St Catherine's School

Year: 12
Subject: Extension II Mathematics
Time Allowed: 3 hours plus 5 mins reading time
Date: August 2004

Exam number: 1416 9652.

Directions to candidates:

- All questions are to be attempted.
- All questions are of equal value.
- All necessary **working** must be shown in every question.
- Full marks may not be awarded for careless or badly arranged work.
- Each section should be started on a **new booklet**.
- Remember to write your examination number at the start of each section
- Approved calculators and geometrical instruments are required.
- Hand in your work in **3 bundles**:
  - Section 1 Questions 1, 2 3 and 4.
  - Section 2 Questions. 5, 6, 7 and 8
  - The question paper

TEACHER'S USE ONLY **Total Marks**
1 15 + 13½ + 12 + 15 $55\frac{1}{2}$ /60
2 15 + 15 + 14 + 13½ $57\frac{1}{2}$ /60
TOTAL 113 **/120**
% mark

Question 1.

(a) Find $\int \frac{1}{9+x^2}dx$ (1m)

(b) Find $\int \frac{\cos^3 x}{\sin^2 x}dx$ (3m)

(c) Find $\int \frac{1}{7-x^2+6x}dx$ (3m)

(d) Evaluate $\int_0^1 \sin^{-1}x \ dx$ using integration by parts (3m)

(e) (i) Show that $(1-\sqrt{x})^{n-1}\sqrt{x} = (1-\sqrt{x})^{n-1} - (1-\sqrt{x})^n$ (1m)

(ii) If $I_n = \int_0^1 (1-\sqrt{x})^n dx$, for $n \geq 0$,

show that $I_n = \frac{n}{n+2} I_{n-1,}$, for $n \geq 1$ (4m)

Question 2

(a) The points P and Q in the complex plane correspond to the complex numbers z and w respectively. O is the origin and the triangle OPQ is isosceles and $\angle POQ = 90^0$.P is in the first quadrant

(i) Show that $z^2 + w^2 = 0$ (2m)

(ii) If R is a point such that OPQR is a parallelogram, show that R represents the complex number $\sqrt{2}$ w cis $\frac{\pi}{4}$ (2m)

(b) Consider the arc given by $\arg[\frac{z-1}{z+1}] = \frac{\pi}{4}$

(i) Explain why the locus of the arc is a major ~~axis~~ arc (2m)

(ii) Find the centre and radius of the circle of which this is the major arc. (3m)

(c) (i) Solve the equation $z^3 = -1$ in the Complex field (2m)

(ii) If α is a complex root of the above equation, show that $\alpha^2 - \alpha + 1 = 0$ (2m)

(iii) Hence simplify $(1-\alpha)^6$ (2m)

Question 3

(a) Consider the hyperbola xy = 4.

(i) State the eccentricity of this hyperbola (1m)

(ii) Find the coordinates of the foci (2m)

(b) For which values of k, will the straight line y = x+ k have no point of intersection with the parabola $y = x^2$ (3m)

(c) `Express $\frac{5x^2+2x+7}{(x+1)(x^2+4)}$ as a sum of partial fractions (3m)

(d) (i) Show that the condition for the line y = mx + c to be a tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is $c^2 = a^2m^2 + b^2$ (3m)

(ii) Hence or otherwise show that the pair of tangents drawn from the point (3,4)to the ellipse $\frac{x^2}{16}+\frac{y^2}{9}=1$ is at right angles to one another. (3m)

Question 4

(a) Sketch the graph of the function $y = x - \dfrac{4}{x}$.

State and show the main features (3m)

(b) Find the turning points of the function defined by $y = \dfrac{x}{x^2+1}$ and sketch its graph

(4m)

(c) Sketch the graph of the function y = (4-x) (x+2) (1m)

Using the above function, sketch

(i) $y = \dfrac{1}{f(x)}$ (2m)

(ii) $y = f|x|$ (2m)

(iii) $y = |f(x)|$ (1m)

(iv) $y = e^{f(x)}$ (2m)

Question 5.

(a) For the following hyperbola $\dfrac{x^2}{9} - \dfrac{y^2}{4} = 1$

(i) the eccentricity (1.5m)

(ii) the equations of the directrices (1m)

(iii) the coordinates of they foci (1m)

(iv) the equations of the asymptotes (1m)

(v) Sketch this hyperbola highlighting the above features (1m)

(vi) Find the angle between the asymptotes (1.5m)

(b) (i) P $(t, \frac{1}{t})$ is point on the rectangular hyperbola xy =1.

Show that the equation of the tangent at P is given by $x + t^2 y = 2t$ (3m)

(ii) OQ is a line perpendicular to this tangent.

Show that the coordinates of Q is $x = \dfrac{2t}{1+t^4}$ $y = \dfrac{2t^3}{1+t^4}$ (3m)

(iii) Show that the locus of Q is given by $4xy = (x^2 + y^2)^2$ (2m)

Question 6.

(a) Evaluate i^{2004} (1m)

(b) (i) Use De Moivre's theorem to show that $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$ (1m)

(iii) Deduce that $8x^3 - 6x + 1 = 0$ has solutions x= $\sin\theta$, where $\sin 3\theta = \dfrac{1}{2}$ (2m)

(iv) Find the roots of $8x^3 - 6x + 1 = 0$ in terms of $\sin\theta$ (2m)

(v) Hence evaluate $\sin\dfrac{\pi}{18}\sin\dfrac{5\pi}{18}\sin\dfrac{13\pi}{18}$ (2m)

(c) The equation $x^3 + px - 1 = 0$ has three non real roots $\alpha,\ \beta$ *and* γ.

(i) By noting the relations between the roots and coefficients, show that $\alpha^2 + \beta^2 + \gamma^2 = -2p$ (2m)

(ii) Show that $\alpha^4 + \beta^4 + \gamma^4 = 2p^2$ (2m)

(iii) Find the monic equation, with coefficients in terms of p, whose roots are $\dfrac{\alpha}{\beta\gamma}$, $\dfrac{\beta}{\gamma\alpha}$ *and* $\dfrac{\gamma}{\alpha\beta}$ (3m)

Question 7.

(a) (i) The area in the first quadrant, bounded by $y = x^2$, the y axis and the line y =4 is rotated about the line x = 3.. Use the method of cylindrical shells to find the volume of the solid formed. (4m)

(b) (i) Show that the area of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is given by the definite integral $\frac{4b}{a}\int_0^a \sqrt{a^2-x^2}\,dx$ (3m)

(ii) Evaluate the integral to show that the area is πab square units. (3m)

(c) A solid figure has a semi circular base of radius 3 cm. The cross sections at right angles to the diameter of this base are semi ellipses, with their major axis on the base of this solid. The semi minor axis of each ellipse is half its semi major axis,

(i) Show that the volume of this solid is given by V= $\frac{\pi}{4}\int_0^3 (9-x^2)dx$ (3m)

(Use the result from part (b)

(iv) Hence find V in terms of π (2m)

Question 8.

(a) State the values of x for which

(i) $\sin^{-1}(\sin x) = x$ (1m)

(ii) $\sin(\sin^{-1} x) = x$ (1m)

(b) Given that $\sin^{-1} x$ and $\cos^{-1} x$ are acute,

(i) Show that $\sin(\sin^{-1} x - \cos^{-1} x) = 2x^2 - 1$ (3m)

(ii) Solve the equation $\sin^{-1} x - \cos^{-1} x = \sin^{-1}(0.5)$ (2m)

(c) Two projectiles are fired from the same point O, with the same speed v and at different angles of inclination α *and* β. They follow different paths and strike the ground at the same point P.

(i) Show that the difference in their time of arrival at P is given by

$$t_2 - t_1 = \frac{2v}{g}(\sin\beta - \sin\alpha)$$ (3m)

(ii) Show that $\beta = \frac{\pi}{2} - \alpha$ (2m)

(iii) Show that $(t_2 - t_1)^2 = \frac{4v^2}{g^2}(1 - \frac{gx}{v^2})$

(3m)

END OF PAPER

STANDARD INTEGRALS

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} \, dx = \ln x, \quad x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax}, \quad a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \quad a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \quad a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1} \frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} \, dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} \, dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

Question 1.

a) $\int \frac{1}{9+x^2}\,dx = \frac{1}{3}\tan^{-1}\frac{x}{3} + c$ $\boxed{1M}$

b) $\int \frac{\cos^3 x}{\sin^2 x}\,dx = \int \frac{(1-\sin^2 x)\cdot\cos x}{\sin^2 x}\,dx$ Let $u = \sin x$, $du = \cos x\,dx$

$= \int \frac{1-u^2}{u^2}\,du$ — $\frac{1}{2}$

$= \int \frac{1}{u^2} - 1\;du$

$= -\frac{1}{u} - u + c$

$= -\frac{1}{\sin x} - \sin x + c$ — $\frac{1}{2}$ $\boxed{3M}$

c) $\cancel{\int \frac{1}{7-x^2+6x}\,dx} = \cancel{\int \frac{1}{16-(x-3)^2}\,dx}$

$\cancel{= \sin^{-1}\frac{(x-3)}{4} + c}$

OR $\int \frac{1}{7-x^2+6x}\,dx = \int \frac{1}{8}\left(\frac{1}{x+1} - \frac{1}{x-7}\right)\,dx.$

$= \frac{1}{8}\ln\frac{x+1}{x-7} + c$ $\boxed{3M}$

d) $\int_0^1 \sin^{-1}x\,dx$ Let $u = \sin^{-1}x$, $v' = 1$; $u' = \frac{1}{\sqrt{1-x^2}}$, $v = x$

$= \left(x\sin^{-1}x\right)_0^1 - \int_0^1 \frac{x\,dx}{\sqrt{1-x^2}}$ (1) $\int uv' = uv - \int u'v$

Let $1-x^2 = u$, $-2x\,dx = du$; $x=0$; $u=1$; $x=1$; $u=0$

$= \sin^{-1}1 + \frac{1}{2}\int_1^0 \frac{du}{\sqrt{u}}$

$= \sin^{-1}1 - \left(\sqrt{u}\right)_0^1$

($1\frac{1}{2}$) $\boxed{3M}$

e) $(1-\sqrt{x})^{n-1}\sqrt{x} = -(1-\sqrt{x})^{n-1}(1-\sqrt{x}-1)$

$= -(1-\sqrt{x})^n + (-\sqrt{x})^{n-1}$

$= (1-\sqrt{x})^{n-1} - (1-\sqrt{x})^n.$ (1M)

$I_n = \int_0^1 (1-\sqrt{x})^n\,dx$ Let $u = (1-\sqrt{x})^n$, $v' = 1$; $u' = n(1-\sqrt{x})^{n-1}\frac{-1}{2\sqrt{x}}$, $v = x$

$\int uv' = uv - \int u'v$ 1

$I_n = \left(x(1-\sqrt{x})^n\right)_0^1 + \frac{n}{2}\int_0^1 (1-\sqrt{x})^{n-1}\cdot\frac{1}{\sqrt{x}}\cdot x\,dx.$

$= \frac{n}{2}\int_0^1 (1-\sqrt{x})^{n-1}\cdot\sqrt{x}\,dx$ 1

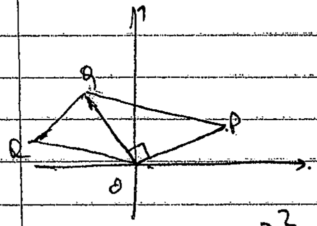
$= \frac{n}{2}\int_0^1 \left((1-\sqrt{x})^{n-1} - (1-\sqrt{x})^n\right)\,dx$ 1

$I_n = \frac{n}{2}\left(I_{n-1} - I_n\right)$

$\left(\frac{n}{2}+1\right)I_n = \frac{n}{2}I_{n-1}$

$I_n = \frac{n}{n+2}I_{n-1}.$ 1

Q. 2



P : (z)

Q : w.

$OP = OQ$

$\therefore\ w = iz$

(i) $z^2 + w^2 = z^2 + (iz)^2$

$= z^2 - z^2$ (2M)

$= 0$.

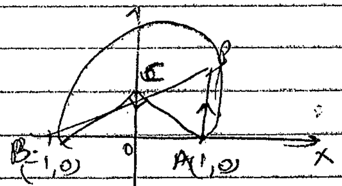
(ii) OPQR is a $\|$gm. $\angle ROQ = \angle PQO$ (alt $\angle$s, $PQ \parallel OR$)

$PQ = \sqrt{2}\,.\,OQ$. $= 45^\circ$ (ΔPOQ is rt$\angle$d. isos)

$\therefore\ OR = \sqrt{2}\,.\,OQ$

$\therefore\ R : \sqrt{2}\,.\,w\,.\,\text{cis}\frac{\pi}{4}$ (2M) for sound explanation.

(b)



Let $A:(1,0)$; $B(-1,0)$

if $P(z)$; $\angle PAX = \arg(z-1)$

$\angle PBX = \arg(z+1)$

$\angle APB = \arg(z-1) - \arg(z+1)$ (ext. $\angle$ of ΔPAB)

$\therefore\ \angle APB = \frac{\pi}{4}$

Thus P moves so that $\angle APB = \pi/4$.

$\therefore$ P traces a major arc of a circle, with AB as a chord. (2M - for sound expl.)

(i) if C is the centre $\angle ACB = \frac{\pi}{2}$ (angle at the centre = 2 angle etc.) (1)

$CA = CB =$ radius $= r$ (say)

$r^2 + r^2 = 2^2$; $2r^2 = 4$, centre $(0,1)$ (1)

$r = \sqrt{2}$. $\therefore\ r = \sqrt{2}$ (1)

also $OC^2 = AC^2 - OA^2$

c). $z^3 = -1$.

$z^3 + 1 = 0$

$(z+1)(z^2 - z + 1) = 0$

$z = -1$; $z = \frac{1 \pm \sqrt{-3}}{2}$

$= \frac{1 \pm \sqrt{3}i}{2}$ (2M)

if $\alpha = \frac{1+\sqrt{3}i}{2}$; $\alpha^2 = \left(\frac{1+\sqrt{3}i}{2}\right)^2$

$= \frac{1}{4}(1 - 3 + 2\sqrt{3}i)$

$= \frac{-2+2\sqrt{3}i}{4}$

$= \frac{-1+\sqrt{3}i}{2} = -\left(\frac{1-\sqrt{3}i}{2}\right)$

Thus $-\alpha^2$ is the other complex root. 1

Sum of roots; $\alpha - \alpha^2 - 1 = 0$

or $\alpha^2 - \alpha + 1 = 0$ 1

$(1-\alpha)^6 = (-\alpha^2)^6$ (1M)

$= \alpha^{12}$

$= (\alpha^3)^4$

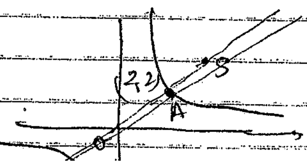
$= (-1)^4$

$= 1$. (1M)

Question 3

a) $e = \sqrt{2}$ $\boxed{1M}$

b)



$OS = OA \times \sqrt{2}$

$= \sqrt{2^2+2^2}\ \sqrt{2}$

$= 4$

also OS is the line $y = x$

$\therefore S = (2\sqrt{2}, 2\sqrt{2})$; $S'(-2\sqrt{2}, -2\sqrt{2})$ (2M)

b) $y = x + k$ meets $y = x^2$ at points given by

$$x^2 = x + k$$

$$x^2 - x - k = 0 \quad \text{——— (1)}$$

if the line does not intersect the curve the discriminant of (1) < 0 (1½)

$$(-1)^2 - 4(1)(-k) < 0$$

$$1 + 4k < 0$$

$$k < \frac{-1}{4}$$ 1½

c) $$\frac{5x^2+2x+7}{(x+1)(x^2+4)} = \frac{a}{x+1} + \frac{bx+c}{x^2+4}$$

$$\therefore 5x^2+2x+7 \equiv a(x^2+4) + (bx+c)(x+1)$$ $\boxed{1M}$

$x = -1$

$$5 - 2 + 7 = 5a$$

$$a = 2$$ ½

coeff of x^2: $5 = a + b \quad \therefore b = 3$ ½

Const term $7 = 4a + c \quad c = -1$ ½

$$\therefore \frac{5x^2+2x+7}{(x+1)(x^2+4)} = \frac{2}{x+1} + \frac{3x-1}{x^2+4}$$ ½

d) $y = mx + c$ meets $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at points given by

$$b^2x^2 + a^2(mx+c)^2 = a^2b^2.$$ $\boxed{1M}$

$$x^2(b^2 + a^2m^2) + x(2ma^2c) + c^2a^2 - a^2b^2 = 0$$

for $y = mx + c$ to be a tgt, the discriminant of the

$$(2mca^2)^2 - 4(b^2+a^2m^2)(a^2c^2 - a^2b^2) = 0$$

which on simplifying reduces to $\boxed{1M}$

$$c^2 = a^2m^2 + b^2$$

(ii) Let $y = mx + c$ be a tgt. to $\frac{x^2}{16} + \frac{y^2}{9} = 1$.

It passes through $(3, 4)$

$$\therefore 4 = 3m + c \quad \text{——— (1)}$$ (1)

also from (1) ; $c^2 = 16m^2 + 9$ ——— (2)

$$\therefore 16m^2 - (4 - 3m)^2 + 9 = 0$$ (1)

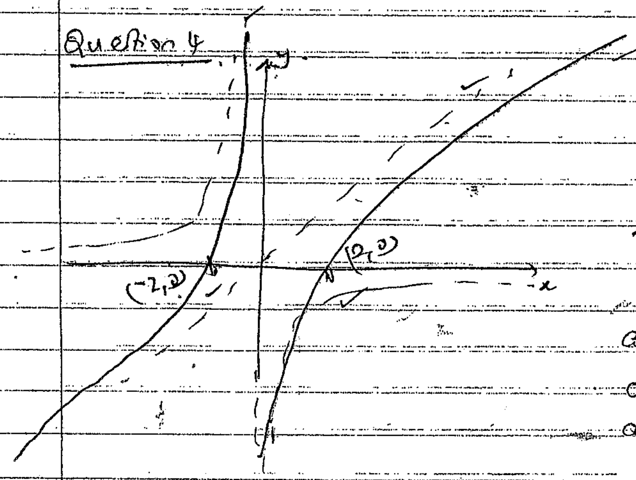
$$16m^2 - (16 - 24m + 9m^2) + 9 = 0$$

$$7m^2 + 24m - 7 = 0$$

The roots m_1 and m_2 are the gradients of the two tgts possible from $(3, 4)$

note $m_1 \times m_2 = -1 \quad \therefore$ The tgts are at right angles.

Question 4



x intercept:

$$y = 0$$

$$x - \frac{4}{x} = 0$$

$$x^2 - 4 = 0$$

$$x = \pm 2$$

as $x \to \infty$; $y \to x^-$

as $x \to -\infty$; $y \to x^+$

as $x \to 0^+$; $y \to -\infty$

as $x \to 0^-$; $y \to \infty$

b) $y = \frac{x}{x^2+1}$

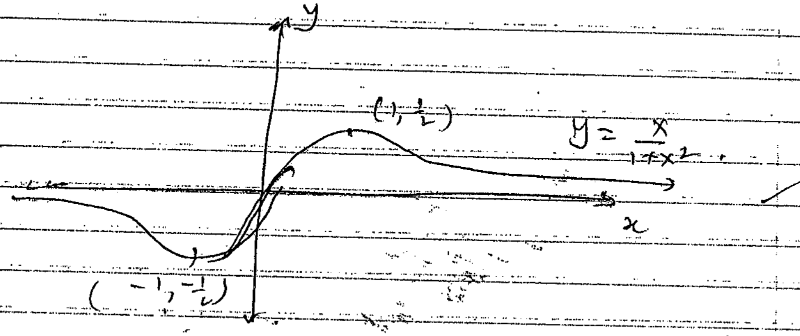
$y' = \frac{(x^2+1)(1) - x(2x)}{(x^2+1)^2}$

$y' = 0 \Rightarrow 1 - x^2 = 0$

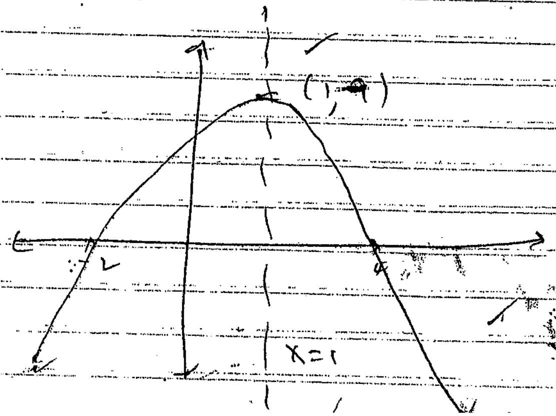
$x = \pm 1$

$x = 1$ $\quad$ $x = -1$
$y = \frac{1}{2}$ $\quad$ $y = -\frac{1}{2}$ $\Big\}$ are the turning points

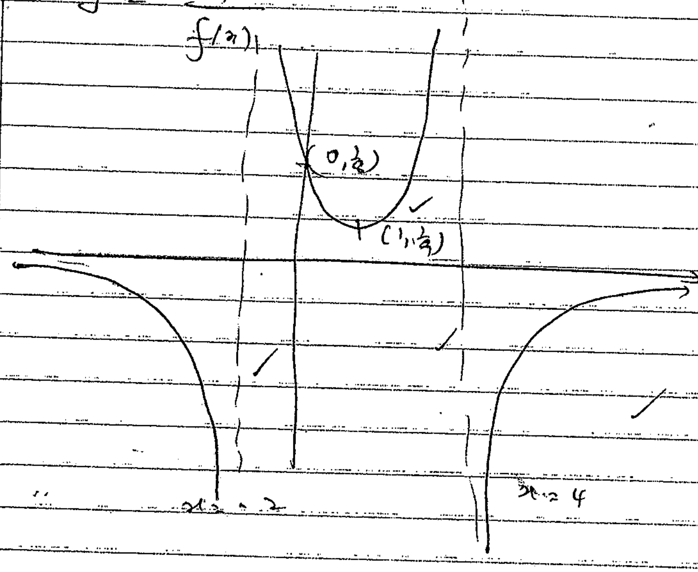
$x \to \infty$; $y \to 0^+$ $\quad$ $x = 0, y = 0$
$x \to -\infty$; $y \to 0^-$



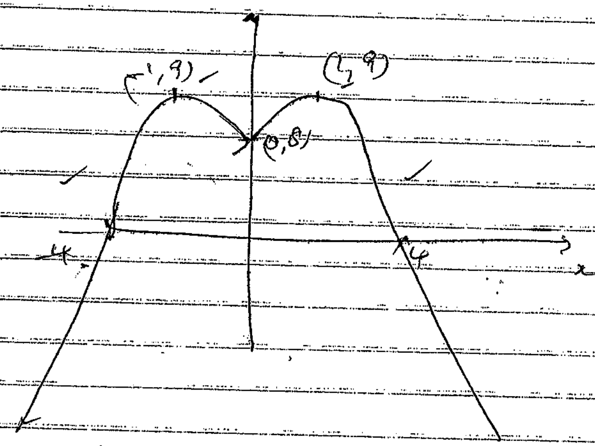
c) $y = (4-x)(x+2)$



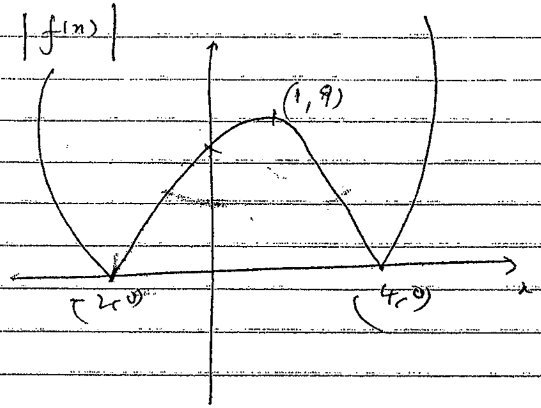
i) $y = \frac{1}{f(x)}$



ii) $y = f|x|$



(iii) $y = |f(x)|$



(iv) $y = e^{f(x)}$

$x = 4\ ;\ f(x) = 0\ ;\ e^{f(x)} = 1$

$x = -2 \qquad e^{f(x)} = 1$

$x = 1\ ;\ f(x) = 9\ ;\ e^{f(x)} = 9$

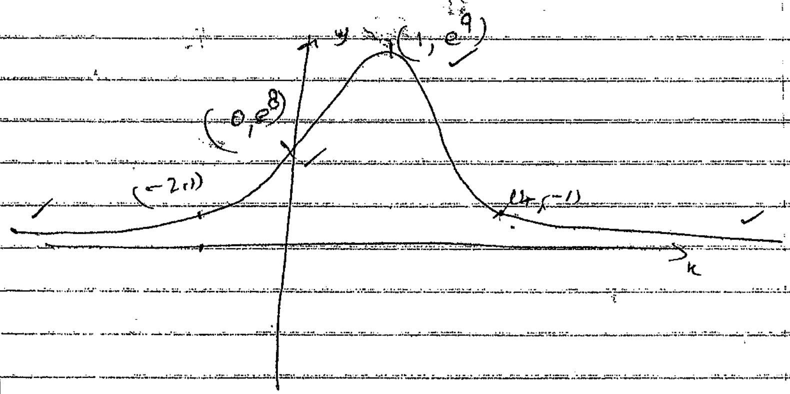
as $x \to \infty$; $f(x) \to -\infty$; $e^{f(x)} \to 0^+$

$x \to -\infty \qquad e^{f(x)} \to 0^+$

max. value of $f(x)$ is 9.

$x = 0,\ f(x) = 8$

$e^{f(x)} = e^8$



Section B

$$\frac{x^2}{9} - \frac{y^2}{4} = 1$$

$b^2 = a^2(e^2 - 1)$

$4 = 9(e^2 - 1)$

$e^2 = \frac{13}{9}$

$e = \frac{\sqrt{13}}{3} \quad (e > 0)$ $1\frac{1}{2}$

(ii) $x = \pm \frac{a}{e}$

$= \pm \frac{9}{\sqrt{13}}$ (1M)

(iii) $S : (\pm ae, 0)$

$= (\pm\sqrt{13}, 0)$ (1M)

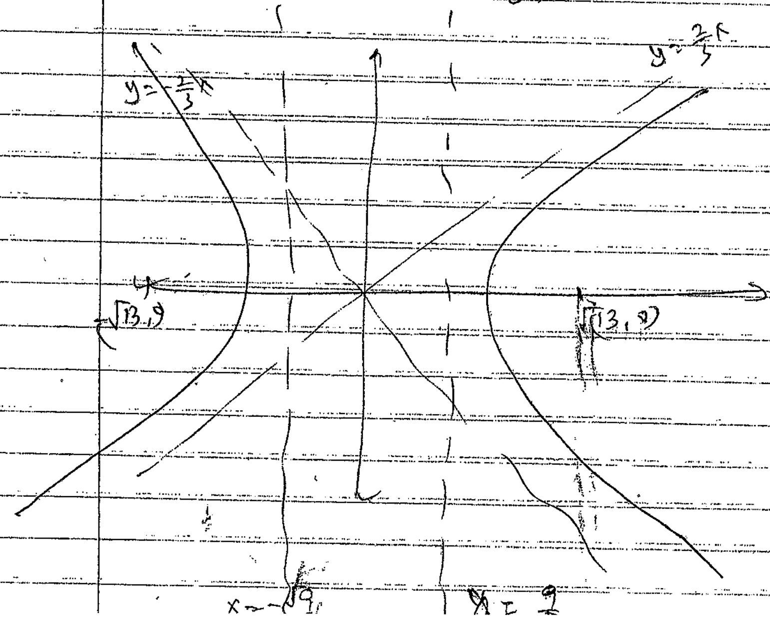
(iv) $y = \pm \frac{b}{a} x$

$= \pm \frac{2}{3} x$ (1M)

(v) angle is $2\tan^{-1}\frac{b}{a}$

$= 2\tan^{-1}\frac{2}{3}$

$= 67^\circ 23'$ ($1\frac{1}{2}$M)



(1M)

b) (i) $xy = 1$

$xy' + y = 0$

$y' = -\frac{y}{x}$

$y'_{(t,\frac{1}{t})} = -\frac{1}{t^2}$ $1\frac{1}{2}$

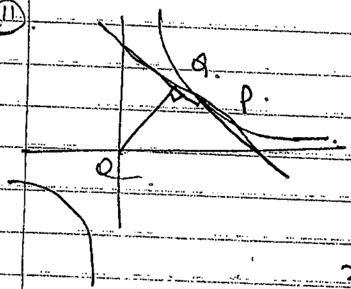
$\therefore$ Equation of tgt. at T is

$y - \frac{1}{t} = -\frac{1}{t^2}(x - t)$

$t^2 y - t = -x + t$

$x + t^2 y = 2t$ $1\frac{1}{2}$

(ii)



grad. of OQ $= t^2$

Equ. of OQ: $\underline{y = t^2 x}$ ($\frac{1}{2}$)

$y = t^2 x$ meets $x + t^2 y = 2t$ at points given by

$x + t^4 x = 2t$

$x = \frac{2t}{1+t^4}$

$y = t^2 \cdot \frac{2t}{1+t^4} = \frac{2t^3}{1+t^4}$. ($1\frac{1}{2}$)

(iii) $4xy$

$= 4 \cdot \frac{2t}{1+t^4} \cdot \frac{2t^3}{1+t^4}$

$= \frac{16t^4}{(1+t^4)^2}$

$(x^2+y^2)^2$

$= \left(\frac{4t^2}{(1+t^4)^2} + \frac{4t^6}{(1+t^4)^2}\right)^2$

$= \left[\frac{4t^2}{(1+t^4)^2}(1+t^4)\right]^2$

$(x^2+y^2)^2 = \left(\frac{4t^2}{1+t^4}\right)^2$

$= \frac{16t^4}{(1+t^4)^2}$

Thus locus of Q is $(x^2+y^2)^2 = 4xy$.

Q. 6

$i^{2004} = (i^2)^{1002}$

$= (-1)^{1002}$

$= 1$ 1

b) $(\cos\theta + i\sin\theta)^3 = \cos 3\theta + i\sin 3\theta$ (De Moivre's thm)

also $(\cos\theta + i\sin\theta)^3$

$= \cos^3\theta + 3\cos^2\theta(i\sin\theta) + 3\cos\theta(i\sin\theta)^2 + (i\sin\theta)^3$ 1M

Equating imag. parts.

$\sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta$

$= 3(1-\sin^2\theta).\sin\theta - \sin^3\theta$

$= 3\sin\theta - 4\sin^3\theta$.

(ii) where $\sin 3\theta = \frac{1}{2}$; $3\sin\theta - 4\sin^3\theta = \frac{1}{2}$

or $6\sin\theta - 8\sin^3\theta = 1$

or $8\sin^3\theta - 6\sin\theta + 1 = 0$

[2M for sound expl.]

Thus $x = \sin\theta$ satisfies $8x^3 - 6x + 1 = 0$, where θ is given by $\sin 3\theta = \frac{1}{2}$.

(iii) $\sin 3\theta = \frac{1}{2}$ $\theta = \frac{\pi}{18}, \frac{5\pi}{18}, \frac{13\pi}{18}$ 1

$3\theta = 30^\circ, 150^\circ, 390^\circ$

$\therefore$ The roots of $8x^3 - 6x + 1 = 0$ are

$\sin\frac{\pi}{18}$, $\sin\frac{5\pi}{18}$ and $\sin\frac{13\pi}{18}$ 1

(iv). Prod. of the roots:-

$\sin\frac{\pi}{18} \cdot \sin\frac{5\pi}{18} \cdot \sin\frac{13\pi}{18} = -\frac{1}{8}$. 2.

c) $x^3 + px - 1 = 0$; α, β, γ.

$\alpha + \beta + \gamma = 0$

$\alpha\beta + \beta\gamma + \gamma\alpha = p$

$\alpha\beta\gamma = 1$

$\alpha^2 + \beta^2 + \gamma^2 = (\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \beta\gamma + \gamma\alpha)$

$= -2p$.

more

(ii). $\alpha^4 + p\alpha^2 - \alpha = 0$

$\beta^4 + p\beta^2 - \beta = 0$

$\gamma^4 + p\gamma^2 - \gamma = 0$

$\therefore \alpha^4 + \beta^4 + \gamma^4 = (\alpha + \beta + \gamma) - p(\alpha^2 + \beta^2 + \gamma^2)$

$= 0 - p(-2p)$

$= 2p^2$.

(iii). $\frac{\alpha}{\beta\gamma} + \frac{\beta}{\gamma\alpha} + \frac{\gamma}{\alpha\beta} = \frac{\alpha^2 + \beta^2 + \gamma^2}{\alpha\beta\gamma} = \frac{-2p}{1}$ $(\frac{1}{2})$

$\frac{\alpha}{\beta\gamma} \cdot \frac{\beta}{\gamma\alpha} + \frac{\beta}{\gamma\alpha} \cdot \frac{\gamma}{\alpha\beta} + \frac{\alpha}{\beta\gamma} \cdot \frac{\gamma}{\alpha\beta} = \frac{1}{\gamma^2} + \frac{1}{\alpha^2} + \frac{1}{\beta^2}$

$\Sigma\alpha^2\beta^2$

(13)

$\Sigma\alpha^2\beta^2 = (\alpha\beta + \beta\gamma + \gamma\alpha)^2 - 2\alpha\beta\gamma(\alpha + \beta + \gamma)$

$= p^2 - 2 \times 1 \times 0$

$= p^2$ (1+1)

$\therefore \Sigma \frac{\alpha}{\beta\gamma} \cdot \frac{\beta}{\gamma\alpha} = \frac{p^2}{1}$ $(\frac{1}{2})$

$\frac{\alpha}{\beta\gamma} \cdot \frac{\beta}{\gamma\alpha} \cdot \frac{\gamma}{\alpha\beta} = \frac{1}{\alpha\beta\gamma} = 1$

$\therefore$ The required equation is (1)

$x^3 + 2px^2 - p^2x + 1 = 0$.

OR

Poly: with $\frac{\alpha}{\beta\gamma}, \frac{\beta}{\gamma\alpha}, \frac{\gamma}{\alpha\gamma}$ as roots is the same as poly.

with roots $\frac{\alpha^2}{\alpha\beta\gamma}, \frac{\beta^2}{\alpha\beta\gamma}, \frac{\gamma^2}{\alpha\beta\gamma}$ $(\alpha\beta\gamma = 1)$

i.e with roots $\alpha^2, \beta^2, \gamma^2$

$\therefore$ The equ is

$P(\sqrt{x}) = 0$ where $P(x)$: $x^3 + px - 1 = 0$

$(\sqrt{x})^3 + p\sqrt{x} - 1 = 0$

$x\sqrt{x} + p\sqrt{x} - 1 = 0$

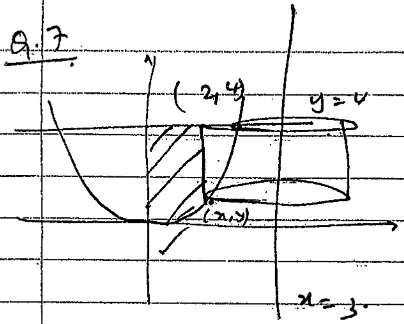
$\sqrt{x}(x + p) = 1$

$x(x+p)^2 = 1$

$x^3 + 2px^2 + p^2x - 1 = 0$

(14)

Q.7

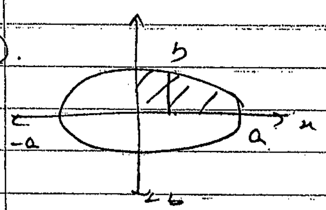


At any pt. $P(x,y)$ a cyl. shell has a vol. $2\pi(3-x)(4-y)\cdot\Delta x$

$$\therefore V = 2\pi\int_0^2 (3-x)(4-x^2)\,dx$$

$$= 4\pi\ u^3$$

(b)



Shaded area $= \int_0^a y\,dx$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$b^2x^2 + a^2y^2 = a^2b^2$$

$$y^2 = \frac{b^2}{a^2}(a^2-x^2)$$

$$y = \pm\frac{b}{a}\sqrt{a^2-x^2}$$

$$= \int_0^a \frac{b}{a}\sqrt{a^2-x^2}$$

Area of ellipse

$$= \frac{4b}{a}\int_0^a \sqrt{a^2-x^2}\,dx$$

Let $x = a\sin\theta$

$dx = a\cos\theta\,d\theta$

$x=0;\ \theta=0$

$x=a;\ \theta=\frac{\pi}{2}$

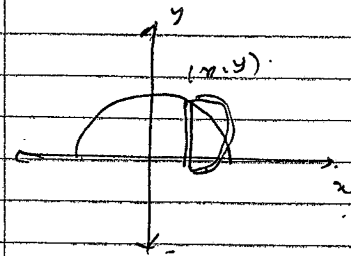
$$= \frac{4b}{a}\int_0^{\pi/2} a\cos\theta\cdot a\cos\theta\,d\theta$$

$$= 4ab\int_0^{\pi/2} \frac{1+\cos 2\theta}{2}\,d\theta$$

$$= 2ab\left[\theta + \frac{\sin 2\theta}{2}\right]_0^{\pi/2}$$

$$= 2ab\cdot\frac{\pi}{2} = \pi ab\ u^2$$

(c).



Semi. circle is $y = \sqrt{9-x^2}$

At a pt. (x,y)

major axis is y

Semi. major axis is $\frac{y}{2}$

" minor axis is $\frac{y}{4}$

Area of the ellipse is $\pi\,\frac{y}{2}\cdot\frac{y}{4}$.

$$\Delta V = \frac{\pi}{8}y^2\cdot\Delta x$$

$$\therefore V = \frac{\pi}{8}\int_{-3}^{3} y^2\,dx$$

$$= \frac{\pi}{8}\int_{-3}^{3}(9-x^2)\,dx$$

$$= \frac{\pi}{4}\int_0^3 (9-x^2)\,dx \quad [\text{being an even fn.}]$$

$$= \frac{\pi}{4}\left(9x - \frac{x^3}{3}\right)_0^3$$

$$= \frac{9\pi}{2}\ u^3$$

Q. 8 $\sin^{-1}(\sin x) = x$, when $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$

$\sin(\sin^{-1}x) = x$ $\quad -1 \le x \le 1$

b) Let $\sin^{-1}x = a$ $\quad \therefore \sin a = x$

$\cos^{-1}x = \frac{\pi}{2} - a$.

$\sin(\sin^{-1}x - \cos^{-1}x)$

$= \sin\left(a - \left(\frac{\pi}{2} - a\right)\right)$

$= \sin\left(-\left(\frac{\pi}{2} - 2a\right)\right)$

$= -\sin\left(\frac{\pi}{2} - 2a\right)$

$= -\cos 2a$

$= 2\sin^2 a - 1$

$= 2x^2 - 1$

(ii) $\sin^{-1}x - \cos^{-1}x = \sin^{-1}(0.5)$

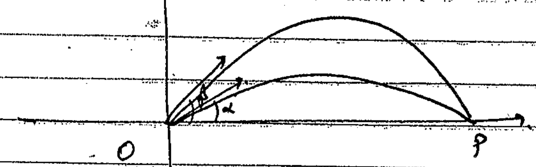
$\therefore \sin(\sin^{-1}x - \cos^{-1}x) = 0.5$

$2x^2 - 1 = \frac{1}{2}$

$4x^2 = 3$

$x = \pm\frac{\sqrt{3}}{2}$.

(c)



Place the axes at the pt. of projection.

For both particle $\ddot{x} = 0$; $\ddot{y} = -g$

For particle with initial angle α;

$\dot{x} = v\cos\alpha$ $\qquad \dot{y} = -gt + c$ $\qquad t=0$, $\dot{y} = v\sin\alpha$

$x = v\cos\alpha\, t$ $\qquad = -gt + v\sin\alpha$

$y = -\frac{gt^2}{2} + vt\sin\alpha$ $\qquad t=0$, $y=0$

$=0$, $= v\cos\alpha$, $=0$, $x=0$

Similarly for the other particle

$\dot{x} = v\cos\beta$ $\qquad \dot{y} = -gt + v\sin\beta$

$x = v\cos\beta\, t$ $\qquad y = -\frac{gt^2}{2} + v\sin\beta\, t$

When they are at P; $y=0$. $-\frac{gt^2}{2} + v\sin\alpha\, t = 0$; at P, $t \ne 0$

$t_1 = \frac{2v\sin\alpha}{g}$ $\qquad t_2 = \frac{2v\sin\beta}{g}$

$t_2 - t_1 = \frac{2v}{g}(\sin\beta - \sin\alpha)$

(ii) also for both particle x is the same at P.

$\therefore v\cos\alpha\, t_1 = v\cos\beta\, t_2$, where

$\therefore v\cos\alpha \cdot \frac{2v\sin\alpha}{g} = v\cos\beta \cdot \frac{2v\sin\beta}{g}$

$\sin 2\alpha = \sin 2\beta$

$2\alpha = 2\beta, (\pi - 2\beta)$

$\alpha = \beta, \left(\frac{\pi}{2} - \beta\right)$;

$\alpha \ne \beta$ $\quad \therefore \alpha = \frac{\pi}{2} - \beta$

(iii) $(t_2 - t_1)^2 = \frac{4v^2}{g^2}\left(\sin\beta - \sin\left(\frac{\pi}{2} - \beta\right)\right)^2$

$= \frac{4v^2}{g^2}(\sin\beta - \cos\beta)^2$

$= \frac{4v^2}{g^2}(\sin^2\beta + \cos^2\beta - 2\sin\beta\cos\beta)$

$$= \frac{4v^2}{g^2}\left(1 - \sin 2\beta\right) \qquad \text{——} \; (A)$$

$$X = V\cos\beta \cdot 2v\frac{\sin\beta}{g}$$

$$= \frac{\cancel{2}v^2}{g} \cdot \sin 2\beta .$$

$$\therefore \sin 2\beta = \frac{gX}{v^2}$$

Sub in (A)

$$(t_2 - t_1)^2 = \frac{4v^2}{g^2}\left(1 - \frac{gx}{v^2}\right).$$