- INDEPENDENT SCHOOLS -

2006
Higher School Certificate
Trial Examination

Mathematics
Extension 1

General Instructions

- Reading time – 5 minutes
- Working time – 2 hours
- Write using black or blue pen
- Board approved calculators may be used
- Write using black or blue pen
- Write your student number and/or name at the top of every page
- All necessary working should be shown in every question
- A table of standard integrals is provided separately

Total marks – 84

Attempt Questions 1 – 7

All questions are of equal value

This paper MUST NOT be removed from the examination room

STUDENT NUMBER/NAME.................................

Student name / number ....

Marks

Question 1 **Begin a new page**

(a) When the polynomial $P(x) = x^3 + ax + 1$ is divided by $(x+2)$ the remainder is 3. Find the value of a. **2**

(b) The acute angle between the lines $y=(m+2)x$ and $y = mx$ is $45°$.

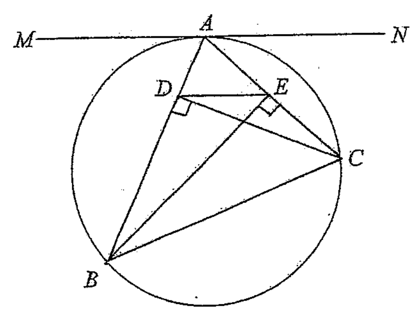
(i) Show that $\left|\dfrac{2}{m^2+2m+1}\right| = 1$. **1**

(ii) Hence find any values of m. **2**

(c)(i) Show that $\cot\theta - \cot 2\theta = \operatorname{cosec} 2\theta$. **2**

(ii) Hence find the exact value of $\cot 15°$. **1**

(d)



ABC is a triangle inscribed in a circle. MAN is the tangent at A to the circle ABC. CD and BE are altitudes of the triangle.

(i) Copy the diagram.

(ii) Give a reason why $BCED$ is a cyclic quadrilateral. **1**

(iii) Hence show that DE is parallel to MAN. **3**

Student name / number ....

Marks

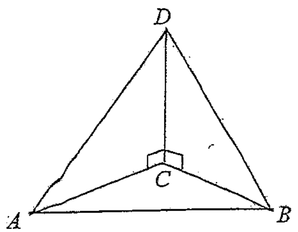
Question 2 **Begin a new page**

(a) $A(-3,4)$ and $B(1,2)$ are two points. Find the coordinates of the point $P(x,y)$ which divides the interval AB externally in the ratio 3 : 1. **2**

(b)(i) Solve the inequality $\frac{1}{1-x}<1$. **2**

(ii) Hence find the set of values of x for which the limiting sum S of the geometric series $1+x+x^2+x^3+...$ is such that $S<1$. **1**

(c)

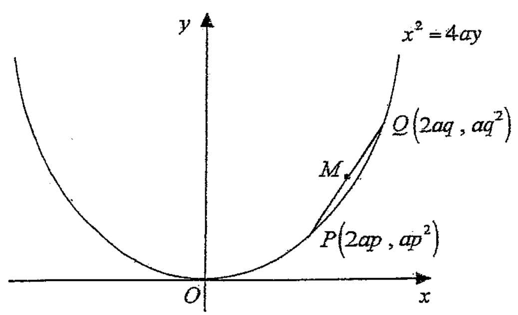


Three points A, B and C lie on horizontal ground. Points A and B are 30 metres apart and $\angle ACB=120°$. A vertical flagpole CD of height h metres stands at C. From each of A and B the angle of elevation of the top D of the flagpole is $30°$.

(i) Show that $AC=BC=h\sqrt{3}$. **1**

(ii) Hence find the value of h. **2**

(d)



(i) Find the coordinates of the point T on the parabola $x^2=4ay$ such that the tangent to the parabola at T is parallel to the line $y=x$. **1**

(ii) $P(2ap, ap^2)$ and $Q(2aq, aq^2)$ are two points that move on the parabola $x^2=4ay$ such that the chord PQ is always parallel to the line $y=x$. M is the midpoint of PQ. Find the equation of the locus of M and state any restrictions on this locus. **3**

Student name / number ....

Marks

Question 3 **Begin a new page**

(a) Consider the function $f(x)=\frac{x-2}{x-1}$.

(i) Show that the function is increasing for all values of x in its domain. **2**

(ii) Sketch the graph of the function showing clearly any intercepts on the coordinate axes and the equations of any asymptotes. **2**

(iii) Find the equation of the inverse function $f^{-1}(x)$. Deduce that the graph of the function $f(x)$ is symmetrical about the line $y=x$. **2**

(b) Consider the function $y=\frac{1}{2}\cos^{-1}(x-1)$.

(i) Find the domain and range of the function. **2**

(ii) Sketch the graph of the function showing clearly the coordinates of the endpoints. **1**

(iii) The region in the first quadrant bounded by the curve $y=\frac{1}{2}\cos^{-1}(x-1)$ and the coordinate axes is rotated through $360°$ about the y axis. Find the volume of the solid of revolution, giving your answer in simplest exact form. **3**

Student name / number ....

Marks

Question 4 **Begin a new page**

(a)(i) Show that the equation $x^3 + 2x - 7 = 0$ has a root α such that $1 < \alpha < 2$. **2**

(ii) If an initial approximation of $1 \cdot 5$ is taken for α, use one application of Newton's method to find the next approximation, rounding your answer to one decimal place. **2**

(b) Use the substitution $x = u^2$, $u \geq 0$, to find the value of $\int_1^3 \frac{1}{(x+1)\sqrt{x}}\,dx$. **4**

Give your answer in simplest exact form.

(c) Five different fair dice are thrown together. Find the probability that

(i) the five scores are all different **2**

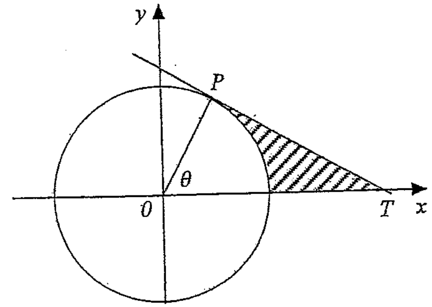
(ii) the five scores include at most one 6 **2**

Student name / number ....

Marks

Question 5 **Begin a new page**

(a)



P is a point on the circle $x^2 + y^2 = 1$ such that the radius OP makes an angle θ with the positive x axis, where $0 < \theta < \frac{\pi}{2}$. The tangent to the circle at P cuts the x axis at T.

(i) Show that the area A of the shaded region is given by $A = \frac{1}{2}(\tan\theta - \theta)$. **2**

(ii) If θ is increasing at a constant rate of $0 \cdot 1$ radians per second find the rate at which A is increasing when $\theta = 1$, giving your answer correct to 2 decimal places. **2**

(b) The number N of individuals in a population at time t years is given by $N = 100 - 60e^{-0 \cdot 1t}$.

(i) Sketch the graph of N as a function of t showing clearly the initial population size and the limiting population size. **2**

(ii) Find the exact time taken for the population to double its initial size and find the rate at which the population is increasing then. **2**

(c) Use Mathematical Induction to show that, for all positive integers $n \geq 1$, **4**

$1^2 + 3^2 + 5^2 + \ldots + (2n-1)^2 = \frac{1}{3}n(2n-1)(2n+1)$.

Student name / number ....

Marks

Question 6 **Begin a new page**

(a) A particle is moving in a straight line with Simple Harmonic Motion. At time t seconds it has displacement x metres from a fixed point O on the line, given by $x = 1 + 3\cos\frac{t}{2}$, velocity v ms^{-1} and acceleration a ms^{-2}.

(i) Show that $a = -\frac{1}{4}(x-1)$. 1

(ii) Find the distance travelled and the time taken by the particle over one complete oscillation of its motion. 2

(iii) Find the time taken by the particle to travel the first 100 metres of its motion, giving your answer in seconds correct to two decimal places. 3

(b) A golfer hits a golf ball from a point O with speed 40 ms^{-1} at an angle θ above the horizontal. The ball travels in a vertical plane where the acceleration due to gravity is 10 ms^{-2}.

(i) Write down expressions for the horizontal displacement x metres, and the vertical displacement y metres, of the golf ball from O after time t seconds. 1

(ii) Hence show that the horizontal range R metres of the golf ball until it returns to ground level is given by $R = 160\sin 2\theta$. 2

(iii) The golfer is aiming over horizontal ground at a circular pond of radius 10 metres with centre 110 metres from O. Find the set of possible values of θ for the golf ball to land directly in the pond, giving your answers correct to the nearest degree. 3

Student name / number ....

Marks

Question 7 **Begin a new page**

(a) A particle is moving in a straight line. At time t seconds it has displacement x metres from a fixed point O on the line, velocity v ms^{-1}, given by $v = (k-x)^2$ for some constant $k > 0$, and acceleration a ms^{-2}. Initially the particle is at O.

(i) Show that $x = \dfrac{k^2 t}{kt+1}$. Hence show that $x < k$ for all values of t. 4

(ii) Express a in terms of x. Deduce that the particle is always moving to the right and always slowing down. 2

(iii) Find the distance travelled and the time taken by the particle for its speed to drop to 1% of its initial value. 2

(b)(i) Show that ${}^{n+1}C_r - {}^{n}C_r = {}^{n}C_{r-1}$, $r = 1, 2, 3, \ldots, n$. 2

(ii) Hence find the value of $\sum_{n=3}^{100} {}^{n}C_2$. 2

Question 1

a. Outcomes assessed : PE3

Marking Guidelines

Criteria	Marks
• uses the remainder theorem to write an equation for a	1
• finds the value of a	1

Answer

$P(x) = x^3 + ax + 1 \qquad \therefore P(-2) = 3 \Rightarrow -8 - 2a + 1 = 3 \qquad \therefore a = -5$

b. Outcomes assessed : P4, H5

Marking Guidelines

Criteria	Marks
i • uses the formula for the angle between two lines to obtain the required equation for m	1
ii • reduces this equation to a quadratic in m	1
• solves this quadratic to find two values of m	1

Answer

i. $\left|\dfrac{(m+2)-m}{1+m(m+2)}\right| = \tan 45°$

$\therefore \left|\dfrac{2}{m^2+2m+1}\right| = 1$

ii. $\left|\dfrac{2}{(m+1)^2}\right| = 1$

$\therefore (m+1)^2 = 2$

$\therefore m + 1 = \pm\sqrt{2}$

$m = -1 \pm \sqrt{2}$

c. Outcomes assessed : H5

Marking Guidelines

Criteria	Marks
i • expresses $\cot 2\theta$ in terms of t where $t = \tan\theta$	1
• simplifies LHS in terms of t and recognizes expression for $\operatorname{cosec} 2\theta$	1
ii • applies identity to evaluate $\cot 15°$	1

Answer

i. Let $t = \tan\theta$. Then

$$\cot\theta - \cot 2\theta = \frac{1}{t} - \frac{1-t^2}{2t} = \frac{2-(1-t^2)}{2t} = \frac{1+t^2}{2t} = \operatorname{cosec} 2\theta$$

ii. $\cot 15° - \cot 30° = \operatorname{cosec} 30°$

$\cot 15° - \sqrt{3} = 2$

$\cot 15° = 2 + \sqrt{3}$

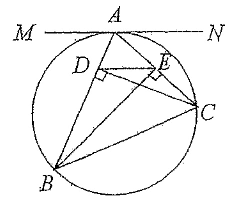
d. Outcomes assessed : PE2, PE3

Marking Guidelines

Criteria	Marks
ii • quotes appropriate test for a cyclic quadrilateral	1
iii • uses equality of exterior and interior opposite angles of $BCED$ with explanation	1
• applies alternate segment theorem with explanation	1
• quotes an appropriate test for parallel lines	1

Answer

i



ii. Interval BC subtends equal angles BDC and BEC at points D, E on the same side of BC. Hence $BCED$ is a cyclic quadrilateral.

iii. $\angle ADE = \angle ACB$ (*exterior $\angle$ of cyclic quad. BCED is equal to interior opp. $\angle$*)

But $\angle MAB = \angle ACB$ (*$\angle$ between tangent and chord AB is equal to $\angle$subtended by AB in the alternate segment*)

Hence $\angle MAB = \angle ADE$

$\therefore DE \parallel MAN$ (*equal alternate $\angle$s on transversal AB*)

Question 2

a. Outcomes assessed : P4

Marking Guidelines

Criteria	Marks
• finds x coordinate	1
• finds y coordinate	1

Answer

$A(-3, 4) \qquad B(1, 2)$

$3 \quad : \quad -1$

$P\left(\dfrac{3\times 1+(-1)\times(-3)}{3+(-1)}, \dfrac{3\times 2+(-1)\times 4}{3+(-1)}\right)$

$\therefore P(3, 1)$ is the required point of external division.

b. Outcomes assessed : H5, PE3

Marking Guidelines

Criteria	Marks
i • applies a method which deals appropriately with the inequality and variable denominator	1
• finds possible values of x	1
ii • uses condition for existence of limiting sum along with result from i. to find x values.	1

Answer

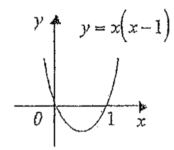
i. $\dfrac{1}{1-x} < 1$

$1 - x < (1-x)^2 \quad , \quad x \neq 1$

$0 < (1-x)^2 - (1-x)$

$0 < (1-x)\{(1-x) - 1\}$

$0 < x(x-1)$



$\therefore x < 0$ or $x > 1$

ii. $1 + x + x^2 + \ldots$ has limiting sum $S = \dfrac{1}{1-x}$ provided $|x| < 1$.

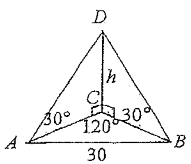
Hence $S < 1$ for $-1 < x < 0$.

c. Outcomes assessed : H5

Marking Guidelines

Criteria	Marks
i • uses right triangle trigonometry to show result	1
ii • uses cosine rule to write equation for h	1
• finds value of h	1

Answer



i. $AC = BC = h\cot 30° = h\sqrt{3}$

ii. Using the cosine rule in ΔABC,

$$30^2 = 3h^2 + 3h^2 - 6h^2\cos 120°$$

$$900 = 6h^2\left(1+\tfrac{1}{2}\right)$$

$$h^2 = 100$$

$$h = 10$$

d. Outcomes assessed : H5, PE3, PE4

Marking Guidelines

Criteria	Marks
i • uses derivative to write equation for x at T then writes coordinates of T.	1
ii • uses gradient of PQ to show sum of p and q is 2	1
• writes coordinates of M in terms of p, q then deduces equation of locus	1
• states restriction $y > a$	1

Answer

i. $x^2 = 4ay$

$$\frac{dy}{dx} = \frac{x}{2a}$$

$$\frac{dy}{dx} = 1 \Rightarrow x = 2a$$

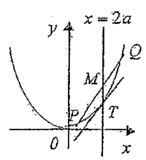
$$\therefore T(2a\,,\,a)$$

ii. PQ has gradient $\dfrac{a(p^2-q^2)}{2a(p-q)} = \dfrac{p+q}{2}$

Hence $p+q=2$

$M\left(a(p+q)\,,\,\frac{a}{2}(p^2+q^2)\right)$

Hence locus of M has equation $x = 2a$.



M lies above T $\therefore y > a$.

Question 3

a. Outcomes assessed : P5, H6, HE4

Marking Guidelines

Criteria	Marks
i • differentiates the function	1
• notes that the derivative is positive throughout the domain	1
ii • sketches hyperbola with correct intercepts on the coordinate axes	1
• shows equations of both asymptotes	1
iii • finds the equation of the inverse function	1
• uses reflection property of graphs of inverse functions to justify required deduction	1

Answer

i.

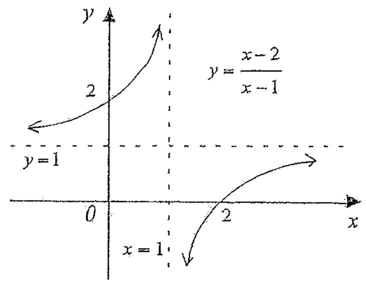
$$f'(x) = \frac{1.(x-1)-(x-2).1}{(x-1)^2}$$

$$= \frac{1}{(x-1)^2}$$

$f(x) = \dfrac{x-2}{x-1}$ has domain $x \neq 1$.

$\therefore f'(x) > 0$ and function is increasing throughout its domain.

ii.



iii.

$$y = \frac{x-2}{x-1}$$

$$(x-1)y = x-2$$

$$xy - y = x-2$$

$$x(y-1) = y-2$$

$$x = \frac{y-2}{y-1}$$

Interchanging $x \leftrightarrow y$, inverse function is

$$y = \frac{x-2}{x-1}$$

$$\therefore f^{-1}(x) = \frac{x-2}{x-1}$$

The graph of $f^{-1}(x)$ is the reflection of the graph of $f(x)$ in the line $y = x$. But the two graphs are identical. Hence the graph of $f(x)$ must be symmetrical in the line $y = x$.

b. Outcomes assessed : H8, HE4

Marking Guidelines

Criteria	Marks
i • states domain of function	1
• states range of function	1
ii • sketches graph of correct shape showing coordinates of endpoints	1
iii • writes volume as integral in terms of y	1
• finds primitive after using appropriate trig. identity	1
• evaluates by substitution of limits	1

Answer

i. $y = \frac{1}{2}\cos^{-1}(x-1)$

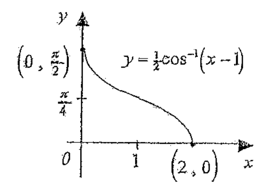
$-1 \le x-1 \le 1$

Domain $\{x : 0 \le x \le 2\}$

$0 \le \cos^{-1}(x-1) \le \pi$

Range $\{y : 0 \le y \le \frac{\pi}{2}\}$

ii



iii. $x = 1 + \cos 2y$

$$V = \pi\int_0^{\frac{\pi}{2}} (1+\cos 2y)^2\,dy$$

$$= \pi\int_0^{\frac{\pi}{2}} \left(1 + 2\cos 2y + \cos^2 2y\right)dy$$

$$= \pi\int_0^{\frac{\pi}{2}} \left(1 + 2\cos 2y + \tfrac{1}{2}(1+\cos 4y)\right)dy$$

$$= \pi\left[\tfrac{3}{2}y + \sin 2y + \tfrac{1}{8}\sin 4y\right]_0^{\frac{\pi}{2}}$$

$$= \pi\left\{\tfrac{3}{2}\left(\tfrac{\pi}{2}-0\right) + (\sin\pi - \sin 0) + \tfrac{1}{8}(\sin 2\pi - \sin 0)\right\}$$

$$= \tfrac{3}{4}\pi^2$$

Question 4

a. Outcomes assessed : PE3

Marking Guidelines

Criteria	Marks
i • shows that values of $f(x)=x^3+2x-7$ at $x=1, x=2$ have opposite signs	1
• uses continuity of f to deduce the existence of a root of the equation between 1 and 2	1
ii • applies Newton's rule for next approximation	1
• calculates this approximation	1

Answer

i. Let $f(x)=x^3+2x-7$

Then f is continuous with

$f(1)=-4<0$ and $f(2)=5>0$

Hence $f(\alpha)=0$ for some $1<\alpha<2$.

ii. $f'(x)=3x^2+2$

$$\alpha_1 = 1\cdot5 - \frac{f(1\cdot5)}{f'(1\cdot5)} = 1\cdot5 - \frac{3\cdot375+3-7}{6\cdot75+2}$$

$\therefore \alpha_1 \approx 1\cdot6$

b. Outcomes assessed : HE6

Marking Guidelines

Criteria	Marks
• converts dx into du and simplifies new integrand	1
• finds u limits	1
• finds primitive function in terms of u	1
• substitutes limits and evaluates in simplest exact form	1

Answer

$x=u^2,\ u\ge 0$

$dx = 2u\,du$

$x=1 \Rightarrow u=1$

$x=3 \Rightarrow u=\sqrt{3}$

$$I=\int_1^3 \frac{1}{(x+1)\sqrt{x}}\,dx$$

$$=\int_1^{\sqrt3} \frac{1}{(u^2+1)u}\,2u\,du$$

$$=2\int_1^{\sqrt3} \frac{1}{(u^2+1)}\,du$$

$$I = 2\left[\tan^{-1}u\right]_1^{\sqrt3}$$

$$=2\left(\tan^{-1}\sqrt3-\tan^{-1}1\right)$$

$$=2\left(\tfrac{\pi}{3}-\tfrac{\pi}{4}\right)$$

$$=\tfrac{\pi}{6}$$

c. Outcomes assessed : HE3

Marking Guidelines

Criteria	Marks
i • counts the number of arrangements of five different scores	1
• selects appropriate denominator and simplifies	1
ii • writes numerical expression for the sum of probabilities of *no 6* and *exactly one 6*	1
• calculates this probability	1

Answer

i. $P(\textit{all different}) = \dfrac{{}^6C_5\times 5!}{6^5} = \dfrac{5}{54}$

ii. $P(\textit{at most one } 6) = {}^5C_0\left(\frac16\right)^0\left(\frac56\right)^5 + {}^5C_1\left(\frac16\right)^1\left(\frac56\right)^4 = 2\left(\frac56\right)^5 \approx 0\cdot804$

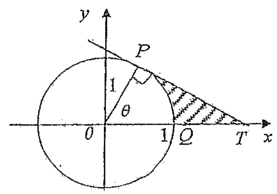
Question 5

a. Outcomes assessed : H4, H5, HE5

Marking Guidelines

Criteria	Marks
i • shows $PT=\tan\theta$	1
• uses difference between area triangle and area of sector to obtain expression for A	1
ii • expresses $\dfrac{dA}{dt}$ in terms of $\dfrac{d\theta}{dt}$	1
• evaluates to find rate of increase of A when $\theta=1$	1

Answer



i. $\angle OPT = 90°$

(tangent ⊥ radius drawn to point of contact)

$\therefore PT=\tan\theta$

$A=\frac12 OP\,.\,PT-\frac12 .1^2 .\,\theta$

$=\frac12(\tan\theta-\theta)$

ii. $\dfrac{dA}{dt}=\frac12\left(\sec^2\theta\,\dfrac{d\theta}{dt}-\dfrac{d\theta}{dt}\right)$

$=\frac12\left(\sec^2\theta-1\right)\dfrac{d\theta}{dt}$

$=\frac12\tan^2\theta\times 0\cdot1$

Hence when $\theta=1$, A is increasing at a rate $0\cdot12$ sq. units per second.

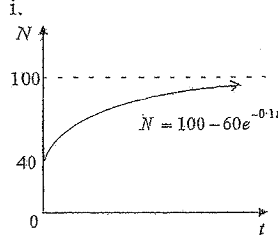
b. Outcomes assessed : HE3

Marking Guidelines

Criteria	Marks
i • sketches graph of correct shape showing initial population size	1
• shows limiting size as horizontal asymptote	1
ii • solves exponential equation for t to find exact time for initial size to double	1
• differentiates then finds rate of increase.	1

Answer

i.



ii. $t=0 \Rightarrow N=100-60e^0=40$

$N=80 \Rightarrow 60e^{-0\cdot1t}=20$

$e^{-0\cdot1t}=\frac13$

$-0\cdot1t=\ln\frac13$

$t=\ln\frac13\div(-0\cdot1)$

Population doubles initial size in $10\ln3$ years

$\dfrac{dN}{dt}=0\cdot1\times60e^{-0\cdot1t}=\frac{1}{10}(100-N)$

Population is then increasing at a rate of 2 individuals per year.

c. Outcomes assessed : HE2

Marking Guidelines

Criteria	Marks
• defines a sequence of statements and shows the first is true	1
• expresses the LHS of $S(k+1)$ in terms of the RHS of $S(k)$ (if true)	1
• rearranges algebraically to give RHS of $S(k+1)$	1
• writes final explanation to complete process of induction	1

Answer

Let $S(n)$ be the sequence of statements $1^2+3^2+5^2+\dots+(2n-1)^2=\frac{1}{3}n(2n-1)(2n+1)$, $n=1,2,3,\dots$

Consider $S(1)$: $LHS=1^2=1$ and $RHS=\frac{1}{3}\,1.(2-1)(2+1)=1$. Hence $S(1)$ is true

If $S(k)$ *is true* : $1^2+3^2+5^2+\dots+(2k-1)^2=\frac{1}{3}k(2k-1)(2k+1)$ **

Consider $S(k+1)$: $LHS=\left\{1^2+3^2+5^2+\dots+(2k-1)^2\right\}+(2k+1)^2$

$$=\tfrac{1}{3}k(2k-1)(2k+1)+(2k+1)^2 \qquad \textit{if } S(k) \textit{ is true, using **}$$
$$=\tfrac{1}{3}(2k+1)\{k(2k-1)+3(2k+1)\}$$
$$=\tfrac{1}{3}(2k+1)\{2k^2+5k+3\}$$
$$=\tfrac{1}{3}(2k+1)(k+1)(2k+3)$$
$$=\tfrac{1}{3}(k+1)\{2(k+1)-1\}\{2(k+1)+1\}$$
$$=RHS$$

Hence if $S(k)$ is true, then $S(k+1)$ is true. But $S(1)$ is true, hence $S(2)$ is true, and then $S(3)$ is true and so on. Therefore $S(n)$ is true for all positive integers $n\ge 1$.

Question 6

a. Outcomes assessed : HE3

Marking Guidelines

Criteria	Marks
i • finds expression for a by differentiation (or by noting centre of oscillation and value of n)	1
ii • uses amplitude to find distance	1
• uses period to find time	1
iii • finds number of complete oscillations and corresponding time	1
• recognizes time taken for extra 4 m is time to first reach O and writes equation for t	1
• adds time for the extra 4m to time for 8 complete oscillations and calculates total time	1

Answer

i. $x=1+3\cos\frac{t}{2}$

$v=-\frac{3}{2}\sin\frac{t}{2}$

$a=-\frac{3}{4}\cos\frac{t}{2}$

$\therefore a=-\frac{1}{4}(x-1)$

ii. Amplitude is 3m and period is 4π s.
Hence distance travelled is 12m and time taken is 4π s.

iii. Initially particle is at its far right extreme where $x=4$.
Also $100=8\times12+4$. Hence time taken for 100m is time for 8 complete oscillations plus the time taken to travel directly from $x=4$ to $x=0$.

$x=0 \Rightarrow \cos\frac{t}{2}=-\frac{1}{3}$. First such t is $2\left(\pi-\cos^{-1}\frac{1}{3}\right)$ seconds.

Hence time taken to travel 100m is $8\times4\pi+2\left(\pi-\cos^{-1}\frac{1}{3}\right)\approx104\cdot35$ s.

b. Outcomes assessed : H4, HE3

Marking Guidelines

Criteria	Marks
i • writes expressions for x and y	1
ii • finds $t>0$ for which $y=0$	1
• substitutes this value of t into expression for x to find R	1
iii • writes inequality for R	1
• finds one interval for θ	1
• finds second interval for θ	1

Answer

i. $x=40\,t\cos\theta$

$y=40\,t\sin\theta-5t^2$

ii. $y=5t(8\sin\theta-t)$

$y=0\Rightarrow t=0,\ 8\sin\theta$

Particle returns to ground level when

$x=40\,(8\sin\theta)\cos\theta$

$=160\,(2\sin\theta\cos\theta)$

$\therefore R=160\sin2\theta$

iii. $100\le R\le120$

$\frac{10}{16}\le\sin2\theta\le\frac{12}{16}$

$38\cdot68°\le2\theta\le48\cdot59°$

or $131\cdot41°\le2\theta\le141\cdot32$

$\therefore\ 20°\le\theta\le24°$

or $66°\le\theta\le70°$

Question 7

a. Outcomes assessed : HE1, HE5

Marking Guidelines

Criteria	Marks
i • writes $\dfrac{dt}{dx}$ as a function of x	1
• integrates to find t as a function of x	1
• rearranges to find x as a function of t	1
• deduces $x<k$	1
ii • finds a in terms of x	1
• notes that $v>0$ and a and v have opposite signs to make required deductions	1
iii • finds x when $v=\frac{4}{100}k^2$	1
• finds corresponding value of t	1

Answer

i. $\dfrac{dx}{dt}=(k-x)^2$

$\dfrac{dt}{dx}=(k-x)^{-2}$

$t=(k-x)^{-1}+c$

$\left.\begin{array}{l}t=0\\x=0\end{array}\right\}\Rightarrow\ 0=k^{-1}+c \quad \therefore c=-\frac{1}{k}$

$\therefore t=\dfrac{1}{k-x}-\dfrac{1}{k}$

$t+\dfrac{1}{k}=\dfrac{1}{k-x}$

$\dfrac{kt+1}{k}=\dfrac{1}{k-x}$

$k-x=\dfrac{k}{kt+1}$

$x=k-\dfrac{k}{kt+1}$

$=\dfrac{k(kt+1)-k}{kt+1}$

$=\dfrac{k^2t}{kt+1}$

$x=k\dfrac{kt}{kt+1}$ where $0<\dfrac{kt}{kt+1}<1$

$\therefore x<k$

ii $a=\frac{1}{2}\dfrac{d}{dx}v^2$

$=\frac{1}{2}\dfrac{d}{dx}(k-x)^4$

$=-2(k-x)^3$

$x<k\Rightarrow v>0$ and $a<0$

Hence particle is always moving right and slowing down.

iii. $v=(k-x)^2$

$=\left(\dfrac{k}{kt+1}\right)^2$

Initially $v=k^2$. Hence particle has 1% of initial speed when

$\dfrac{1}{(kt+1)^2}=\dfrac{1}{100}$

$kt+1=10$

$\therefore t=\frac{9}{k}$ and $x=\frac{9k}{10}$.

Hence particle has travelled $\frac{9k}{10}$ m and taken $\frac{9}{k}$ s.

b. **Outcomes assessed : PE3, HE3, HE7**

Marking Guidelines

Criteria	Marks
i • writes expression for ${}^{n+1}C_r - {}^{n}C_r$	1
• rearranges to obtain required result	1
ii • uses result from i. to write required sum as difference of two sums	1
• cancels out terms to simplify and evaluate	1

Answer

i. $${}^{n+1}C_r - {}^{n}C_r = \frac{(n+1)!}{r!\,(n+1-r)!} - \frac{n!}{r!\,(n-r)!}$$
$$= \frac{n!}{r!\,(n+1-r)!}\left\{(n+1)-(n+1-r)\right\}$$
$$= \frac{n!\,.\,r}{r!\,(n+1-r)!}$$
$$= \frac{n!}{(r-1)!\,(n+1-r)!}$$
$$= {}^{n}C_{r-1}$$

ii. $${}^{n}C_2 = {}^{n+1}C_3 - {}^{n}C_3$$
$$\sum_{n=3}^{100} {}^{n}C_2 = \sum_{n=3}^{100} {}^{n+1}C_3 - \sum_{n=3}^{100} {}^{n}C_3$$
$$= \sum_{n=4}^{101} {}^{n}C_3 - \sum_{n=3}^{100} {}^{n}C_3$$
$$= {}^{101}C_3 - {}^{3}C_3$$
$$= 166\,649$$