St Catherine's School
Waverley, Sydney

Student Number:...

Year 12
Assessment Task 1

Mathematics Extension II

Time allowed: 55 minutes

General Instructions

- Attempt ALL questions
- Write your Student NUMBER at the top of this page and on the writing paper used

Question	Marks
Total marks	

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1},\quad n \neq -1;\quad x \neq 0,\ \text{if } n<0$$

$$\int \frac{1}{x}\,dx = \ln x,\quad x>0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax},\quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax,\quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax,\quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax,\quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax,\quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a},\quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a},\quad a>0,\ -a<x<a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right),\quad x>a>0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE: $\ln x = \log_e x,\quad x>0$

Question 1.

If $z=1-\sqrt{3}i$ and $w=-1+i$, find the Modulus and Argument of the following complex numbers

(a) z (1m)

(b) w (1m)

(c) $\dfrac{z^5}{w^4}$ (2m)

Question 2

(a) Solve for z: $z^5=1$, where z is a complex number (2m)

(b) Show that if w is a complex root of $z^5=1$, then $w^2, w^3\ and\ w^4$ are the other complex roots. (2m)

(c) Factorise z^5-1 in the field of Complex Numbers (1m)

(d) Factorise z^5-1 in the field of Real Numbers (2m)

(e) Represent the five roots on an Argand diagram (1m)

(f) Deduce that $\cos\dfrac{2\pi}{5}+\cos\dfrac{4\pi}{5}=-\dfrac{1}{2}$ (1m)

Question 3

Sketch the region on the Argand diagram containing all of the points representing the complex number z, such that

(a) $-\dfrac{\pi}{4}\le|z-i|\le\dfrac{\pi}{4}$ (2m)

(b) $z\bar{z}-3(z+\bar{z})\le 0$ (2m)

(c) $Arg\,z=Arg(z-1)$ (2m

$Arg\,z = Arg\,(z-(1+i))$

Question 4

Let $\alpha, \beta\ and\ \gamma$ be the roots of the equation $P(x)=0$, where $P(x)=x^3-4x+1$

Write down the cubic equation, whose roots are

(a) $\alpha^2, \beta^2\ and\ \gamma^2$ (2m)

(b) $\dfrac{1}{\alpha}, \dfrac{1}{\beta}\ and\ \dfrac{1}{\gamma}$ (1m)

Also find the value of

(c) $\alpha^3+\beta^3+\gamma^3$ (2m)

Question 5

If α is a root of multiplicity 3 of the polynomial equation $P(x)=0$, (3m)

where $P(x)=x^4+x^3-9x^2+11x-4$, factorise $P(x)$

Question 6

(a) Show that $1+i$ is a root of $x^3-3x^2+4x-2=0$ (2m)

(b) Hence or otherwise find all the roots of the polynomial equation. (2m)

Question 7

If the roots of $x^3+kx^2+lx+m=0$ are in geometric progression, find the relationship between k, l and m. (3m)

Question 8

One of the vertices of an equilateral triangle is the point represented by $1+\sqrt{3}i$. (4m)

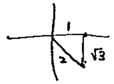
If the vertices of the equilateral triangle lie on a circle with centre the origin, find the other vertices and draw a neat sketch of the equilateral triangle clearly labelling the vertices.

END OF PAPER

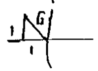
Solutions	Marks	Comments

Question 1: $z = 1-\sqrt{3}i \quad w = -1+i$

a) $|z| = 2 \quad \operatorname{Arg} z = -\frac{\pi}{3}$ — 1



b) $|w| = \sqrt{2} \quad \operatorname{Arg} w = \frac{3\pi}{4}$ — 1



c) $z^5 = 32\,\mathrm{cis}\,\frac{-5\pi}{3} \quad w^4 = 4\,\mathrm{cis}\,3\pi$

$\therefore \left|\frac{z^5}{w^4}\right| = \frac{|z^5|}{|w^4|} = \frac{32}{4} = 8$ — 1

$\arg\left(\frac{z^5}{w^4}\right) = \arg z^5 - \arg w^4$

$= -\frac{5\pi}{3} - 3\pi$

$= -\frac{14\pi}{3}$

$= -\frac{2\pi}{3}$ — 2

Question 2:

a) $z^5 = 1$

let $z = r\,\mathrm{cis}\,\theta$

$\therefore r^5\,\mathrm{cis}\,5\theta = \mathrm{cis}\,0$

$\therefore r^5 = 1 \quad r = 1$ — 1

$\cos 5\theta = 1 \quad \sin 5\theta = 0$

$\therefore 5\theta = 2k\pi \quad k = 0,1,2,3,4$

$\therefore \theta = \frac{2k\pi}{5} \quad k = 0,1,2,3,4$

$\therefore$ solutions are $z = \mathrm{cis}\,0,\ \mathrm{cis}\,\frac{2\pi}{5},\ \mathrm{cis}\,\frac{4\pi}{5},\ \mathrm{cis}\,\frac{6\pi}{5},\ \mathrm{cis}\,\frac{8\pi}{5}$

or $z = 1,\ \mathrm{cis}\left(\pm\frac{2\pi}{5}\right),\ \mathrm{cis}\left(\pm\frac{4\pi}{5}\right)$ — 2

Solutions	Marks	Comments

b) let $\omega = \mathrm{cis}\,\frac{2\pi}{5}$

then $\omega^2 = \mathrm{cis}\,\frac{4\pi}{5}$

$\omega^3 = \mathrm{cis}\,\frac{6\pi}{5}$

$\omega^4 = \mathrm{cis}\,\frac{8\pi}{5}$ which are the other complex roots — 2

c) $z^5 - 1 = (z-1)\left(z - \mathrm{cis}\frac{2\pi}{5}\right)\left(z - \mathrm{cis}\frac{4\pi}{5}\right)\left(z - \mathrm{cis}\frac{6\pi}{5}\right)\left(z - \mathrm{cis}\frac{8\pi}{5}\right)$ — 1

$= (z-1)\left(z - \mathrm{cis}\frac{2\pi}{5}\right)\left(z - \mathrm{cis}\frac{-2\pi}{5}\right)\left(z - \mathrm{cis}\frac{4\pi}{5}\right)\left(z - \mathrm{cis}\frac{-4\pi}{5}\right)$ — 1

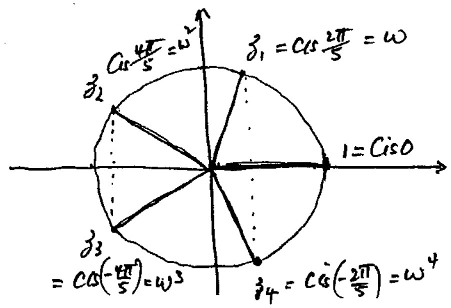
$= (z-1)\left[z - \mathrm{cis}\left(\pm\frac{2\pi}{5}\right)\right]\left[z - \mathrm{cis}\left(\pm\frac{4\pi}{5}\right)\right]$ — 1

d) $z^5 - 1 = (z-1)\left(z - \mathrm{cis}\frac{2\pi}{5}\right)\left(z - \mathrm{cis}\frac{-2\pi}{5}\right)\left(z - \mathrm{cis}\frac{4\pi}{5}\right)\left(z - \mathrm{cis}\frac{-4\pi}{5}\right)$ — 1

$= (z-1)\left(z - \mathrm{cis}\frac{2\pi}{5}\right)\left(z - \overline{\mathrm{cis}\frac{2\pi}{5}}\right)\left(z - \mathrm{cis}\frac{4\pi}{5}\right)\left(z - \overline{\mathrm{cis}\frac{4\pi}{5}}\right)$

$= (z-1)\left(z^2 - 2\cos\frac{2\pi}{5}z + 1\right)\left(z^2 - 2\cos\frac{4\pi}{5}z + 1\right)$ — 2

e)

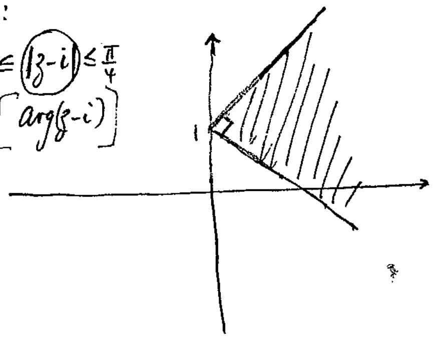
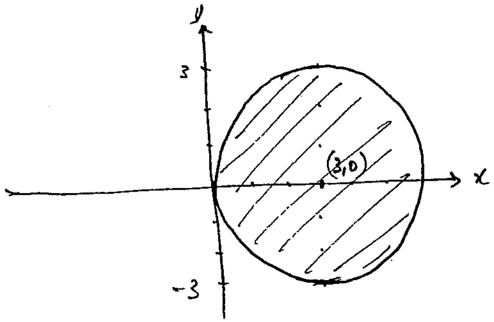


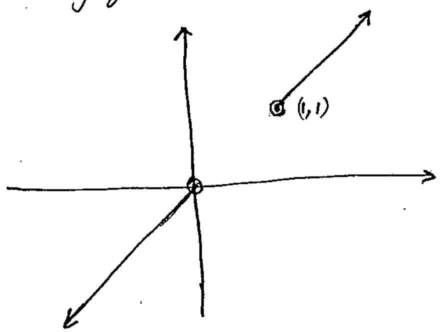
— 1

f) Sum of roots $= 0$ $\quad \therefore 1 + \mathrm{cis}\frac{2\pi}{5} + \mathrm{cis}\frac{4\pi}{5} + \mathrm{cis}\frac{6\pi}{5} + \mathrm{cis}\frac{8\pi}{5} = 0$

$\therefore 1 + \mathrm{cis}\frac{2\pi}{5} + \mathrm{cis}\frac{4\pi}{5} + \overline{\mathrm{cis}\frac{2\pi}{5}} + \overline{\mathrm{cis}\frac{4\pi}{5}}$

$\therefore 2\cos\frac{2\pi}{5} + 2\cos\frac{4\pi}{5} = -1 \quad \therefore \cos\frac{2\pi}{5} + \cos\frac{4\pi}{5} = -\frac{1}{2}$ — 1

Solutions	Marks	Comments
Question 3:		
a) $-\frac{\pi}{4} \le \arg(z-i) \le \frac{\pi}{4}$ $[\arg(z-i)]$ 	2	
b) $z\bar{z} - 3(z+\bar{z}) \le 0$ $x^2+y^2-3(2x) \le 0$ $x^2+y^2-6x \le 0$ $x^2-6x+9+y^2 \le 9$ $(x-3)^2+y^2 \le 9$	1	
	2	

Solutions	Marks	Comments
Question 3 c) $\arg z = \arg[z-(1+i)]$ 	2	
Question 4: $P(x) = x^3 - 4x + 1$ a) let $y = x^2 \quad \therefore x = \sqrt{y} = y^{\frac{1}{2}}$ $\therefore P(y) = (y^{\frac{1}{2}})^3 - 4(y^{\frac{1}{2}}) + 1 = 0$	1	
$y^{3/2} - 4y^{\frac{1}{2}} + 1 = 0$ $y^{3/2} - 4y^{\frac{1}{2}} = -1$ $y^{\frac{1}{2}}(y-4) = -1$ $y(y-4)^2 = 1$ $y(y^2-8y+16) = 1$ $y^3 - 8y^2 + 16y - 1 = 0$	2	
b) let $y = \frac{1}{x} \quad \therefore x = \frac{1}{y}$ $\therefore P(y) = \left(\frac{1}{y}\right)^3 - 4\left(\frac{1}{y}\right) + 1 = 0$ $\times y^3 \qquad 1 - 4y^2 + y^3 = 0$ or $x^3 - 4x^2 + 1 = 0$	1	

Solutions	Marks	Comments
(c) $\alpha^3 - 4\alpha + 1 = 0$ — ① $\beta^3 - 4\beta + 1 = 0$ — ② $\gamma^3 - 4\gamma + 1 = 0$ — ③	1	
①+②+③ $\alpha^3+\beta^3+\gamma^3 - 4(\alpha+\beta+\gamma) + 3 = 0$ $\therefore \alpha^3+\beta^3+\gamma^3 = 4(\alpha+\beta+\gamma) - 3$ but $\alpha+\beta+\gamma = 0$ $\therefore \alpha^3+\beta^3+\gamma^3 = {}^-3$.	2	
Question 5: α root of multiplicity 3 $P(x) = x^4 + x^3 - 9x^2 + 11x - 4$ $P(1) = 0$	1	
$P'(x) = 4x^3 + 3x^2 - 18x + 11$ $P'(1) = 0$ $\therefore x = 1$ is the multiple root.	2	
$\therefore P(x) = (x-1)^3(x-\alpha)$ Since product of roots is -4 $\alpha = {}^-4$ $\therefore P(x) = (x-1)^3(x+4)$	3	
Question 6: a) $(1+i)^3 = 1 + 3i - 3 - i$ $= -2 + 2i$ $(1+i)^2 = 1 + 2i - 1$ $= 2i$	1	
$\therefore x^3 - 3x^2 + 4x - 2$ is $(1+i)^3 - 3(1+i)^2 + 4(1+i) - 2$ $= -2 + 2i - 6i + 4 + 4i - 2$ $= 0$ $\therefore (1+i)$ is a root.	2	

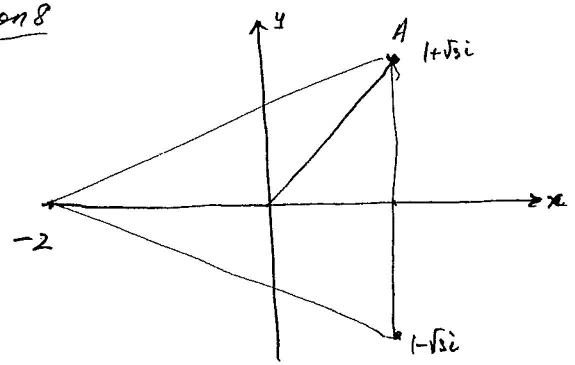
Solutions	Marks	Comments
Question 6 b) Since coefficients are real if $1+i$ is a root then $1-i$ is also a root.	1	
$\therefore (x-1-i)(x-1+i)$ is a factor $(x-1)^2 + 1$ is a factor $x^2 - 2x + 2$ is a factor.	$1\frac{1}{2}$	
$\therefore x^3 - 3x^2 + 4x - 2 = (x^2 - 2x + 2)(x - 1)$ $\therefore$ roots are $1+i, 1-i, 1$	2	
Question 7 let roots be $\frac{\alpha}{c}, \alpha, \alpha c$ $\therefore$ product of roots $= \alpha^3 = {}^-m$ — ① Sum of roots 2x $= \frac{\alpha^2}{c} + \alpha^2 c + \alpha^2 = l$ — ② Sum of roots $= \frac{\alpha}{c} + \alpha + \alpha c = {}^-k$ — ③	$1\frac{1}{2}$	
now from ② $\alpha\left(\frac{\alpha}{c} + \alpha c + \alpha\right) = l$ $\therefore \alpha(-k) = l$ from ③ but $\alpha = -m^{1/3}$ from ① $\therefore -m^{\frac{1}{3}} \cdot {}^-k = l$ cubing $mk^3 = l^3$ or $l^3 - mk^3 = 0$	3	

St Catherines's School Waverley

Course:

Marking Scheme for Task:

Academic Year: 2007-8

Solutions	Marks	Comments
Question 8 		
angle subtended at centre by sides of an equilateral triangle is 120°	1	
$\therefore$ next vertex $= 1+\sqrt{3}i \times \operatorname{cis} 120^\circ$ $= (1+\sqrt{3}i)\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)$ $= -\frac{1}{2}+\frac{\sqrt{3}}{2}i-\frac{\sqrt{3}}{2}i-\frac{3}{2}$ $= -2$	2	
next vertex $= -2 \times \operatorname{cis} 120^\circ$ $= -2\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)$ $= 1-\sqrt{3}i$	3	
$\therefore$ vertices are $1+\sqrt{3}i$, $1-\sqrt{3}i$ and -2	4	