Student Number:________________

St. Catherine's School
Waverley

February 2007
HSC ASSESSMENT TASK
EXAMINATION

Extension I Mathematics

Time allowed: *55 minutes*

INSTRUCTIONS

- There are 3 sections of value 13, 13 and 12
- Marks for each part of a question are indicated
- All questions should be attempted on the separate paper provided
- All necessary working should be shown
- Start each question on a new page
- Approved scientific calculators and drawing templates may be used badly arranged work.

Section A **13 Marks**

1. Two of the roots of the polynomial equation $x^3 - 13kx^2 + 13kx - 1 = 0$ are k and $\frac{1}{k}$ where $k > 0$.

   (i) Find the third root [1]

   (ii) Find the possible value(s) of k [2]

2. A polynomial $P(x)$ is of degree 3. It is also odd. Ie. $f(-x) = -f(x)$.

   (i) Show that x is a factor of $P(x)$. [2]

   (ii) This polynomial has $(x + 3)$ as a factor and when it is divided by $(x - 2)$ the remainder is 10. Find $P(x)$. [2]

3. The polynomial $P(x) = ax^3 + bx + c$ (where $a \neq 0,\ c \neq 0$) has one factor in the form $\left(x^2 + px + 1\right)$ where p is real and $p \neq 0$.

   (i) Find the other factor. [1]

   (ii) Hence or otherwise show that $a^2 - c^2 = ab$ [2]

4. (i) Show that $f(x) = x^3 - 3x^2 + x - 4 = 0$ has a root between 3 and 4. [1]

   (ii) Apply Newton's Method once to find a more accurate approximation of the root given that given $x = 3.5$ is an approximate root. Give your answer correct to 2 decimal places. [2]

Section B – Start a new page **13 Marks**

5. (i) Show that $\sqrt{3}\sin x - \cos x$ can be rewritten as $2\sin\left(x - \frac{\pi}{6}\right)$ [2]

   (ii) Hence or otherwise solve $\sqrt{3}\sin x - \cos x = 1$ for $0 \leq x \leq 2\pi$ [2]

Section B Continued on Next Page

Section B Continued

6. (i) Show that the equation $\sqrt{3}\sec^2\theta - 2\tan\theta - 2\sqrt{3} = 0$ can be rewritten as $\left(\sqrt{3}\tan\theta + 1\right)\left(\tan\theta - \sqrt{3}\right) = 0$ [2]

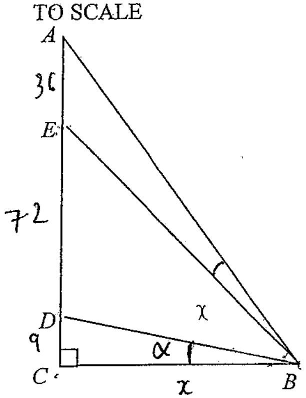
(ii) Hence or otherwise find the general solutions to the equation

$$\sqrt{3}\sec^2\theta - 2\tan\theta - 2\sqrt{3} = 0$$ [2]

7. (i) Write down the expansion $\tan(\alpha + \beta)$ [1]

The diagram shows a building, AC, with windows at D and E. From a point, B, on horizontal ground, it is found that $\angle ABE = \angle DBC$.

DIAGRAM NOT TO SCALE



$DC = 9m$, $DE = 72m$, $AE = 36m$
$BC = x$ metres
$\angle CBD = \alpha$, $\angle EBC = \beta$

(ii) Show that $\dfrac{117}{x} = \dfrac{90x}{x^2 - 729}$ [2]

Hint: First find $\tan(\alpha+\beta)$ in ΔABC

(iii) Hence show that $x = 56.2m$ to 3 significant figures [2]

Section C – Start a new page **12 Marks**

8. Prove that $\dfrac{\sin 8A}{1+\cos 8A} = \tan 4A$ [2]

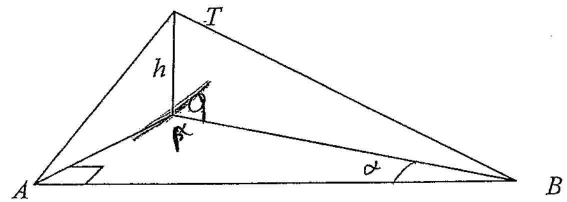
9. Show that $\dfrac{2\cos A}{\operatorname{cosec}A - 2\sin A} = \tan 2A$ [3]

Section C Continued on Next Page

Section C Continued

10. Prove that $\dfrac{\sin 3\theta}{\sin\theta} - \dfrac{\cos 3\theta}{\cos\theta}$ is independent of θ. [2]

11. A surveyor stands at a point A, which is due south of a tower OT of height h m. The angle of elevation of the top of the tower from A is 45°. The surveyor then walks 100 m due east to point B, from where she measures the angle of elevation of the top of the tower to be 30°.



(i) Express the length of OA and OB in terms of h. [1]

(ii) Show that $h = 50\sqrt{2}$. [2]

(iii) Calculate the bearing of B from the base of the tower. [2]

End of Task

		Marks	Comments+Criteria
			SOLUTIONS
1.	$\alpha\beta\gamma = \frac{-d}{a} = 1$ Let α, β, γ be the roots		
	$\therefore k \times \frac{1}{k} \times \gamma = 1$	1	
	$\therefore$ The third root is 1		
ii	$\alpha + \beta + \gamma = \frac{-b}{a} = 13k$		
	$k + \frac{1}{k} + 1 = 13k$	1	
	$k^2 + 1 + k = 13k^2$		
	$12k^2 - k - 1 = 0$		
	$12k^2 - 4k + 3k - 1 = 0$		
	$4k(3k-1) + (3k-1) = 0$		
	$(4k+1)(3k-1) = 0$	1	
	$\therefore k = -1/4$ or $k = 1/3$		
2	$P(x)$ is odd		
	$\therefore P(x) = -P(x)$		
	$\therefore P(0) = -P(0)$	1	
	Hence: $P(0) = 0$		
	$\therefore x$ is a factor	1	
i	If $(x+3)$ is a factor so is $(x-3)$ (as odd)	1	
	$\therefore P(x) = ax(x-3)(x+3)$		
	$P(2) = 10$		
	$10 = 2a \times -1 \times 5$		
	$10 = -10a$		
	$a = -1$		
	$\therefore P(x) = -x(x-3)(x+3)$	1	
3 i	$P(x) = ax^3 + bx + c$	1	Alternatively & more concisely
	$$\begin{array}{r} ax - ap \\ x^2+px+1 \,\big)\, ax^3 \qquad + bx + c \\ ax^3 + apx^2 + ax \\ \hline -apx^2 + (b-a)x + c \\ -apx^2 - ap^2x - ap \\ \hline (b-a+ap^2)x + c + ap \end{array}$$		$ax^3+bx+c = (x^2+px+1)(ax+c)$ (observation) (2) Equate coeffs. of x^2: $0 = c + ap$ (1) of x: $b = cp + a$ (2) eliminate p. (1)
	$\therefore$ other factor is $ax - ap$	1	$p = -\frac{c}{a}$ from (1) subs in (2) $b = c\left(-\frac{c}{a}\right) + a$
	$\therefore (b - a + ap^2)x + c + ap = 0$		
	$\therefore a - b = ap^2$ (1) & $-c = ap$ (2) $p = -\frac{c}{a}$	1	$\therefore ab = -c^2 + a^2$ (1)
	Subst (2) into (1)		
	$a - b = \cancel{a} \times \frac{c^2}{a^{\cancel{2}}}$		
	$\therefore a^2 - ba = c^2$		
	$\therefore a^2 - c^2 = ab$	1	
4(a)	$f(3) = -1$, $f(4) = 16$. Note: Given that the function is a polynomial by definition it is continuous for all values of x. Hence as $f(3) < 0$ & $f(4) > 0$ there must be a zero between $x = 3$ & $x = 4$.	1	Don't penalise if they do not state the city.
(b)	$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$ $\quad f'(x) = 3x^2 - 6x + 1$	0.5	I don't employ leniency & ignore mechanical errors.
	$x_2 = 3.5 - \frac{5.625}{16.75}$ $\quad f'(3.5) = 16.75$	0.5	
	$f(3.5) = 5.625$	0.5	
	$= 3.16$ (to 2 d.p)	0.5	

$= 2\sin(x - \alpha)$

$2\sin\left(x - \frac{\pi}{6}\right)$

$= 2\left(\sin x \cos\frac{\pi}{6} - \cos x \sin\frac{\pi}{6}\right)$ 1

$= 2\left(\frac{\sqrt{3}}{2}\sin x - \frac{1}{2}\cos x\right)$

$= \sqrt{3}\sin x - \cos x$ 1

$\therefore \sqrt{3}\sin x - \cos x = 2\sin\left(x - \frac{\pi}{6}\right)$

or $r = \sqrt{\sqrt{3}^2 + 1^2}$

$= 2$

$\sqrt{3}\sin x - \cos x$

$= 2(\sin x \cos\alpha - \cos x \sin x)$

$\therefore \sqrt{3} = 2\cos\alpha$

$1 = 2\sin\alpha$

$\tan\alpha = \frac{1}{\sqrt{3}}$

$\alpha = \frac{\pi}{6}$

$\frac{\pi}{6}$ is 1M

ii

$2\sin\left(x - \frac{\pi}{6}\right) = 1$

$\sin\left(x - \frac{\pi}{6}\right) = \frac{1}{2}$

$x - \frac{\pi}{6} = \frac{\pi}{6}, \frac{5\pi}{6}$

$0 \le x \le 2\pi$

$\therefore x = \frac{\pi}{3}, \pi$ for $0 \le x \le 2\pi$ 1

6i

$\sqrt{3}\sec^2\theta - 2\tan\theta - 2\sqrt{3} = 0$

$(\sqrt{3}\tan\theta + 1)(\tan\theta - \sqrt{3})$

$= \sqrt{3}\tan^2\theta - 3\tan\theta + \tan\theta - \sqrt{3}$ 1

$= \sqrt{3}(\sec^2\theta - 1) - 2\tan\theta - \sqrt{3}$ 1

$= \sqrt{3}\sec^2\theta - 2\tan\theta - 2\sqrt{3}$

$\therefore \sqrt{3}\sec^2\theta - 2\tan\theta - 2\sqrt{3} = 0$ can be rewritten as

$(\sqrt{3}\tan\theta + 1)(\tan\theta - \sqrt{3}) = 0$

Alternatively

$\sqrt{3}(1 + \tan^2\theta) - 2\tan\theta - 2\sqrt{3} = 0$

$\sqrt{3}\tan^2\theta - 2\tan\theta - \sqrt{3} = 0$

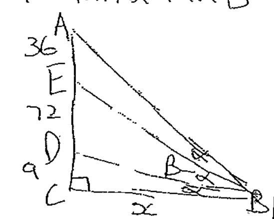
on factorising

$(\sqrt{3}\tan\theta + 1)(\tan\theta - \sqrt{3}) = 0$

$\therefore \tan\theta = -\frac{1}{\sqrt{3}}$ or $\tan\theta = \sqrt{3}$ 1

$\therefore \theta = \pi n - \frac{\pi}{6}$ or $\theta = \pi n + \frac{\pi}{3}$ 1

$1 - \tan\alpha\tan\beta$ 1



ii

$\tan\alpha = 9/x$

$\tan(\beta + \alpha) = \frac{117}{x}$

$\tan\beta = \frac{81}{x}$ 1

$\therefore \tan(\alpha + \beta) = \frac{117}{x} = \frac{9/x + 81/x}{1 - 9/x \times 81/x}$

$\frac{117}{x} = \frac{90/x}{1 - \frac{729}{x^2}}$

$\therefore \frac{117}{x} = \frac{90x}{x^2 - 729}$ 1

iii

$117x^2 - 85293 = 90x^2$ 1

$27x^2 = 85293$

$x^2 = 3159$

$\therefore x = 56.2$ m (to 3 s.f) 1

(note: $x > 0$)

8

$\frac{\sin 8A}{1 + \cos 8A} = \tan 4A$

LHS $= \frac{\cancel{2}\sin 4A\cos 4A}{\cancel{1} + \cancel{2}\cos^2 4A - \cancel{1}}$ 1

$= \tan 4A$ 1

$=$ RHS

$\therefore$ true

$$\frac{2\cos A}{\operatorname{cosec} A - 2\sin A} = \tan 2A$$

$$LHS = \frac{2\cos A}{\frac{1}{\sin A} - 2\sin A}$$

$$= \frac{2\cos A}{\frac{1-2\sin^2 A}{\sin A}}$$ 1

$$= \frac{2\sin A\cos A}{\cos 2A}$$ 1

$$= \frac{\sin 2A}{\cos 2A} = \tan 2A = RHS$$ ∴ true. 1

10 $$\frac{\sin 3\theta}{\sin\theta} - \frac{\cos 3\theta}{\cos\theta}$$

$$= \frac{\sin 3\theta\cos\theta - \cos 3\theta\sin\theta}{\sin\theta\cos\theta}$$ 1

$$= \frac{\sin(3\theta - \theta)}{\frac{1}{2}\sin 2\theta}$$

$$= \frac{\sin 2\theta}{\frac{1}{2}\sin 2\theta}$$

$= 2$ which is indept of θ. 1

11 i In ΔBOT

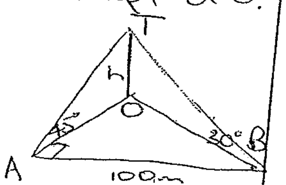
$\tan 30° = \frac{h}{OB}$

$OB = \sqrt{3}h$

In ΔAOT

$\tan 45° = \frac{h}{OA}$

$OA = h$



…st have the value of tan 30° & tan 45° to get 1/1 otherwise 0/1

In ΔAOB, rt ∠d. at A. 1

$h^2 + 100^2 = 3h^2$ (By Pythag) 1

$2h^2 = 10000$

$h^2 = 5000$

$h = \sqrt{25} \times \sqrt{100} \times \sqrt{2}$

$= 50\sqrt{2}$ m 1

(Note: $h > 0$)

Solutions	Marks	Comments+Criteria
iii $\tan\angle AOB = \frac{100}{OA} = \frac{100}{50\sqrt{2}}$		
$\therefore \angle AOB = 54°44'$	1	
$180° - 54°44' = 125°16'$ T	1	