ST CATHERINE'S COLLEGE - 2002 - EXT 1. ½ YRLY

SECTION A

Question 1

a) P(x) : $x^3 - x^2 - 10x - 8$ is a polynomial.

(i) Show that x = -1 is a zero of P(x). (1m)

(ii) Factorise P(x) fully. (2m)

(iii) Sketch the graph of y= P(x). (1m)

b) b) Prove by Mathematical Induction that

$$\frac{1}{1\times 2}+\frac{1}{2\times 3}+\frac{1}{3\times 4}+\ldots\ldots\ldots\ldots\ldots\ldots\frac{1}{n\times(n+1)}=\frac{n}{n+1}$$

is true for all n>1. (4m)

c) P(x): $ax^3 + bx^2 + 2x - 4$ is a polynomial with real coefficients.

When P(x) is divided by x-1 the remainder is 3 and P(x) is divisible by (x+2).

Find the values of a and b. (4m)

Question 2

a) (i) Find the Cartesian equation of the parabola whose parametric equation is given by

$$x = t - 2 \text{ and } y = \frac{t^2+1}{2}$$

(1m)

(ii) Sketch this parabola. (3m)

b) The line y = ax -2 is given to be a tangent to the parabola $x^2 = 4y$. Find the possible value(s) of a. (3m)

c) P:(2,1) is a point on the parabola $x^2 = 4y$.

(i) Find the equation of the tangent to the parabola at P (2m)

(ii) This tangent meets the x and y axes at M and N respectively.
Find the coordinates of the points M and N. (2m)

(iii) If PQ is drawn perpendicular to the directrix meeting it at Q, find the coordinates of Q. (1m)

~~SECTION B~~

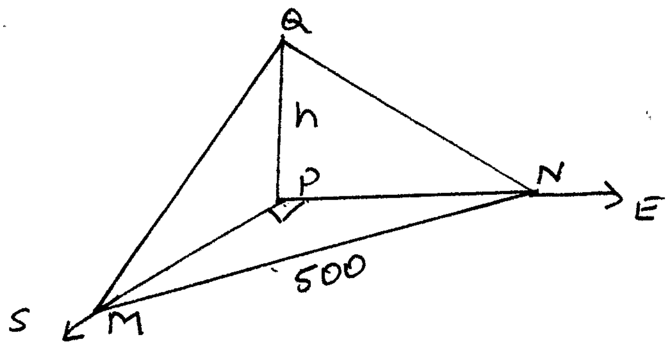
Question 3

a) Show that the quadratic equation
$x^2 - (m-3)x - 4(m+1) = 0$ has real roots for all values of m. (3m)

b) From a point M due south of a tower PQ, the angle of elevation of the top of a tower is 12^0.

From a point N due east of the tower, the angle of elevation of the same tower is 15^0

The distance between the points M and N is 500 metres.



Let the height of the tower be h metres.

(i) Use the triangle PQM to find an expression for the side PM in terms of h. (1m)

(ii) Use the triangle PQN to find the an expression for the side PN in terms of h. (1m)

(iii) Use the triangle PMN to evaluate the height of the tower to the nearest metre. (3m)

c) If the roots of the polynomial

$x^3+6x^2+3bx-10=0$ are in arithmetic progression,

Consider the sum and product of roots and

(i) Find the roots. (3m)

(ii) Find the value of b. (1m)

Question 4

a) Use Mathematical Induction to show that 5^n-1 is divisible by 4 for all values of n: $n \geq 1$ (4m)

b) P(x) is an even polynomial (i.e. P(-x) = P(x))

(i) Show that if b is a zero of P(x), -b is also a zero of P(x). (1m)

(ii) A polynomial P(x) is an even polynomial of degree 4 and has (x-2) and (x+1) as two of its factors,. Also P(0) = 48. Find P(x). (3m)

c) P: $(2ap, ap^2)$ and Q: $(2aq, aq^2)$ are two points on a parabola $x^2=4ay$. PQ is drawn parallel to the line y = 2x+3.

(i) Show that (p+q) = 4. (2m)

(ii) Find the locus of the point of the mid point of PQ. (2m)

Question 5.

a) Solve for x: $\dfrac{x+1}{x-1} \leq 3$ (4m)

b) If α and β are the roots of the equation $x^2-4x+1=0$, find

(i) $\alpha+\beta$ and $\alpha\beta$ (1m)

(ii) $\alpha^2+\beta^2$ (2m)

(iii) Find the quadratic equation whose roots are α^2 and β^2 (1m)

c) The acute angle between the lines x -2y +3 =0 and y = mx is 45^0

(i) Show that $\left|\dfrac{2m-1}{m+2}\right|=1$ (2m)

(ii) Find the possible values of m. (2m)

SECTION C

Question 6.

a) Solve for x: $5x-x^2>0$ (1m)

b) If two roots of the equation $x^3+qx+r=0$ are equal,

show that $4q^3+27r^2=0$. (4m)

c) (i) Carefully sketch the continuous curve using the following information.

f(0) = -5, for 0< x <1, $f'(x) > 0$ and $f''(x) < 0$.

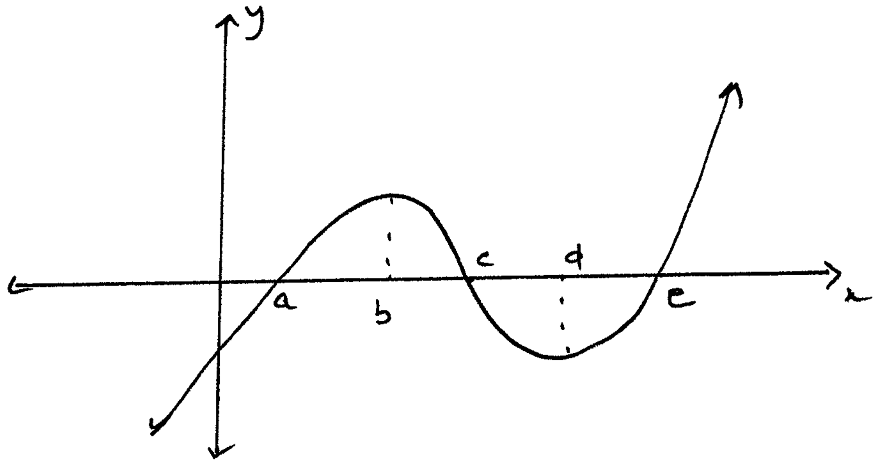
$f(1)= f'(1) = f''(1) = 0$, for $1 < x < 2$, $f'(x) > 0$ anf $f''(x) > 0$

f(2) = 5 and $f'(2) = 0$.

for x > 2, $f'(x) < 0$ and $f''(x) < 0$. (3m)

(ii) What type of point is x = 1. (1m)

c) The sketch shows the graph of y= f (x)



(i) Copy the diagram onto your paper.

(ii) Sketch the graph its derivative function (1m)

(iii) Sketch the graph of its primitive function. (2m)

(In both sketches label the key points clearly)

Question 7.

a) (i) Show that $y=6-3x-x^3$ is a monotonic decreasing function. (2m)

(ii) Explain why this function has only one real root. (1m)

(iii) Show that there is a root between x = 1 and x = 2 (1m)

(iv) Start with x =0.5 and use Newton's method once to find a better approximation for the root. (2m)

v) By halving the interval, show that the root lies between 1.25 and 1.5. (2m)

b) P: $(2ap, ap^2)$ and Q: $(2aq, aq^2)$ are two points on a parabola $x^2 = 4ay$. Chord PQ subtends a right angle at the origin.

(i) Show that $pq = -4$. (1m)

(ii) Find the locus of the midpoint of PQ. (3m)

END OF PAPER.

where they meet,

$x^2 = 4(ax - 2)$

$x^2 - 4ax + 8 = 0$ (1M)

Since the line is a tgt, there is only one point of intersection

$\therefore \Delta = 0$. (1M)

$16a^2 - 32 = 0$

$a^2 = 2$ (1M)

$a = \pm\sqrt{2}$

c) $x^2 = 4y$

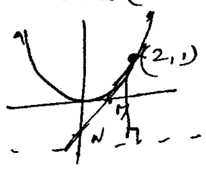
$y' = \frac{2x}{4} = \frac{x}{2}$ (1M)

y' at $(2,1)$ $= \frac{2}{2} = 1$

Eqn. of tangent at $(2,1)$ is

$y - 1 = 1(x - 2)$ (1M)

$y = x - 1$



(ii) M: $y = 0$ $x = 1$

$(1, 0)$ (1M)

N: $x = 0$; $y = -1$

$(0, -1)$ (1M)

(iii) directrix is $y = -1$.

$Q: (2, -1)$ (1M)

Qn 3.

a) $x^2 - (m-3)x - 4(m+1) = 0$ Δ; (1M)

real roots. $\Rightarrow \Delta \geq 0$. ✓ (1M) Conclusion (1M)

$\Delta = (m-3)^2 + 16(m+1)$

$= m^2 - 6m + 9 + 16m + 16$

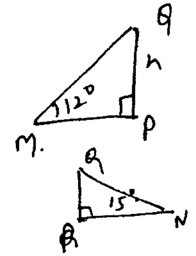
$= m^2 + 10m + 25$ ✓

$= (m+5)^2$

$(m+5)^2$, being a perfect square is ≥ 0 ✓

Thus $\Delta \geq 0$ $\therefore$ real roots $\forall m$.

b)



In ΔPQM; $\tan 12° = \frac{h}{MP}$

$\therefore MP = h \cot 12.$ 1M

In ΔPQN; $\tan 15° = \frac{h}{NP}$.

$NP = h \cot 15°$ 1M

In ΔPMN $\angle MPN = 90°$

$MN^2 = MP^2 + PN^2$

$500^2 = h^2(\cot^2 15 + \cot^2 12)$ 1M

$h = \frac{500}{\sqrt{\cot^2 15 + \cot^2 12}}$ 1M

$= 83.261...$

$= \boxed{83 \text{ m}}$ (nearest m) 1M

c) Let $a-d$, a, $a+d$ be the roots. $(-\frac{1}{2}$ for accuracy$)$

Sum of roots is $3a = -6$

$a = -2$ (1M)

Product $a(a^2 - d^2) = 10$ $d^2 = 9$ (1M)

$a(a-d) + a(a+d) + (a-d)(a+d) = 3b$

$2a^2 + a^2 - d^2 = 3b$

$3(-2)^2 - 9 = 3b$

$\boxed{b = 1}$ 1M

Year 12 ...

a)

$P(x) = x^3 - x^2 - 10x - 8$

(i) $P(-1) = (-1)^3 - (-1)^2 - 10(-1) - 8$ (1M)

$= 0$

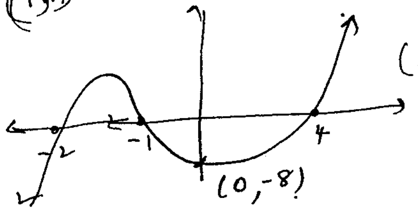
(ii) $\therefore x = -1$ is a zero.

$$\begin{array}{r} x^2 - 2x - 8 \\ x+1 \overline{) x^3 - x^2 - 10x - 8} \\ x^3 + x^2 \\ \hline -2x^2 - 10x - 8 \\ -2x^2 - 2x \\ \hline -8x - 8 \\ -8x - 8 \\ \hline 0 \end{array}$$

(1m)

$\therefore P(x) = (x+1)(x^2 - 2x - 8)$

$= (x+1)(x-4)(x+2)$ (1m)



(1m)

y intercept not necessary.

b) Let $P(n)$: $\frac{1}{1\times 2} + \frac{1}{2\times 3} + \frac{1}{3\times 4} \cdots + \frac{1}{n(n+1)} = \frac{n}{n+1}$

Consider $P(1)$

LHS $\frac{1}{1\times 2} = \frac{1}{2}$

RHS: $\frac{1}{1+2} = \frac{1}{2}$ (1M)

$\therefore P(1)$ is true

Let $P(n)$ be true for $n = k$ —— (A) ($\frac{1}{2}$m)

$\frac{1}{1\times 2} + \frac{1}{2\times 3} + \frac{1}{3\times 4} + \cdots + \frac{1}{k(k+1)} = \frac{k}{k+1}$

Consider $P(k+1)$: $\left[\frac{1}{1\times 2} + \cdots + \frac{1}{(k+1)(k+2)} = \frac{k+1}{k+2}\right]$

LHS $= \frac{1}{1\times 2} + \frac{1}{2\times 3} + \cdots + \frac{1}{k(k+1)} + \frac{1}{(k+1)(k+2)}$ (2m)

$= \frac{k}{k+1} + \frac{1}{(k+1)(k+2)}$ (by (A))

$= \frac{k(k+2)+1}{(k+1)(k+2)} = \frac{k^2+2k+1}{(k+1)(k+2)} = \frac{(k+1)^2}{(k+1)(k+2)} = \frac{k+1}{k+2}$

$=$ RHS.

$\therefore P(k+1)$ ...

Thus $P(1)$ is true and if $P(k)$ is true, $P(k+1)$ is ...

$\therefore$ By the principle of Mathematical Induction ($\frac{1}{2}$m)

$P(n)$ is true for all $n \geq 1$.

c) $P(x)$: $ax^3 + bx^2 + 2x - 4$

$P(1) = 3$ and $P(-2) = 0$ [1M]

$a + b + 2 - 4 = 3$

$a + b = 5$ —— (1) [1M]

$-8a + 4b - 4 - 4 = 0$

$-8a + 4b = 8$ [1M]

$-2a + b = 2$ —— (2)

Solve (1) & (2) simultaneously

$3b = 12$

$b = \frac{12}{3} = 4$

$a = 1$ $b = 4$ [1M]

(Some may not state explicitly $P(1) = 3$ and $P(-2) = 0$ marks may not be deducted if they do not state. Marks be given if they state it.)

Q 2

$x = t - 2$ $y = \frac{t^2+1}{2}$

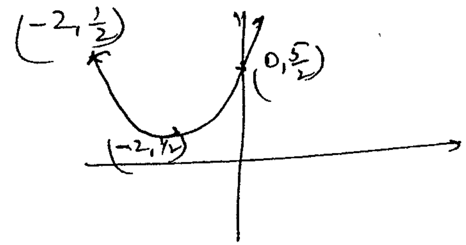
eliminate t; $y = \frac{(x+2)^2+1}{2}$ (1m)

(unsimplified version can fetch a mark)

$2y - 1 = (x+2)^2$

$(x+2)^2 = 2(y - \frac{1}{2})$

Vertex $(-2, \frac{1}{2})$



Vertex ($1\frac{1}{2}$m)

- Sketch may or may not show directrix/vertex.
- It should however show one more point say the y intercept. ($1\frac{1}{2}$m)

b). $y = ax - 2$ meets

$x^2 = 4y$ at points given by.

Solving them simultaneously.

4 Let P(n): ...

P(1): $5^1 - 1$
$= 4$ is divisible by 4. [1M]

Let P(n) be true for $n = k$.

i.e. $5^k - 1$ is divisible by 4
∴ There is an integer M:
s.t. $5^k - 1 = 4M$. —— (A) [1M]

Consider P(k+1):
$5^{k+1} - 1$

$= 5 \cdot 5^k - 1$ by (A)
$= 5(4M+1) - 1$
$= 20M + 4$ [1M]
$= 4(5M+1)$
M is an integer ∴ 5M+1 is also an integer. [½M]

∴ $5^{k+1} - 1$ is divisible by 4

∴ P(1) is true and P(k+1) is true if P(k) is true ∴ By the principle of Mathematical Induction, P(n) is true ∀n≥1 [½M]

b is a zero of P(x) ⇒ P(3)=0 [1M]

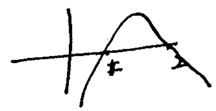
b (i) now P(-3) = P(3) = 0 (P(x) is even)
(x+3) is also a factor [1M]

(ii) (x-2) is a factor
Similarly (x-1) is also a factor [1M]
(x+2) is also a factor
∴ P(x) = A(x-2)(x+1)(x+2)(x-1) [½M]
∴ P(x) = 12(x-2)(x+1)(x+2)(x-1)
P(0) = 48
⇒ 48 = 4A
A = 12 [1M]

Q.2

a) $\frac{x+1}{x-1} \le 3$ $x \ne 1$

Multiply by $(x-1)^2$
$(x+1)(x-1) \le 3(x-1)^2$
$(x+1)(x-1) - 3(x-1)^2 \le 0$
$(x-1)(x+1-3x+3) \le 0$
$(x-1)(4-2x) \le 0$
$2(x-1)(2-x) \le 0$
$x \le 1$ or $x \ge 2$.



b) $x^2 - 4x + 1 = 0$
$\alpha + \beta = 4$ [½M]
$\alpha\beta = 1$ [½M]
$\alpha^2 + \beta^2 = (\alpha+\beta)^2 - 2\alpha\beta$ [1M]
$= 16 - 2$ [1M] [1M]
$= 14$

(ii) $X^2 - (\alpha^2+\beta^2)X + \alpha^2\beta^2 = 0$
$X^2 - 14x + (-1)^2 = 0$
$X^2 - 14x + 1 = 0$ [1M] [1M]

grad of $x - 2y + 3 = 0$
is $\frac{1}{2}$
grad of $y = mx$ is m
∴ $\left|\frac{m - \frac{1}{2}}{1 + \frac{1}{2}m}\right| = \tan 45°$
$\left|\frac{2m-1}{2+m}\right| = 1$
∴ $2m - 1 = \pm(2+m)$
$m = 3$ or $m = \frac{-1}{3}$

Q.4 (c) grad of PQ = grad. of $y = 2x+3$
$\frac{aq^2 - ap^2}{2aq - 2ap} = 2$
$\frac{q+p}{2} = 2$
∴ $q + p = 4$

mid pt. of PQ:
X: $a(p+q)$
Y = $\frac{a(p^2+q^2)}{2}$
also $p + q = 4$
∴ $X = 4a$ is the locus

a) $5x - x^2 > 0$

$x(5-x) > 0$

$0 < x < 5$ (1M)

b) if α, α, γ are the roots.

$\alpha + \alpha + \gamma = 0$ $2\alpha + \gamma = 0$ — ①

$\alpha\alpha + \alpha\gamma + \gamma\alpha = +q$ $\alpha^2 + 2\alpha\gamma = q$ — ②

$\alpha^2\gamma = -r$ $\alpha^2\gamma = -r$ — ③

[1M]

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from ① $\gamma = -2\alpha$

Sub in ② $\alpha(\alpha + 2\gamma) = q$

$\alpha(\alpha - 4\alpha) = q$

$-3\alpha^2 = q$

$\alpha^2 = -\frac{q}{3}$ — Ⓐ

[1M]

sub in ③ $\alpha^2(-2\alpha) = -r$

$\alpha^3 = \frac{r}{2}$ — Ⓑ [1M]

$\alpha^6 = \left(-\frac{q}{3}\right)^3 = \left(\frac{r}{2}\right)^2$ (from Ⓐ & Ⓑ)

$\therefore -\frac{q^3}{27} = \frac{r^2}{4}$ [1M]

$\therefore 4q^3 + 27r^2 = 0$

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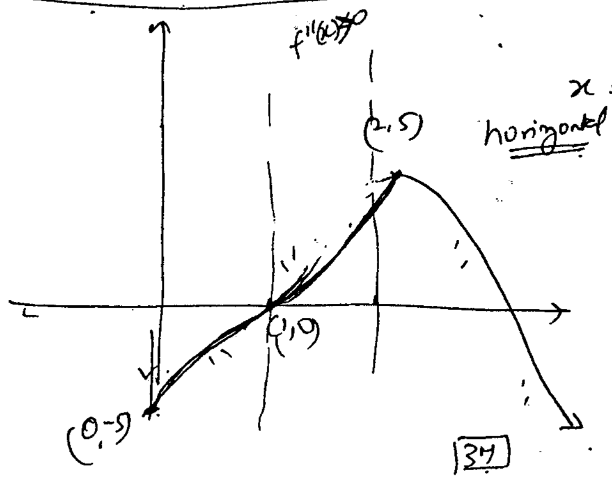
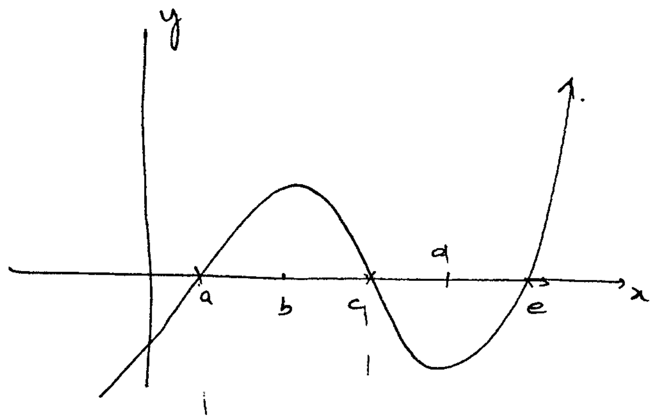
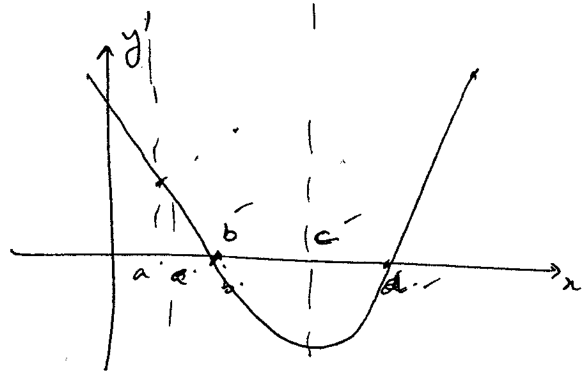
c)



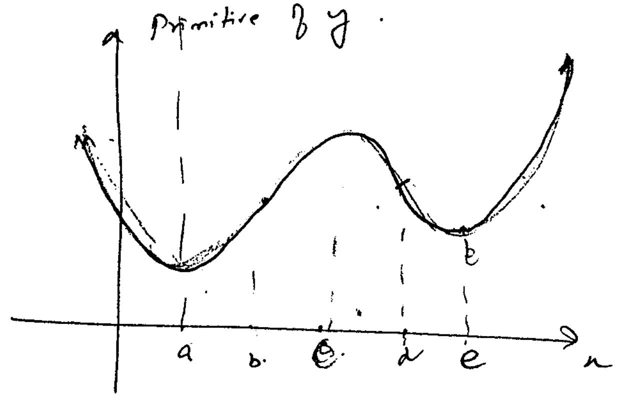
$x = 1$ is a horizontal pt. of inflexion [1M]

[3M]





[1M]



[1M] for the General Shap

$\frac{1}{2}$M for a, c, e.

$\frac{1}{2}$M b d.

v) $p(1.5) = 6 - 3\times1.5 - 1.5^3$

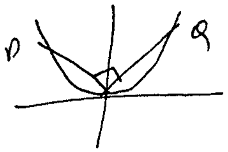
< 0 ①

root lies between 1 and 1.5

$p(1.25) = 6 - 3\times1.25 - 1.25^3$ ①

> 0

root lies between 1.25 and 1.5.



grad. of $OP = \frac{ap^2}{2ap} = \frac{p}{2}$] $\frac{1}{2}$

grad of $OQ = \frac{q}{2}$

$OP \perp OQ$

$\therefore \frac{p}{2} \times \frac{q}{2} = -1$] $\frac{1}{2}$ [1M]

$pq = -4$

mid. pt. of PQ: $x = a(p+q)$ — ①

$y = \frac{a}{2}(p^2+q^2)$ — ②

$= \frac{a}{2}(p+q)^2 - 2pq$ — ③ ✓ [1M]

also $pq = -4$

Now $(p+q)^2 = p^2 + q^2 + 2pq$ [1M]

from ①, ② & ③;

$\left(\frac{x}{a}\right)^2 = \frac{2y}{a} + 8$. [1M]

or $x^2 = 2ay - 8a^2$

$= 2a(y - 4a)$.

Q.7

(i) $y = 6 - 3x - x^3$

$y' = -3 - 3x^2$

$= -3(1+x^2)$

$1 + x^2 > 0 \quad \forall x$

$\therefore -3(1+x^2) < 0$ for all x

$\therefore y$ is a monotonic decreasing function.

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iii) Let $P(x) = 6 - 3x - x^3$

$P(1) = 6 - 3 - 1$

> 0 ($\frac{1}{2}$M)

$P(2) = 6 - 6 - 8$

< 0 ($\frac{1}{2}$M)

$P(x)$ is a continuous function.

$\therefore$ There is a root between 1 and 2

(ii) $y' \neq 0$, $\therefore$ there is no turning point. can cross the x axis only once.

(iv) $P(x) = 6 - 3x - x^3$ $\qquad P'(x) = -3(1+x^2)$

$P(0.5) = 6 - \frac{3}{2} - \frac{1}{8}$ $\qquad P'(0.5) = \frac{-15}{4}$

$= \frac{35}{8}$

a better approximation is

$0.5 - \frac{P(0.5)}{P'(0.5)}$

$= 0.5 + \frac{35/8}{15/4} = \frac{5}{3} = 1.\dot{6}$