Student Number:..

Teacher:.

St Catherine's School

Waverley, Sydney

Year 12 Extension 1 Mathematics

Assessment Task

Task #1

February 2009

Time allowed: *55 minutes*

General Instructions

- Working time – 55 minutes
- Write using black or blue pen
- Board-approved calculators may be used.

Total marks – 41

- Attempt Questions 1–2
- Start each question on a new page
- All necessary working should be shown in every question

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}}\,dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}}\,dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

Question 1 – 22 Marks

a) If α, β, and γ are the roots of the equation $2x^3-3x^2+4x-6=0$ find the value of:

- i. $\alpha+\beta+\gamma$ 0.5m
- ii. $\alpha\beta+\alpha\gamma+\beta\gamma$ 1m
- iii. $\alpha\beta\gamma$ 0.5m
- iv. $\alpha^2+\beta^2+\gamma^2$ 2m
- v. $(\alpha+1)(\beta+1)(\gamma+1)$ 2m

b) Use the remainder theorem to find a factor of: $P(x)=x^3-7x-6$, then factorise completely. 3m

c) Solve the equation $x^3+6x^2+3x-10=0$ if the roots form consecutive terms of an arithmetic series. 3m

d) Show there is a root of the equation $2x^3-x+3=0$ in the interval $-2\le x\le -1$. Hence find an approximate root of the equation by halving the interval **twice** correct to 2 decimal places. 3m

e) Use Newton's method once to calculate $\sqrt{17}$ correct to two decimal places.
Hint: Let $f(x)=x^2-17$. 3m

f) Prove by induction that the sum of n terms of a geometric progression whose first term is a and common ration r is given by: $\dfrac{a(r^n-1)}{r-1}$.

That is, prove: $S_n=a+ar+ar^2+ar^3+......+ar^{n-1}=\dfrac{a(r^n-1)}{r-1}$; $n\ge 1$. 4m

PTO

Question 2 – 19 Marks – Start a new page

a) By expressing 105^0 as a sum of two suitable angles, find the exact value of $\tan 105^o$.
Do not rationalise your solution. 2m

b) If $\cos A=\dfrac{12}{13}$ and $\cos B=\dfrac{3}{5}$ where $0<A<90^o$ and $0<B<90^o$ find the value of $\sin(A+B)$. 3m

c) Show that $\dfrac{1+\cos\theta}{\sin\theta}=\cot\dfrac{\theta}{2}$ 3m

d) Show that $\dfrac{\sin 5\theta}{\sin\theta}-\dfrac{\cos 5\theta}{\cos\theta}=4\cos 2\theta$ 3m
(Hint: Use the result $\sin(A-B)=\sin A\cos B-\cos A\sin B$)

e) i. Write $\cos x-\sqrt{3}\sin x$ in the format $A\cos(x+\alpha)$. 2m

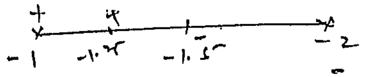
ii. Hence find the general solutions of the equation $\cos x-\sqrt{3}\sin x=1$ 2m

f) Express $\sin\theta$ in terms of $\tan\dfrac{\theta}{2}$ and by choosing a suitable value of θ, show that:

$$\tan 15^o=2-\sqrt{3}$$ 4m

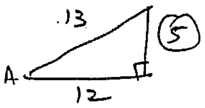
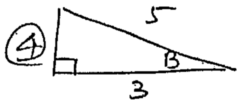
END OF PAPER

Qn	Solutions	Marks	Comments+Criteria
1. a.	$2x^3-3x^2+4x-6=0$		
	$\alpha+\beta+r = \frac{3}{2}$	0.5	
	$\alpha\beta+\beta r+\alpha r = \frac{4}{2}=2$	1	
	$\alpha\beta r = \frac{6}{2} = 3.$	0.5	no penalty for subsequent errors.
	$\alpha^2+\beta^2+r^2 = (\alpha+\beta+r)^2 - 2\Sigma\alpha\beta$	1	
	$= \left(\frac{3}{2}\right)^2 - 2(2)$		
	$= -\frac{7}{4}.$	1	
	$(\alpha+1)(\beta+1)(r+1)$		
	$= \alpha\beta r + \Sigma\alpha\beta + \Sigma\alpha + 1$	1	
	$= 3 + 2 + \frac{3}{2} + 1$	1	
	$= \frac{15}{2}$		
1b	$P(x) = x^3-7x-6$		
	$P(-1) = -1+7-6$		
	$= 0$		
	$(x+1)$ is a factor.	1M	
	$x+1 \overline{)\, x^3-7x-6}$ with quotient x^2-x-6	1M	
	$\therefore P(x) = (x+1)(x^2-x-6)$	1M.	
	$= (x+1)(x-3)(x+2)$		

Qn	Solutions	Marks	Comments+Criteria
c)	$x^3+6x^2+3x-10=0$		
	Let the roots be		If the information is not used to find the roots; $\frac{2}{3}$
	$a-d,\ a,\ a+d.$		
	$a-d+a+a+d = -6$		
	$3a = -6$		
	$a = -2$	1M	
	$(a-d)a(a+d) = 10$		
	$a(a^2-d^2) = 10$		
	$-2(4-d^2) = 10$		
	$4-d^2 = -5$		
	$d^2 = 9$		
	$d = \pm 3$	1M	
	The roots are		
	$-5, -2, 1$	1M.	
d)	Let $P(x):\ 2x^3-x+3=0$		
	$P(-2) = -16+4+3$	½M	
	< 0		
	$P(-1) = -2+1+3$	½M	
	> 0		
	$P(x)$ is continuous	½M.	
	∴ There is a root between -2 and -1		
			
	Consider $P(-1.5) = -6.75+1.5+3$		
	$< 0.$		

Qn	Solutions	Marks	Comments+Criteria
	∴ There is a root between -1 and -1.5	$\frac{1}{2}$M	
	Consider $p(-1.25)$ $= 0.3475$ > 0	$\frac{1}{2}$	
	∴ There is a root between -1.25 and -1.5	$\frac{1}{2}$	✓
	Approximation after 2 iteration is -1.375		
e)	$f(x) = x^2 - 17$ $f'(x) = 2x$ $f(4) < 0$; $f(5) > 0$; $f(x)$ is continuous.	①M	Accept. Any value of x_1 between 4 and 5 no need to penalise for not writing $f(x)$ is continuous.
	Take $x_1 = 4.5$ $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$	①M	
	$= 4.5 - \frac{4.5^2 - 17}{2(4.5)}$ $= 4.14$ 2.d.p	①M	Some have used calc. to conclude that there is a root between 4 & 5 Students who have worked with $\sqrt{17}$, max. marks is 2.

Qn	Solutions	Marks	Comments+Criteria
	Let $P(n)$: $a + ar + ar^2 + \ldots + ar^{n-1} = \frac{a(r^{n}-1)}{r-1}$		
	Consider $P(1)$ LHS: a ; RHS: $\frac{a(r^1 - 1)}{r-1}$ $= a$ ∴ $P(1)$ is true	①M	
	Let $P(k)$ be true. i.e. $a + ar + \ldots + ar^{k-1} = \frac{a(r^k - 1)}{r-1}$ —— Ⓐ		
	Consider $P(k+1)$: $\boxed{a + ar + \ldots + ar^k = \frac{a(r^{k+1}-1)}{r-1}}$	($\frac{1}{2}$M)	
	LHS $= a + ar + \ldots + ar^k$ $= \underbrace{a + ar + \ldots + ar^{k-1}} + ar^k$ $= \frac{a(r^k - 1)}{r-1} + ar^k$ by Ⓐ $= \frac{a}{r-1}(r^k - 1 + r^{k+1} - r^k)$ $= \frac{a}{r-1}(r^{k+1} - 1)$	($\frac{1}{2}$M)	
	∴ $P(k+1)$ is true if $P(k)$ is true $P(1)$ is true; $P(k+1)$ is true if $P(k)$ is true ∴ By the principle of mathematical induction $P(n)$ is true for all $n \geq 1$.	($\frac{1}{2}$M)	

Qn	Solutions	Marks	omments+Criteria
2 a	$105^0 = 60+45$		
	$\tan 105^{\circ} = \tan(60+45)$		
	$= \frac{\tan 60 + \tan 45}{1 - \tan 60 \tan 45}$	1M	
	$= \frac{\sqrt{3}+1}{1-\sqrt{3}}$	1M.	
b)	$\cos A = \frac{12}{13}$ $\quad \cos B = \frac{3}{5}$		
	 		
	$\therefore \sin A = \frac{5}{13}$ $\quad \sin B = \frac{4}{5}$	1M	
	$\sin(A+B) = \sin A \cos B + \cos A \sin B$	1M	
	$= \frac{5}{13} \times \frac{3}{5} + \frac{12}{13} \times \frac{4}{5}$		
	$= \frac{15+48}{65} = \frac{63}{65}$	1M	
c)	$\frac{1+\cos\theta}{\sin\theta} = \frac{1+2\cos^2\theta/2 - 1}{2\sin\theta/2 \cos\theta/2}$		$\cos\theta = 2\cos^2\theta - 1$ 1M $\sin\theta = 2\sin\theta/2 \cos\theta/2$ M
	$= \text{cor}\,\theta/2$		Ans: 1M

Qn	Solutions	Marks	Comments+Criteria
d)	$\frac{\sin 5\theta}{\sin\theta} - \frac{\cos 5\theta}{\cos\theta}$		
	$= \frac{\sin 5\theta \cos\theta - \cos 5\theta \sin\theta}{\sin\theta \cos\theta}$	1M	
	$= \frac{\sin(5\theta - \theta)}{\sin\theta\cos\theta}$	1M.	
	$= \frac{2 . 2\sin 2\theta \cos 2\theta}{2 . \sin\theta \cos\theta .}$	1M	
	$= \frac{4 \sin 2\theta \cos 2\theta}{\sin 2\theta}$	$\frac{1}{2}$M.	
	$= 4\cos 2\theta$.		
e)	$\cos x - \sqrt{3}\sin x = A\cos(x+\alpha)$		
	$= A(\cos x \cos\alpha - \sin x \sin\alpha)$		
	$\therefore 1 = A\cos\alpha$ $\quad \therefore A = \sqrt{4}$		
	$\sqrt{3} = A\sin\alpha$ $\quad = 2$	1M	
	$\tan\alpha = \sqrt{3}$	1M	
	$\alpha = 60^0$		
	$\therefore A\cos(x+\alpha) = 2\cos(x+60^\circ)$		
(ii)	$\cos x - \sqrt{3}\sin x = 1$		
	$2\cos(x+60) = 1$		
	$\cos(x+60) = \frac{1}{2}$	$\frac{1M}{2}$	

Qn	Solutions	Marks	Comments+Criteria
	$x + 60° = 360n \pm 60$	1M.	Gave 1 out of 2 if not general.
	$x = 360n\ ;\ 360n - 120°$	½M.	
f)	$\sin\theta = \frac{2\tan\theta/2}{1+\tan^2\theta/2}$	1M.	
	$\sin 30 = \frac{2\tan 15}{1+\tan^2 15}$	½M.	
	$\frac{1}{2} = \frac{2t}{1+t^2}$, where $t = \tan 15°$		(−1) for wrong approach but if still show $\sin\theta = \frac{2t}{1+t^2}$
	$1+t^2 = 4t$		
	$t^2 - 4t + 1 = 0$		
	$t = \frac{4 \pm \sqrt{16-4}}{2}$	½	
	$= \frac{4 \pm 2\sqrt{3}}{2}$		
	$= 2 \pm \sqrt{3}$		
	$\tan 15° < \tan 45°$ ($y = \tan x$ is an increasing curve)	1M.	
	$\therefore \tan 15 = 2 - \sqrt{3}$		
	or Concluding using Calculator		