St Catherine's School

Year: 12
Subject: Extension 1 Mathematics
Time allowed: 55 minutes

Assessment Task No: 3
Date: June 2005

TOPICS: INT. BY SUBSTITUTION; APPL^{N}. OF CALCULUS TO PHY. WORLD

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Teacher's Name: Dr Bose

Directions to candidates:

- All questions are to be attempted.
- Marks may be deducted for careless or badly arranged work
- All necessary **working** must be shown in every question.
- Approved calculators and rulers are required.

Q1 Integration	12 /12
Q2 Projectile motion	7 /7
Q3 SHM	6 /6
Q4 Exponential Change	6 /6
Total	31 /31

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n<0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x>0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a>0, \quad -a<x<a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x>a>0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x>0$

Year 12 June 2005 Assessment Task Extension 1

Question 1 (12 marks)

i) Find $\int x(x-3)^4 dx$ using the substitution $u = x - 3$ (3)

ii) Find $\int \sin^3 x \cos x dx$ using the substitution $u = \sin x$ (2)

iii) Find $\int \frac{dx}{(4-x^2)^{\frac{3}{2}}}$ using the substitution $x = 2\sin\theta$ (4)

iv) Find $\int_0^4 \frac{e^{\sqrt{x}}}{4\sqrt{x}} dx$ using the substitution $u = \sqrt{x}$ (3)

Question 2 (7 marks)

A golf-ball is projected with velocity 20 m/sec at an angle of 30^0 to the horizontal. It lands on a level surface 15 m below its starting point.
Using $g = -10 m/s^2$, find

i) the time of flight (2)
ii) the horizontal distance travelled (2)
iii) the magnitude of the velocity with which it lands (3)

+ or −

Question 3 (6 marks)

A particle is moving so that its velocity at point x is given by

$v^2 = 36 - 4x^2$

i) Show that the particle is undergoing Simple Harmonic Motion (2)
ii) What is the period of the motion? (1)
iii) What is the amplitude of the motion? (1)
iv) Sketch position as a function of time if the particle starts from the origin with negative velocity. (2)

Question 4 (6 marks)

Newton's Law of Cooling states that the rate of change of temperature is proportional to (T-A) where A is the temperature of the surrounding air.

$$\text{so } \frac{dT}{dt} = k(T-A)$$

i) Show that $T = A + Ce^{kt}$ (where C and k are constants) satisfies Newton's Law of Cooling. (1)
ii) A bar of iron at 400^0C is brought into a room where the temperature A is a constant 30^0C It cools to 320^0in 20 minutes.
Find the values of the constants C and k (3)
iii) At what time will it reach a temperature of 100^0C? (2)

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Question 1.

1.) $\int x(x-3)^4\,dx$

let $u = x-3$

$\frac{dy}{dx} = 1$

$\therefore du = dx$

$\therefore \int (u+3)(u)^4\,du$

$= \int u^5 + 3u^4\,du$

$= \frac{u^6}{6} + \frac{3u^5}{5} + c$

$= \frac{(x-3)^6}{6} + \frac{3(x-3)^5}{5} + c$

ii) $\int \sin^3 x \cos x\,dx$

let $u = \sin x$

$\frac{dy}{dx} = \cos x$

$du = \cos x \,.\, dx$

$\int u^3\,dy$

$= \frac{u^4}{4} + c$

$= \frac{\sin^4 x}{4} + c$

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iii) $\int \frac{dx}{(4-x^2)^{\frac{3}{2}}}$

let $x = 2\sin\theta$

$\frac{dx}{d\theta} = 2\cos\theta$

$dx = 2\cos\theta \,.\, d\theta$

$\therefore \int \frac{2\cos\theta \,.\, d\theta}{\sqrt{(4-4\sin^2\theta)^3}}$

$= \int \frac{2\cos\theta \,.\, d\theta}{\sqrt{4^3(1-\sin^2\theta)^3}}$

$= \int \frac{2\cos\theta\,d\theta}{\sqrt{64\cos^6\theta}}$

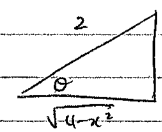
$= \int \frac{\cancel{2\cos\theta}\,d\theta}{4\cancel{8}\cos^{\cancel{3}2}\theta}$

$= \frac{1}{4}\int \sec^2\theta\,d\theta$

$= \frac{1}{4}\tan\theta + c$ [in terms of θ]

now $\frac{x}{2} = \sin\theta$

$\therefore \theta = \sin^{-1}\frac{x}{2}$



$\therefore = \frac{1}{4}\tan\left(\sin^{-1}\frac{x}{2}\right) + c$ [in terms of x]

$= \frac{1}{4}\frac{x}{\sqrt{4-x^2}} + c = \frac{x}{4\sqrt{4-x^2}} + c$

iv) $\int_0^4 \frac{e^{\sqrt{x}}}{4\sqrt{x}}\,dx$

let $u = \sqrt{x}$

$= x^{\frac{1}{2}}$

$\frac{dy}{dx} = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$

$du = \frac{1}{2\sqrt{x}} \cdot dx$

limits:

$u_1 = \sqrt{x} = \sqrt{4} = 2$

$u_2 = \sqrt{0} = 0$

$\frac{1}{2}\int_0^4 \frac{e^{\sqrt{x}}}{2\sqrt{x}}\,.dx$

$\frac{1}{2}\int_0^2 e^u\,.du$

$= \frac{1}{2}\left[e^u\right]_0^2$

$= \frac{1}{2}\left(e^2 - e^0\right)$

$= \frac{1}{2}\left(e^2 - 1\right)$

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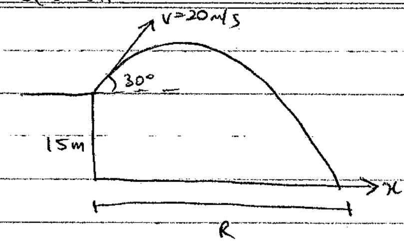
Question 2.

i)



accel: $\boxed{\ddot{x} = 0}$ $\boxed{\ddot{y} = -g = -10}$

when $t=0$,

$\dot{x} = v\cos\theta$, $\dot{y} = v\sin\theta$

$= 20\cos 30$ $= \cancel{20\sin\theta}\ 20\sin 30$

$= 10\sqrt{3}$ $= 10$

Integrate w.r.t t

$\therefore \dot{x} = c_1$ $\dot{y} = -10t + c_2$

vel: $\boxed{\therefore \dot{x} = 10\sqrt{3}}$ $\boxed{\dot{y} = -10t + 10}$

Integrate w.r.t t

$x = 10\sqrt{3}\,t + c_3$ $y = -5t^2 + 10t + c_4$

when $t=0$, $x=0$, $y=15$

displ: $\boxed{\therefore x = 10\sqrt{3}.t}$ $\boxed{y = -5t^2 + 10t + 15}$

t_{flight} when $y=0$

$\therefore 0 = -5t^2 + 10t + 15$

$= t^2 - 2t - 3$

$= (t-3)(t+1)$

$\therefore t = -1, 3$

$t > 0$

$\therefore t_{fl} = 3$ secs.

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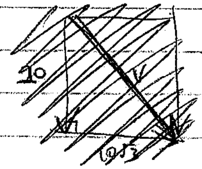
ii) From (i)

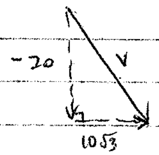
$x = 10\sqrt{3}\,t$

sub $(t=3)$ in

$x = 10\sqrt{3}(3)$

$= 30\sqrt{3}$ m ✓

iii) 



$\boxed{\begin{array}{l}\text{From(i):}\\ \dot{x} = 10\sqrt{3}\\ \dot{y} = -10t+10\end{array}}$

when $t=3$:

$\dot{x} = 10\sqrt{3}$

$\dot{y} = -10(3)+10$

$= -20$ ✓

$\therefore V = \sqrt{(10\sqrt{3})^2 + 20^2}$

$= \sqrt{300+400}$

$= \sqrt{700} = 26.457\ldots$ ✓

$= 26.46$ m/s (2dp)

(magnitude only)

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Question 3.

i) $v^2 = 36 - 4x^2$

$\ddot{x} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$ ✓

$= \frac{d}{dx}\left(\frac{1}{2}(36-4x^2)\right)$

$= \frac{d}{dx}(18-2x^2)$

$= -4x$ ✓

$\therefore \ddot{x} = -n^2x$

$\therefore$ SHM. 2

ii) $\ddot{x} = -4x$

$\therefore n = 2$

$T = \frac{2\pi}{n}$

$= \frac{2\pi}{2}$ 1

$= \pi$ secs. ✓

iii) ~~centre of motion~~

~~when x = 0~~

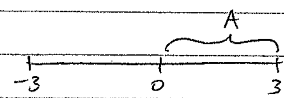
~~velocity at centre~~

velocity at end pts = 0 ✓ 1

$\therefore 0 = 36 - 4x^2$

$= 9 - x^2$

$\therefore x = \pm 3$



$\therefore$ amplitude = 3 m

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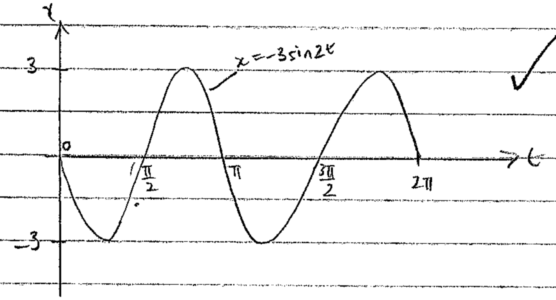
iv) to start with negative velocity ~~from~~ from the origin,

$x = -a\sin nt$

$\therefore\ x = -3\sin 2t$

check: $\dot{x} = -6\cos 2t$ which is neg.



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Question 4.

i) $\dfrac{dT}{dt} = k(T-A)$

let $T = A + Ce^{kt}$ —— ①

$\dfrac{dT}{dt} = Cke^{kt}$

$= k(Ce^{kt})$

sub ① in

$\dfrac{dT}{dt} = k(T-A)$

$\therefore$ true.

ii) $T = A + Ce^{kt}$

$A = 30$

when $t = 0$, $T = 400$

$400 = 30 + C$

$\therefore C = 370$

$\therefore T = 30 + 370e^{kt}$

Now, when $t = 20$, $T = 320$

$320 = 30 + 370e^{20k}$

$\ln\dfrac{29}{37} = 20k$

$\therefore k = \dfrac{\ln\frac{29}{37}}{20} = -0.01218\ldots$

$= -0.0122$ (4dp)

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iii) $T = A + Ce^{kt}$

$= 30 + 370e^{-\frac{\ln\frac{29}{37}}{20}t}$

$\therefore$ when $T = 100$:

$100 = 30 + 370e^{\frac{\ln\frac{29}{37}}{20}t}$

$\ln\frac{7}{37} = \frac{-\ln\frac{29}{37}}{20}t$

$\therefore t = 136.68\ldots$

$= 136.7$ mins (1dp)