St Catherine's School

Year: 12
Subject: Extension II Mathematics
Time Allowed: 55 minutes

Date: June 2006

Exam number: ______

Directions to candidates:

- All questions are to be attempted.
- All necessary **working** must be shown in every question.
- Full marks may not be awarded for careless or badly arranged work.
- Approved calculators and geometrical instruments are required.
- Hand in your work in **1** **bundle**:
- Attach the question paper

Extension II Mathematics

Q.1 Find (i) $\int x\tan^{-1}x\,dx$ (3m)

(ii) $\int \cos^2 x\sin^3 x\,dx$ (2m)

(iii) $\int \frac{2x+5}{x^2+2x+5}dx$ (3m)

(iv) $\int \frac{dx}{x^2\sqrt{16-x^2}}$ (4m)

Q.2. If $I_n = \int x^n \sin x\,dx$, show that

$$I_n = -x^n\cos x + nx^{n-1}\sin x - n(n-1)I_{n-2}$$ (4m)

Q.3 The base of a solid is given by the circle $x^2+y^2=4$. Every cross section perpendicular to x axis is an equilateral triangle, one side of which lies in the base of the solid. Find the volume of the solid. (4m)

Q.4. The area bounded by the ellipse $\frac{x^2}{9}+\frac{y^2}{4}=1$ and the lines x=1 and x=3 is rotated about the y axis

(i) Use the method of cylindrical shells to show that the volume of the solid is given by $\frac{8\pi}{3}\int_1^3 x\sqrt{9-x^2}\,dx$ (2m)

check (ii) Hence find the exact volume (3m)

Q.5. The area bounded by $y=2x-x^2$ and the x axis is rotated about the y axis. Considering slices perpendicular to the axis of rotation,

(i) Show that the volume is given by $4\pi\int_0^1 \sqrt{1-y}\,dy$ (4m)

(ii) Hence find the exact volume of the solid. (2m)

Q-i

(i) $\int x \tan^{-1} x \, dx$

$u = \tan^{-1} x \qquad v' = x$

$u' = \frac{1}{1+x^2} \qquad v = \frac{x^2}{2}$

$\int uv' = uv - \int u'v$

$\therefore \int x \tan^{-1} x \, dx$

$= \frac{x^2}{2} \tan^{-1} x - \int \frac{x^2}{2(1+x^2)} dx$

$= \frac{x^2}{2} \tan^{-1} x - \frac{1}{2} \int \frac{x^2+1-1}{x^2+1} dx$

$= \frac{x^2}{2} \tan^{-1} x - \frac{1}{2} \int \left(1 - \frac{1}{1+x^2}\right) dx$

$= \frac{x^2}{2} \tan^{-1} x - \frac{1}{2} (x - \tan^{-1} x) + c.$

ii) $\int \cos^2 x \sin^3 x \, dx$

$= \int \cos^2 x (1 - \cos^2 x) \sin x \, dx$

$= \int \cos^2 x \sin x \, dx - \int \cos^4 x \sin x \, dx$

$= -\frac{\cos^3 x}{3} + \frac{\cos^5 x}{5} + C$

$$\int \frac{2x+5}{x^2+2x+5}\,dx$$

$$= \int \frac{2x+2}{x^2+2x+5}\,dx + \int \frac{3}{(x+1)^2+4}\,dx$$

$$= \ln(x^2+2x+5) + \frac{3}{2}\tan^{-1}\frac{x+1}{2} + c.$$

iv) $$\int \frac{dx}{x^2\sqrt{16-x^2}}$$

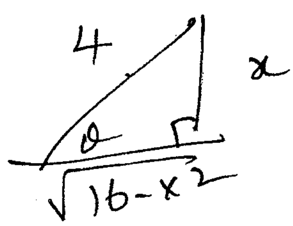
$x = 4\sin\theta$

$dx = 4\cos\theta\,d\theta$

$16-x^2 = 16-16\sin^2\theta$

$= 16\cos^2\theta$

$$= \int \frac{4\cos\theta\,d\theta}{16\sin^2\theta \cdot 4\cos\theta}$$



$$= \frac{1}{16}\int \operatorname{cosec}^2\theta\,d\theta$$

$$= -\frac{1}{16}\cot\theta + c$$

$$= -\frac{1}{16}\cdot\frac{\sqrt{16-x^2}}{x} + c.$$

Q.2

$$I_n = \int x^n \sin x\,dx$$

$u = x^n$ $\quad v' = \sin x$

$u' = n x^{n-1}$ $\quad v = -\cos x$

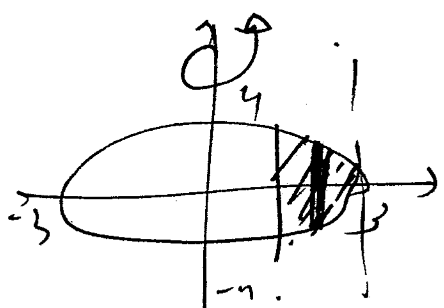
$\int uv' = uv - \int u'v$

$$V = \sqrt{3}\int_{-2}^{2}(4-x^2)\,dx \qquad y^2 = 4-x^2$$

$$= 2\sqrt{3}\left[\left(4x - \frac{x^3}{3}\right)\right]_0^2 \qquad \text{(even fn.)}$$

$$= 2\sqrt{3}\left(8 - \frac{8}{3}\right) = \frac{32\sqrt{3}}{3}\ u^3.$$

Q.4



Consider a cyl. shell at a dist. x from the origin, parallel to the y axis. the vol. of the cyl. shell is ΔV,

$$\Delta V = 2\pi \cdot x \cdot 2y \cdot \Delta x$$

$$= 4\pi\, xy\, \Delta x$$

$$V = 4\pi\int_1^3 xy\,dx \qquad \frac{y^2}{4} = 1 - \frac{x^2}{9}$$

$$y^2 = \frac{4}{9}(9-x^2)$$

$$= 2\pi\int_1^3 x\left(\frac{4}{3}\sqrt{9-x^2}\right)dx \qquad y = \pm\frac{2}{3}\sqrt{9-x^2}.$$

$$2y = \frac{4}{3}\sqrt{9-x^2}.$$

$$= \frac{8\pi}{3}\int_1^3 x\sqrt{9-x^2}\,dx$$

$$= -\frac{4\pi}{3}\int_1^3 (9-x^2)^{\frac{1}{2}}(-2x)\,dx = -\frac{4\pi}{3}\left(9-x^2\right)^{3/2}\cdot\frac{2}{3}\Big|_1^3 =$$

$$\therefore \int x^n \sin x \, dx$$

$$= -x^n \cos x + n \int x^{n-1} \cos x \, dx.$$

$u = x^{n-1}$ $\quad v' = \cos x$

$u' = (n-1)x^{n-2}$ $\quad v = \sin x$

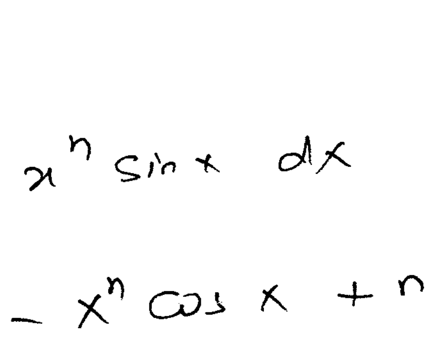
$$\int uv' = uv - \int u'v$$

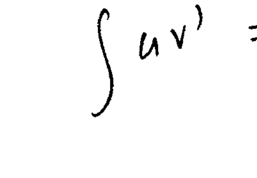
$$\therefore \int x^n \sin x \, dx$$

$$= -x^n \cos x + n \left[x^{n-1} \sin x. - (n-1) \int x^{n-2} \sin x \, dx \right]$$

$$= -x^n \cos x + n x^{n-1} \sin x - n(n-1) I_{n-2}.$$

Q.3





Take a cross section at a dist. x from the origin. length of every side of the equilateral triangle is 2y

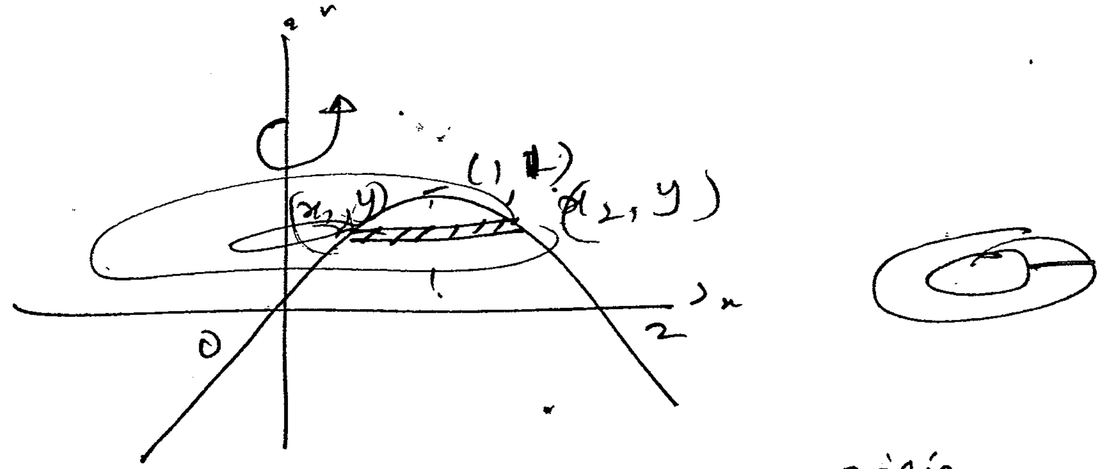
Area of cross section is $\frac{1}{2} \times 2y \times 2y \times \sin 60$

$$= \sqrt{3} y^2$$

$$\Delta v = \sqrt{3} y^2 \Delta x$$

$$\therefore v = \sqrt{3} \int_{-2}^{2} y^2 \, dx$$

Q-5



Take a slice a dist. y from the origin.

The area of cross section is

$$\pi\left(x_2^2 - x_1^2\right)$$

$$\Delta V = \pi\left(x_2^2 - x_1^2\right).\Delta y$$

$$x^2 - 2x + y = 0$$

$$x_1 + x_2 = 2$$

$$(x_1 - x_2)^2 = (x_1 + x_2)^2 - 4x_1x_2$$
$$= 4 - 4y.$$

$$|x_1 - x_2| = 2\sqrt{1-y}$$

$$\therefore V = \pi \int_0^1 2.2\sqrt{1-y}\, dy$$

$$= 4\pi \int_0^1 \sqrt{1-y}\, dy$$

$$= 4\pi \left[(1-y)^{3/2}\right]_0^1 .\left(-\frac{2}{3}\right)$$

$$= -\frac{8\pi}{3}.(0-1)$$

$$= \frac{8\pi}{3} u^3$$