\frac{\pi}{4}$

$\therefore \tan^{-1}x + \tan^{-1}\frac{1-x}{1+x} = \frac{\pi}{4}.$

c)



$-1 \le 4x \le +1.$

$-\frac{1}{4} \le x \le \frac{1}{4}$

$-\frac{\pi}{2} \le \frac{y}{2} \le \frac{\pi}{2}$

d) $\sin 2\theta(\tan\theta + \cot\theta) = 2\sin\theta\cos\theta\left(\frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta}\right)$

$= 2\sin^2\theta + 2\cos^2\theta$

$= 2(\sin^2\theta + \cos^2\theta)$

$= 2$

Question 4

a) $\int_0^3 \frac{3x+1}{\sqrt{x+1}}\,dx$

$u = x+1 \quad \therefore \quad x =$

$du = dx$

at $x = 3$ $u = 4$

at $x = 0$ $u = 1$.

$= \int_1^4 \frac{3u-2}{\sqrt{u}}\,du$

$= \int_1^4 3u^{\frac{1}{2}} - 2u^{-\frac{1}{2}}\,du$

$= \left[\frac{3u^{1\frac{1}{2}}}{1\frac{1}{2}} - \frac{2u^{\frac{1}{2}}}{\frac{1}{2}}\right]_1^4$

$= (2\times 8 - 4\times 2) - (2 -$

$= 10$

b) $\int_0^{\frac{1}{2}} \frac{dx}{1+4x^2} = \frac{1}{4}\int_0^{\frac{1}{2}} \frac{dx}{\frac{1}{4}+x^2}$ $\qquad \int \frac{1}{a^2+x^2} = \frac{1}{a}$

$= \left[\frac{1}{4}\cdot\frac{1}{\frac{1}{2}}\tan^{-1}\frac{x}{\frac{1}{2}}\right]_0^{\frac{1}{2}}$

$= \left[\frac{1}{2}\tan^{-1}2x\right]_0^{\frac{1}{2}}$

$= \frac{1}{2}\tan^{-1}1 - \frac{1}{2}\tan^{-1}0 = \frac{\pi}{8}$

c) c(i) $^{12}C_5 = 792$

S & P chosen $\therefore$ choose 3 others.

No inc S & P : $\quad ^{10}C_3 = 120$

$P(S \,\&\, P \text{ inc}) = \frac{120}{792} = \frac{5}{33}.$

d) $\dfrac{^nC_7}{^nC_4} = \dfrac{n!}{(n-7)!\,7!} \div \dfrac{n!}{(n-4)!\,4!}$

$= \dfrac{(n-4)!\,4!}{(n-7)!\,7!} = \dfrac{(n-4)(n-5)(n-6)(n-7)!}{(n-7)! \times 7\times 6\times 5\times 4!}$

$= \dfrac{(n-4)(n-5)(n-6)}{210}$

Q5

a) $P(x) = x^3 + 3x^2 + 6x - 5$ $\quad P(\frac{1}{2}) = -1.125$

$P'(x) = 3x^2 + 6x + 6$ $\quad P'(\frac{1}{2}) = 9.75$

$$x_2 = \alpha - \frac{P(\alpha)}{P'(\alpha)}$$

$$= 0.5 - \frac{-1.125}{9.75}$$

$$\doteqdot 0.62$$

check $P(0.62) = 0.01$

b) (i) $\sqrt{3}\sin x + \cos x = R\sin(x+\alpha)$

$LHS = 2\left(\frac{\sqrt{3}}{2}\sin x + \frac{1}{2}\cos x\right)$

$= 2(\sin x + \alpha)$ $\quad$ where $\sin\alpha = \frac{1}{2}$

$= 2\sin(x + \frac{\pi}{6})$ $\quad$ $\cos\alpha = \frac{\sqrt{3}}{2}$ $\therefore \alpha = \frac{\pi}{6}$

(ii) $2\sin(x + \frac{\pi}{6}) = 1$

$\sin(x + \frac{\pi}{6}) = \frac{1}{2}$

$x + \frac{\pi}{6} = n\pi \pm (-1)^n \frac{\pi}{6}$

$x = n\pi + (-1)^n\frac{\pi}{6} - \frac{\pi}{6}$ $\quad n \in I$

($\therefore n\pi$ for even n, $\quad n\pi - \frac{\pi}{3}$ for odd n)

c) $\frac{dx}{dt} = 5x$ $\quad$ $x = 5$ when $t = 0$

$\frac{dt}{dx} = \frac{1}{5x}$

$t = \frac{1}{5}\ln x + c$

at $t = 0$, $x = 5$ $\therefore c = -\frac{1}{5}\ln 5$

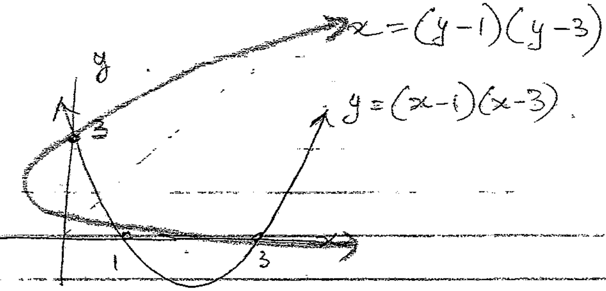
$t = \frac{1}{5}\ln x - \frac{1}{5}\ln 5$

$= \frac{1}{5}\ln\frac{x}{5}$

$\therefore 5t = \ln\frac{x}{5}$

$\frac{x}{5} = e^{5t}$ $\quad \therefore x = 5e^{5t}$

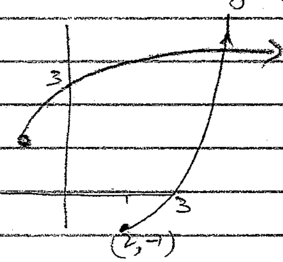
d) (i)



(ii) $x = (y-1)(y-3)$ is not a function as for $x > -1$, there are 2 y values (see red diagram)

$y = (x-1)(x-3)$, $x \geq 2$

(iii)



If the domain is restricted to $x \geq 2$, $f(x)$ has an inverse $f^{-1}x$

Q6 a) i) $P(2ap, ap^2)$ $Q(2aq, aq^2)$ on $x^2 = 4ay$

Tangent at P: $y = \frac{x^2}{4a}$

$\frac{dy}{dx} = \frac{2x}{4a} = \frac{4ap}{4a} = p$

Eq. of tang $y - ap^2 = p(x - 2ap)$

$y = px - ap^2$ so $px - y = ap^2$

ii) ∴ tangent at Q is $qx - y = aq^2$

Intersection of tangents $px - y = ap^2$

$qx - y = aq^2$

$px - qx = ap^2 - aq^2$

$x(p-q) = a(p+q)(p-q)$

∴ $x = a(p+q)$

∴ pt of intersection $R(a(p+q), apq)$

(iii) PQ is a focal chord ∴ $(0, a)$ lies on PQ

Eq. of PQ: grad $= \frac{ap^2 - aq^2}{2ap - 2aq} = \frac{a(p+q)(p-q)}{2a(p-q)}$

$= \frac{(p+q)}{2}$

∴ $y - ap^2 = \frac{p+q}{2}(x - 2ap)$

$y = \frac{p+q}{2} \cdot x - apq$

now $(0, a)$ satisfies so

$a = 0 - apq$

$a + apq = 0$

$a(1 + pq) = 0$ ∴ $pq = -1$

(iv) $R(a(p+q), apq) = (a(p+q), -a)$

which has $y = -a$ for all p, q.

∴ R lies on directrix $y = -a$

6b) i) $v = \sqrt{-12 + 8x - x^2}$

$v = 0$ when $-12 + 8x - x^2 = 0$

$x^2 - 8x + 12 = 0$

$(x-6)(x-2) = 0$

∴ at rest at $x = 6$ & $x = 2$

ii) Amplitude $= \frac{1}{2}(6-2) = 2$

(iii) $a = \frac{d}{dx}(\frac{1}{2}v^2)$

$\frac{1}{2}v^2 = \frac{1}{2}(-12 + 8x - x^2)$

$= -6 + 4x - \frac{1}{2}x^2$

$\frac{d}{dx}(\frac{1}{2}v^2) = 4 - x$

∴ $\ddot{x} = 4 - x = -1(x - 4)$

∴ $n = 1$

iv) Period $= \frac{2\pi}{n} = 2\pi$

Q7 a) $\frac{dV}{dt} = 100$ cm³/s. find $\frac{dA}{dt}$.

$V = \frac{4}{3}\pi r^3$ $\qquad$ $A = 4\pi r^2$

$\frac{dV}{dr} = 4\pi r^2$ $\qquad$ $\frac{dA}{dr} = 8\pi r$.

$$\frac{dA}{dt} = \frac{dA}{dr} \cdot \frac{dr}{dt}$$

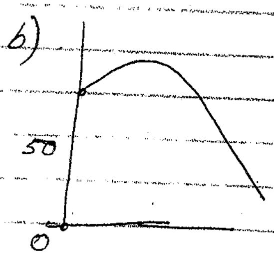
$$= \frac{dA}{dr} \cdot \frac{dV}{dt} \cdot \frac{dr}{dV}$$

$$= 8\pi r \cdot 100 \cdot \frac{1}{4\pi r^2}$$

$$= \frac{200}{r}$$

when $r = 10$ $\quad \frac{dA}{dt} = \frac{200}{10} = 20$ cm²/sec

b)



$\ddot{x} = 0$ $\qquad$ $\ddot{y} = -10$.

$\dot{x} = c_1$ $\qquad$ $\dot{y} = -10t + c_2$

at $t = 0$, $\dot{x} = 40\cos\theta$ $\qquad$ $\dot{y} = 40\sin\theta$

$\therefore \dot{x} = 40\cos\theta$ $\qquad$ $\therefore \dot{y} = -10t + 40\sin\theta$

so $x = 40t\cos\theta + c_3$ $\qquad$ $y = -5t^2 + 40t\sin\theta + c_4$

at $t = 0$ $x = 0$ $y = 50$.

$\therefore x = 40t\cos\theta$ $\qquad$ $y = -5t^2 + 40t\sin\theta + 50$

The range is 200 so at $x = 200$ $y = 0$.

$\therefore 200 = 40t\cos\theta$ $\qquad$ $\therefore 0 = -5t^2 + 40t\sin\theta + 50$

This is at $t = \frac{200}{40\cos\theta}$ $\qquad$ $0 = \frac{-125}{\cos^2\theta} + \frac{40 \times 5}{\cos\theta}\sin\theta + 50$

$= \frac{5}{\cos\theta}$

$0 = -125\sec^2\theta + 200\tan\theta + 50$

$0 = 5(1 + \tan^2\theta) - 8\tan\theta - 2$

$= 5\tan^2\theta - 8\tan\theta + 3$

(iii)

$(5\tan\theta - 3)(\tan\theta - 1) = 0$.

$\tan\theta = \frac{3}{5}$ $\qquad$ $\tan\theta = 1$

$\theta \doteqdot 30°58'$. $\qquad$ $\theta = 45°$

at least 3 = 3 or 4.

d) $(y+n)^4 \rightarrow y^4 + 4y^3n + 6y^2n^2 + 3yn^3 + n^4$

$P(\text{at least } 3) = (.4)^4 + 4 \times (.4)^4 \times (.6)$

$= 0.1792$