Student Number:________________

St. Catherine's School
Waverley

24 February 2009

Assessment Task 1

Extension II Mathematics

Time allowed: 60 minutes *HSC assessment weighting: 15%*

INSTRUCTIONS

- Write your STUDENT NUMBER on each page
- Marks for each part of a question are indicated
- All questions should be attempted on your own paper
- All necessary working should be shown
- Start each question on a NEW page
- Approved scientific calculators and drawing templates may be used

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE: $\ln x = \log_e x, \quad x > 0$

Question 1 (12 marks)

(a) Let $w_1 = 8-2i$ and $w_2 = -5+3i$. Find in the form $x+iy$:

(i) $w_1 + \overline{w_2}$ **1**

(ii) $\dfrac{1}{w_1 w_2}$ **2**

(b) (i) Show that $(1-2i)^2 = -3-4i$ **1**

(ii) Hence solve the equation $z^2 - 5z + (7+i) = 0$ **2**

(c) (i) Express $1-i\sqrt{3}$ in modulus-argument form. **2**

(ii) Express $\left(1-i\sqrt{3}\right)^5$ in modulus-argument form **2**

(iii) Hence express $\left(1-i\sqrt{3}\right)^5$ in the form $x+iy$ **1**

Question 2 (12 Marks)

a) Find all solutions to the equation $z^3 = -1$ in modulus-argument form. **3**

b) Sketch the region in the Argand diagram where the two inequalities **3**

$|z-i| \le 2$ and $0 \le \arg(z+1) \le \frac{\pi}{4}$ hold simultaneously.

c) Describe the locus of Z on the Argand diagram if

$\arg(z-1) - \arg(z+1) = \dfrac{\pi}{2}$, giving its Cartesian equation. **3**

d) Sketch the region in the Argand diagram that satisfies the inequality **3**

$z\overline{z} + 2(z+\overline{z}) \le 0$

Question 3 (12 marks)

(a) If $z = \cos\theta + i\sin\theta$

(i) Show that $z^n + \dfrac{1}{z^n} = 2\cos n\theta$ **2**

(ii) Hence show that $\cos^4\theta = \dfrac{1}{8}(\cos 4\theta + 4\cos 2\theta + 3)$ **3**

(b) The roots of $x^3 + 5x^2 + 11 = 0$, are α, β and γ.

(i) Find the polynomial equation whose roots are α^2, β^2 and γ^2. **2**

(ii) Find the polynomial equation whose roots are $\dfrac{1}{\alpha}$, $\dfrac{1}{\beta}$ and $\dfrac{1}{\gamma}$ **2**

(c) Show that $\dfrac{x^3-4x-10}{x^2-x-6} = x+1+\dfrac{3x-4}{x^2-x-6}$. **2**

(d) Using part (c) express $\dfrac{x^3-4x-10}{x^2-x-6}$ as the sum of partial fractions. **2**

Question 4 (12 marks)

a) Determine the complex roots of $z^6 = 1$ in the form $\cos\theta + i\sin\theta$ and hence factorise $z^6 - 1$ over:

(i) The complex field **2**

(ii) The real field using linear and quadratic factors **4**

b) When a polynomial $P(x)$ is divided by $(x-2)$ and $(x-3)$ the respective remainders are 4 and 9.

Determine what the remainder must be when $P(x)$ is divided by $(x-2)(x-3)$ **3**

c) If ω is a complex root of $z^3 = 1$

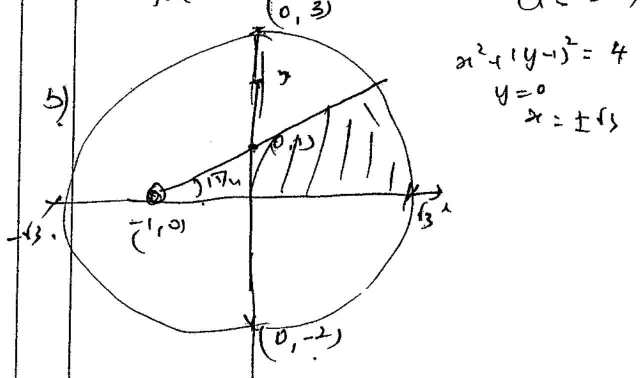
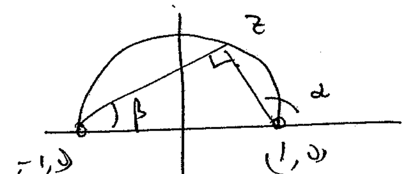
(i) Show that $1 + \omega + \omega^2 = 0$ **1**

(ii) If k is a positive integer, find two possible values of $1 + \omega^k + \omega^{2k}$ **2**

Qn	Solutions	Marks	Comments+Criteria
1. (i)	$\omega_1 = 8-2i \;;\; \omega_2 = -5+3i$ $\omega_1 + \overline{\omega_2} = 8-2i+-5-3i$ $= 3-5i$		
(ii)	$\dfrac{1}{\omega_1\omega_2}\;;$ $\omega_1\omega_2 = (8-2i)(-5+3i)$ $= (-40-6i^2)+i(24+10)$ $= -34+34i$ $= 34(-1+i)$ $\dfrac{1}{\omega_1\omega_1} = \dfrac{1}{34}\;\; \dfrac{-1-i}{}$	(2M)	$\dfrac{1}{34(-1+i)} \cdot \dfrac{-1-i}{-1-i}$ $= \dfrac{-1-i}{34(1+1)}$ $\cancel{\dfrac{(-1)^2-(i)^2}{1+1}}$ $= -\dfrac{1}{68}(1+i)$
b)	$(1-2i)^2 = 1+4i^2-4i$ $= -3-4i$ $z^2-5z+7+i=0$ $z = \dfrac{5\pm\sqrt{25-4(7+i)}}{2}$ $= \dfrac{5\pm\sqrt{-3-4i}}{2}$ $= \dfrac{5\pm(1-2i)}{2}$ $= \dfrac{6-2i}{2} \;;\; \dfrac{4+2i}{2}$ $= 3-i \;;\; 2+i$		
c)	$1-i\sqrt{3} =$ $2\,\text{cis}(-\pi/3)$ $32\,\text{cis}(-5\pi/3) = 32\,\text{cis}\,\pi/3$	2 2	$1\tfrac{1}{2}$ if left as $\text{cis}(-5\pi/3)$

$32(\cos 60 + i\sin 60)$

1.

Qn	Solutions	Marks	Comments+Criteria
2.	$z^3 = -1$ Let $z = r\,\text{cis}\,\theta \;;\; \lvert z\rvert = r$ $\lvert z^3\rvert = \lvert -1\rvert$ $\lvert z\rvert^3 = 1$ $r^3 = 1$ $r = 1 \;;\; r$ is Real $(\text{cis}\,\theta)^3 = -1$ $\cos 3\theta = -1 \;;\; \sin 3\theta = 0$ $3\theta = \pi, 3\pi, 5\pi$ $\theta = \dfrac{\pi}{3}, \pi, 5\pi/3$ The roots are $\text{cis}\,\pi/3$, $\text{cis}\,\pi = -1$; $\text{cis}\,5\pi/3$.		
b)	 $x^2+(y-1)^2 = 4$ $y=0$ $x = \pm\sqrt{3}$		
c)	 α: $\text{Arg}(z-1)$; $\beta = \text{Arg}(z+1)$ $\alpha-\beta = \dfrac{\pi}{2}$ given. $\therefore$ Locus is a semi circle, Centre (0,0) & radius 1, whose equation $y = \sqrt{1-x^2}$.		

Qn	Solutions	Marks	Comments+Criteria

d) Let $z = x + iy$

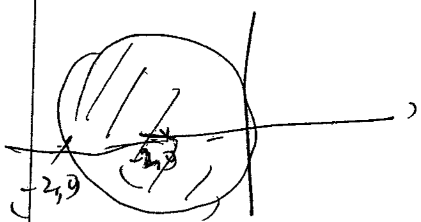
$\bar{z} = x - iy$

$z\bar{z} = x^2 + y^2$; $z + \bar{z} = 2x$

$\therefore z\bar{z} + 2(z + \bar{z}) \le 0$

$x^2 + y^2 + 4x \le 0$

$(x+2)^2 + y^2 \le 4$



Q.3 Let $z = \cos\theta + i\sin\theta$

$\frac{1}{z} =$

$z^n = (\cos\theta + i\sin\theta)^n$

$= \cos n\theta + i\sin n\theta$ (De Moivre's th^m)

$\frac{1}{z^n} = \frac{1}{\cos n\theta + i\sin n\theta} \times \frac{\cos n\theta - i\sin n\theta}{\cos n\theta - i\sin n\theta}$

$= \cos n\theta - i\sin n\theta$

$\therefore z^n + \frac{1}{z^n} = 2\cos n\theta$

So $z + \frac{1}{z} = 2\cos\theta$

Consider $(z + \frac{1}{z})^4 = z^4 + 4z^3\cdot\frac{1}{z} + 6z^2\cdot\frac{1}{z^2} + 4\cdot z\cdot\frac{1}{z^3} + \frac{1}{z^4}$

Qn	Solutions	Marks	Comments+Criteria

$(2\cos\theta)^4 = (z^4 + \frac{1}{z^4}) + 4(z^2 + \frac{1}{z^2}) + 6$

$16\cos^4\theta = 2\cos 4\theta + 4(2\cos 2\theta) + 6$

$\cos^4\theta = \frac{1}{8}(\cos 4\theta + 4\cos 2\theta + 3)$

(b) $x^3 + 5x^2 + 11 = 0$

The roots are α, β, γ

Let $P(x): x^3 + 5x^2 + 11$

The polynomial with roots $\alpha^2, \beta^2, \gamma^2$ is $P(\sqrt{x}) = 0$ ✓ 1

$(\sqrt{x})^3 + 5(\sqrt{x})^2 + 11 = 0$

$x\sqrt{x} = -5x - 11$

$x^3 = (5x - 11)^2$ ✓ 1

or $x^3 - 25x^2 - 110x - 121 = 0$

The polynomial with roots $\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}$ is $P(\frac{1}{x}) = 0$ 1M

$\frac{1}{x^3} + \frac{5}{x^2} + 11 = 0$

$1 + 5x + 11x^3 = 0$ 1M.

c)
$$\begin{array}{r} x + 1 \\ x^2 - x - 6 \,\big)\, x^3 - 4x - 10 \\ x^3 - x^2 - 6x \\ \hline x^2 + 2x - 10 \\ x^2 - x - 6 \\ \hline 3x - 4 \end{array}$$

Qn	Solutions	Marks	Comments+Criteria

$$\therefore \frac{x^3-4x-10}{x^2-x-6} = x+1 + \frac{3x-4}{x^2-x-6}$$

Consider

$$\frac{3x-4}{x^2-x-6} \equiv \frac{A}{x-3} + \frac{B}{x+2}$$

$$3x-4 \equiv A(x+2) + B(x-3)$$

sub $x=3$; $\quad 5 = 5A \quad \therefore A = 1$

sub $x=-2$ $\quad -10 = -5B \quad \therefore B = 2$

$$\therefore \frac{x^3-4x-10}{x^2-x-6} = x+1 + \frac{1}{x-3} + \frac{2}{x+2}$$

Q.4

$z^6 = 1$

Let $z = r\,\text{cis}\,\theta$; $|z| = r$

$|z^6| = |1|$

$|z|^6 = 1$

$r^6 = 1$

$r = 1$ (r is Real

$(\text{cis}\,\theta)^6 = 1$

$\text{cis}\,6\theta = 1$

$\cos 6\theta = 1$; $\sin 6\theta = 0$

$\therefore 6\theta = 0, 2\pi, 4\pi, 6\pi, 8\pi, 10\pi$

$\therefore \theta = 0, \frac{\pi}{3}, \frac{2\pi}{3}, \pi, \frac{4\pi}{3}, \frac{5\pi}{3}$

$\therefore z^6 - 1 = 0$ has as roots

$\text{cis}\,0 = 1$; $\text{cis}\frac{\pi}{3}$; $\text{cis}\frac{2\pi}{3}$; $\text{cis}\,\pi = -1$

$\text{cis}\frac{4\pi}{3} = \overline{\text{cis}\frac{2\pi}{3}}$; $\text{cis}\frac{5\pi}{3} = \overline{\text{cis}\frac{\pi}{3}}$

Qn	Solutions	Marks	Comments+Criteria

$$\therefore z^6 - 1 = (z-1)(z+1)\left(z - \text{cis}\frac{\pi}{3}\right)\left(z - \overline{\text{cis}\frac{\pi}{3}}\right)\left(z - \text{cis}\frac{2\pi}{3}\right)\left(z - \overline{\text{cis}\frac{2\pi}{3}}\right)$$

In Complex.

Consider $\left(z - \text{cis}\frac{\pi}{3}\right)\left(z - \overline{\text{cis}\frac{\pi}{3}}\right)$

$$= z^2 - z\left(\text{cis}\frac{\pi}{3} + \overline{\text{cis}\frac{\pi}{3}}\right) + \text{cis}\frac{\pi}{3}\,\overline{\text{cis}\frac{\pi}{3}}$$

$$= z^2 - 2z\cos\frac{\pi}{3} + 1$$

Similarly

$$\left(z - \text{cis}\frac{2\pi}{3}\right)\left(z - \overline{\text{cis}\frac{2\pi}{3}}\right)$$

$$= z^2 - 2z\cos\frac{2\pi}{3} + 1.$$

$$\therefore z^6 - 1 = (z-1)(z+1)\left(z^2 - 2z\cos\frac{\pi}{3} + 1\right)\left(z^2 - 2z\cos\frac{2\pi}{3} + 1\right)$$

(b) When $P(x)$ is divided by $(x-2)(x-3)$, a quadratic expression, the remainder is atmost a linear expression

$\therefore$ $P(x) = Q(x)(x-2)(x-3) + ax + b$

$P(2) = 4 \qquad 4 = 2a + b$

$P(3) = 9 \qquad 9 = 3a + b$

$\therefore a = 5$

$b = -6$

The remainder is $5x - 6$.

c)

Qn	Solutions	Marks	Comments+Criteria
c)	$z^3 = 1$ $(z-1)(z^2+z+1) = 0$ ω is complex ∴ is a root of $z^2+z+1=0$ ∴ $\omega^2+\omega+1=0$ when k is a multiple of 3 $\omega^k = 1$ $\omega^{2k} = 1$ ∴ $1+\omega^k+\omega^{2k} = 1+1+1$ $= 3$ when k is not a multiple of 3, $\omega^k = \omega$ or ω^2 $\omega^{2k} = \omega^2$ or ω ∴ $1+\omega^k+\omega^{2k} = \cancel{1+0+0}\ 1+\omega+\omega^2$ $= 0$. if ω^k : $\omega^4 = \omega$ $\omega^5 = \omega^3 \times \omega^2 = \omega^2$ $\omega^6 = 1$ if $\omega^k = \omega$: $\omega^{2k} = \omega^2$ if $\omega^k = \omega^2$: $\omega^{2k} = (\omega^2)^2 = \omega^4 = \omega^3 \times \omega = \omega$.	1 M	$\omega^7 = \omega^6 \times \omega = \omega$.