St Catherine's School

Year: 12
Subject: Mathematics Extension II
Time allowed: 55 minutes

Assessment Task No: 1
Date: 24/2/05

Student Number: 15227508

Directions to candidates:

- All questions are to be attempted.
- Marks may be deducted for careless or badly arranged work
- All necessary **working** must be shown in every question.
- Approved calculators are required.
- Arrange your work in two sections.

Section A	25/25 /
Section B	13 /
	TOTAL 38/38 /

100%

excellent.

Section A

Marks

1. Let $z = -1 + \sqrt{3}i$ and $w = 1 - i$ be two complex numbers.

(a) Find the modulus and the arguments of the following complex numbers:

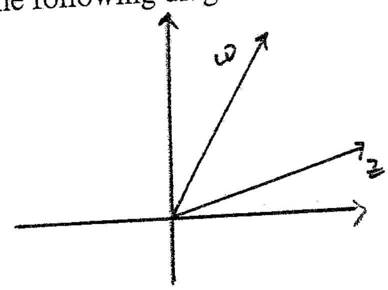
(i) z — 1m

(ii) w — 1m

(iii) $\dfrac{z^2}{w^4}$ — 2m

(b) Find the smallest integer value of n such that z^n is real. — 1m

2. Copy the following diagram onto your answer sheet



Indicate on the diagram the following vectors:

(i) $z + w$ — 1m

(ii) $z - w$ — 1m

3. Sketch on separate Argand Diagrams the locus of z described by each of the following conditions.

(i) $|z - 3i| < |z + 3|$ — 2m

(ii) $z\bar{z} = z + \bar{z}$ — 2m

(iii) $\arg\left(\dfrac{z - i}{z + 1}\right) = \dfrac{\pi}{3}$ — 2m

P.T.O

4. (i) Solve $z^5 = 1$ over the field of complex numbers and show that if w is the complex root with the smallest possible argument that w^2, w^3 and w^4 are also roots. 3m

(ii) Factorise $z^5 - 1$ over the complex field. 1m

(iii) By grouping the roots in complex conjugate pairs, factorise $z^5 - 1$ over the real field. 2m

(iv) Using parts (ii) and (iii), find the value of $\cos\frac{2\pi}{5} + \cos\frac{4\pi}{5}$. 2m

5. (i) On an Argand diagram, sketch the locus of z: such that $|z| = 1$. 1m

(ii) Using the Argand diagram or otherwise, show that

$$-\frac{\pi}{6} \le \arg(z+2) \le \frac{\pi}{6}$$ 2m

(iii) Write down the greatest and the least values of $|z+2|$, where z lies on this locus 1m

Section B

1. $\alpha,\ \beta\ and\ \gamma$ are the roots of $x^3 + 7x - 11 = 0$.
Write down the polynomial whose roots are $\alpha^2,\ \beta^2\ and\ \gamma^2$.
2m

2. 1+i is a root of the polynomial equation P (x) $x^3 + x^2 - 4x + 6 = 0$.

Factorise P (x) fully in the Field of Complex Numbers 3m

3. Factorise fully the polynomial P (x), given that
P (x): $3x^4 + 17x^3 + 30x^2 + 12x - 8 = 0$ has a root of multiplicity 3. 4m

4. If $ax^3 + bx^2 + d = 0$ has a non zero double root, show that $27a^2d + 4b^3 = 0$
4m

End of paper

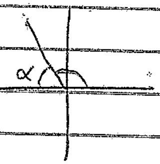
SECTION A. 15227508

1a. $z = -1 + \sqrt{3}i$

$w = 1 - i$

i) $|z| = \sqrt{(-1)^2 + \sqrt{3}^2}$

$= 2$



arg z:

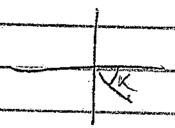
$\tan\alpha = \frac{\sqrt{3}}{1}$

$\alpha = \frac{\pi}{3}$

$\therefore \arg z = \frac{2\pi}{3}$

ii) $|w| = \sqrt{1^2 + (-1)^2}$

$= \sqrt{2}$



arg w:

$\tan\alpha = 1$

$\alpha = \frac{\pi}{4}$

$\therefore \arg w = -\frac{\pi}{4}$

iii) $\frac{|z^2|}{|w^4|} = \frac{2^2}{(\sqrt{2})^4} = \frac{4}{4}$

$= 1$

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$\frac{\arg(z)^2}{\arg(w)^4} = \frac{\left(\frac{2\pi}{3}\right)^2}{\left(-\frac{\pi}{4}\right)^4}$

using De Moivre's

$\frac{4\pi}{3} - \pi$

$= \frac{7\pi}{3}$

$= \frac{\pi}{3}$

4

b. $z = -1 + \sqrt{3}i$

$= 2 \operatorname{cis} \frac{2\pi}{3}$

$z^n = \left(2 \operatorname{cis} \frac{2\pi}{3}\right)^n$

now, $z^1 = 2 \operatorname{cis} \frac{2\pi}{3}$

1

$z^2 = \left(2 \operatorname{cis} \frac{2\pi}{3}\right)^2$

$= 4 \operatorname{cis} \frac{4\pi}{3}$ (using De Moivre's)

$z^3 = 8 \operatorname{cis} \frac{6\pi}{3}$ (using De Moivre's)

$= 8 \operatorname{cis} 2\pi$

$= 8(\cos 2\pi + i \sin 2\pi)$

$= 8$ which is real

$\therefore$ smallest integer value of $n = 3$

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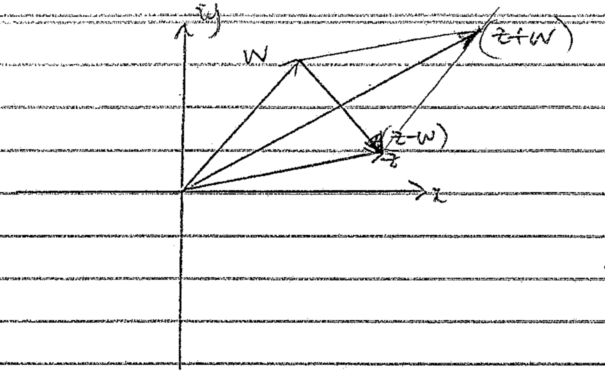
2.

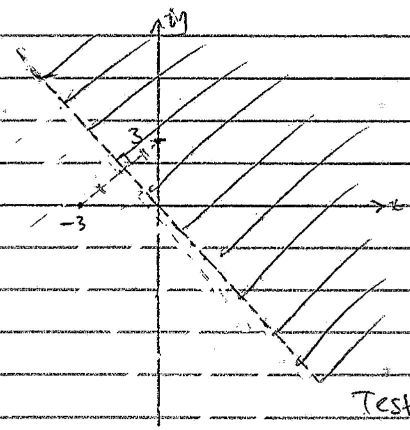


(z-w)
head - tail

2/

3. i) $|z - 3i| < |z + 3|$

$z - (3i)$ $z - (-3)$



2/

Test (-3, 0)

$|-3-3i| < |0|$

$\sqrt{3^2+|3|^2} < 0$

$= \sqrt{18} \not< 0$

ii) $z\bar{z} = z + \bar{z}$

let $z = x+iy$

$(x+iy)(x-iy) = x+iy+x-iy$

$(x^2+y^2) = 2x$

2/

$x^2 - 2x + y^2 = 0$

$x^2 - 2x + (-1)^2 + y^2 = 1$

$(x-1)^2 + y^2 = 1$

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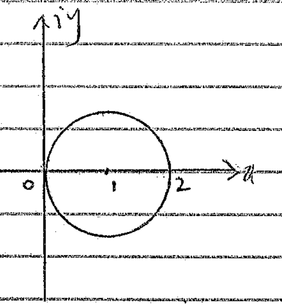
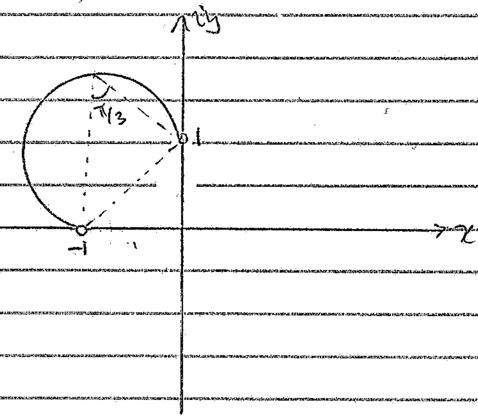
ii) $(x-1)^2 + y^2 = 1$



iii) $\arg\left(\frac{z-i}{z+1}\right) = \frac{\pi}{3}$



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$\frac{\pi}{3} > 0$, ∴ above the arc

$\frac{\pi}{3}$ is acute, ∴ major arc

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4i) $z^5 = 1$

$= \text{cis}\, 0$

$= \text{cis}\, 2k\pi \quad , k \in \mathbb{Z}$

$\therefore z = (\text{cis}\, 2k\pi)^{\frac{1}{5}}$

$= \text{cis}\, \frac{2k\pi}{5}$, ~~at~~ using De Moivre's Theorem

$z_0 = \text{cis}\, 0$

$z_1 = \text{cis}\, \frac{2\pi}{5}$

$z_2 = \text{cis}\, \frac{4\pi}{5}$

$z_3 = \text{cis}\, \frac{6\pi}{5}$

$z_4 = \text{cis}\, \frac{8\pi}{5}$

roots of $z^5 = 1$

Let ω be the complex root: ~~[illegible]~~ $\text{cis}\, \frac{2\pi}{5}$

$\omega = \text{cis}\, \frac{2\pi}{5}$

$\omega^2 = \text{cis}\, \frac{4\pi}{5}$ (De Moivre's)

$= \omega_2$

$\omega^3 = \text{cis}\, \frac{6\pi}{5}$

$= \omega_3$

$\omega^4 = \text{cis}\, \frac{8\pi}{5}$

$= \omega_4$

$\therefore$ if ω is a root of $z^5 = 1$, ω^2, ω^3 and ω^4 are also roots.

3

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ii) $z^5 - 1 = (z - \text{cis}\, 0)(z - \text{cis}\, \frac{2\pi}{5})(z - \text{cis}\, \frac{4\pi}{5})(z - \text{cis}\, \frac{6\pi}{5})(z - \text{cis}\, \frac{8\pi}{5})$

$= (z-1)(z - \text{cis}\, \frac{2\pi}{5})(z - \text{cis}\, \frac{4\pi}{5})(z - \text{cis}\, \frac{6\pi}{5})(z - \text{cis}\, \frac{8\pi}{5})$

1

iii) Note:

~~[illegible]~~ $\text{cis}\, \frac{8\pi}{5} = \text{cis} - \frac{2\pi}{5} = \overline{\text{cis}\, \frac{2\pi}{5}}$

$\text{cis}\, \frac{6\pi}{5} = \text{cis} - \frac{4\pi}{5} = \overline{\text{cis}\, \frac{4\pi}{5}}$

~~[illegible]~~ Now:

$(z - \text{cis}\, \frac{2\pi}{5})(z - \overline{\text{cis}\, \frac{2\pi}{5}})$

$= z^2 - 2\cos\frac{2\pi}{5} z + 1$

$\therefore z^5 - 1 = (z-1)(z^2 - 2\cos\frac{2\pi}{5} z + 1)(z^2 - 2\cos\frac{4\pi}{5} z + 1)$

2

iv). From (ii) and (iii),

we know that the sum of the roots of $z^5 = 1$ are:

$$1 + \text{cis}\frac{2\pi}{5} + \text{cis}-\frac{2\pi}{5} + \text{cis}\frac{4\pi}{5} + \text{cis}-\frac{4\pi}{5} = -\frac{b}{a}$$

$$= 0$$

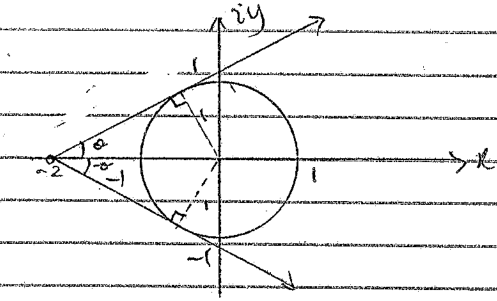
$$\therefore \left(\text{cis}\frac{2\pi}{5} + \text{cis}-\frac{2\pi}{5}\right) + \left(\text{cis}\frac{4\pi}{5} + \text{cis}-\frac{4\pi}{5}\right) = -1$$

$$2\cos\frac{2\pi}{5} + 2\cos\frac{4\pi}{5} = -1$$

$$\therefore \cos\frac{2\pi}{5} + \cos\frac{4\pi}{5} = -\frac{1}{2}$$

2

5 i) $|z| = 1$



ii) On the above diagram, construct the lines off $(-2, 0)$ that are also tgts to the circle.

From the Right Led Δs constructed:

$$\sin\theta = \frac{1}{2}$$

$$\therefore \theta = \frac{\pi}{6}$$

$$\therefore -\theta = -\frac{\pi}{6}$$

2 forced

$$\therefore -\frac{\pi}{6} \leq \arg(z-2) \leq \frac{\pi}{6}$$

~~... is given by:~~

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(iii) From diagram,

- the shortest distance is from $(-2,0)$ to $(-1,0)$

$\therefore$ ~~ans~~

min. $|z+2| = 1$ unit

- the longest distance is from $(-2,0)$ to $(1,0)$

$\therefore$ max $|z+2| = 3$ units.

good.

1.

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SECTION B.

1. Roots needed: $\alpha^2+\beta^2+\gamma^2$

let $X = x^2$

$\therefore x = \sqrt{X}$

Sub this into eqtn:

$(\sqrt{X})^3 + 7(\sqrt{X}) - 11 = 0$

$X\sqrt{X} + 7\sqrt{X} - 11 = 0$

$X\sqrt{X} + 7\sqrt{X} = 11$

$\sqrt{X}(X+7) = 11$

sq. both sides 2/

$X(X^2+14X+49) = 121$

$X^3+14X^2+49X = 121$

$\therefore X^3+14X^2+49X-121 = 0$

or, it can be written as:

$x^3+14x^2+49x-121=0$

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2. If $1+i$ is a root,

$\therefore$ $1-i$ is a root too

($P(x)$ has real coeffs)

$\therefore$ $(x-1-i)(x-1+i)$ is a factor of $P(x)$

$= x^2 - x + ix - x + 1 - i - ix + i - i^2$

$= x^2 - 2x + 2$

$\therefore P(x) = (x^2 - 2x + 2)\,Q(x)$

3

By inspection:

$P(x) = (x^2 - 2x + 2)(x+3)$

$= (x-(1+i))(x-(1-i))(x+3)$ over $\mathbb{C}$

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3. $P(x) = 3x^4 + 17x^3 + 30x^2 + 12x - 8 = $

Due to multiplicity of 3:

$P'(x) = 12x^3 + 51x^2 + 60x + 12 = 0$

$P''(x) = 36x^2 + 102x + 60 = 0$

$\therefore 6x^2 + 17x + 10 = 0$

$x = \frac{-17 \pm \sqrt{17^2 - 4(6)(10)}}{2(6)}$

$= \frac{-17 \pm 7}{2(6)}$

$= -\frac{5}{6}, -2$

~~$\therefore P(x) = (x+8)(x+12)\,Q(x)$~~

~~$= (x^2 + 17x$~~

~~$\therefore P(x) = (x+5)(x+2)\,Q(x)$~~

~~$= (6x^2 + 17x + 10)\,Q(x)$~~

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Test the roots:

$P(-\frac{5}{6}) \neq 0$

$P(-2) = 0$
$P'(-2) = 0$ ✓
$P''(-2) = 0$ ✓

$\therefore$ triple zero at -2 ✓

$\therefore P(x) = (x+2)^3 \, Q(x)$

$= (x^3 + 3(2)x^2 + 3(4)x + 2^3)\,Q(x)$

$= (x^3 + 6x^2 + 12x + 8)(Q(x))$

By inspection:

$P(x) = (x^3 + 6x^2 + 12x + 8)(3x-1)$ ✓

$= (x+2)^3(3x-1)$

✓✓

4. $P(x) = ax^3 + bx^2 + d = 0$ — ①

$P'(x) = 3ax^2 + 2bx = 0$ — ②

$\therefore 3ax + 2b = 0$ (when $x \neq$

$x = \frac{-2b}{3a}$ — ③

Sub ③ into ① ✓

$a\left(\frac{-2b}{3a}\right)^3 + b\left(\frac{-2b}{3a}\right)^2 + d = 0$

$a\left(\frac{-8b^3}{27a^3}\right) + b\left(\frac{4b^2}{9a^2}\right) + d = 0$

$\frac{-8b^3}{27a^2} + \frac{4b^3}{9a^2} + d = 0$

$-8b^3 + 12b^3 + 27a^2d = 0$ ✓

$\therefore 27a^2d + 4b^3 = 0$ ✓✓