Student Number

St. Catherine's School Waverley

2011

HIGHER SCHOOL CERTIFICATE
ASSESSMENT TASK 1 – 15%
THURSDAY 24th FEBRUARY 2011

Mathematics

General Instructions

- Working time – 55 minutes
- Write using black or blue pen
- Board approved calculators may be used
- All necessary working should be shown
- Start each question in a new booklet

Total marks - 48

Attempt questions 1-3
All questions are of equal value

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1},\quad n\neq -1;\quad x\neq 0,\ \text{if } n<0$$

$$\int \frac{1}{x}\,dx = \ln x,\quad x>0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax},\quad a\neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax,\quad a\neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax,\quad a\neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax,\quad a\neq 0$$

$$\int \sec ax\tan ax\,dx = \frac{1}{a}\sec ax,\quad a\neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a},\quad a\neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a},\quad a>0,\ -a<x<a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right),\quad x>a>0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE: $\ln x = \log_e x,\ x>0$

6300 - 3

Total marks – 48
Attempt Questions 1-3
All questions are of equal value

Begin each question in a **NEW** booklet. Extra writing booklets are available.

Question 1 (16 Marks) **Marks**

(a) Differentiate with respect to x:

(i) $(7-6x)^4$ 1

(ii) $\dfrac{1}{6\sqrt{x}}$ 1

(iii) $\dfrac{x^3}{x-2}$ 2

(iv) $x^3(2-x)^4$ 3

(b) (i) Find the gradient of the tangent to the curve $y = x^3 - 9x^2 + 20x - 8$ at the point $(1, 4)$. 2

(ii) Hence state the gradient of the normal at the point (1.4). 1

(iii) Find the equation of the tangent at $(1, 4)$. 1

(iv) At what points on this curve are the tangents parallel to the line 3

$y = -4x + 3$?

(c) If $f(x) = \sqrt{8x+1}$, evaluate $f'(1)$. 2

Question 2 (16 Marks) **Start a NEW booklet.** **Marks**

(a) For the curve $y = x^3 - 3x^2 - 12$,

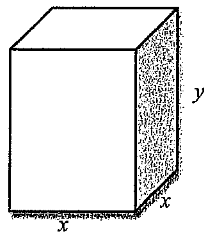
(i) Find the stationary points and determine their nature. 3

(ii) Show that a point of inflexion exists and determine its coordinates. 2

(iii) Hence or otherwise, sketch the curve, showing clearly all essential features. 2

(b) The curve $y = 3x^2 + \dfrac{a}{x^2}$ has a turning point at $x = 3$. Find the value of a. 3

(c) A rectangular box, open at the top, is to be constructed out of thin sheet metal on a square base of side x units, as shown in the diagram.



(i) If the box holds a volume of 500 cubic units, and its height is y units, find a formula for y in terms of x. 1

(ii) Show that the area, A square units of sheet metal required is given by 2

$$A = x^2 + \frac{2000}{x}.$$

(iii) Find the least area of sheet metal required to make the box. 3

Question 3 (16 Marks) **Start a NEW booklet.** **Marks**

(a) Evaluate $\sum_{r=0}^{4}(2^{r-1}+3r)$ **2**

(b) For the arithmetic sequence, 1, 4, 7, …….

(i) Find the 40^{th} term. **2**

(ii) Calculate the sum of the first 40 terms. **2**

(c) The first two terms of a geometric sequence are 9 and 6.

(i) What is the third term? **1**

(ii) Calculate the exact sum of the first 5 terms. **2**

(d) The sum of the first seven terms of an arithmetic series is five times the seventh term. The sum of the sixth and seventh term is 40. Find the sum of the first ten terms of the series. **3**

(e) A plant is 50 cm high when first observed. In the first week of observation, it grows 10 cm, and in each succeeding week the growth in height is 80% of the previous week's growth.

If this pattern of growth continues, what will be its ultimate height? **2**

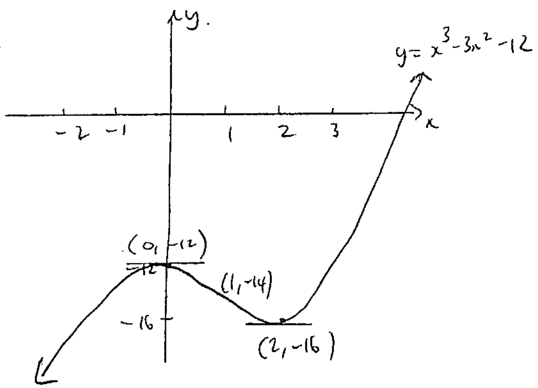
(f) Can there be an infinite geometric series with a limiting sum of $\frac{5}{8}$ and a first term of 2? Show all working and reasoning. **2**

End of paper

FEBRUARY 2011 HSC MATHEMATICS TASK 1 (152) SOLUTIONS

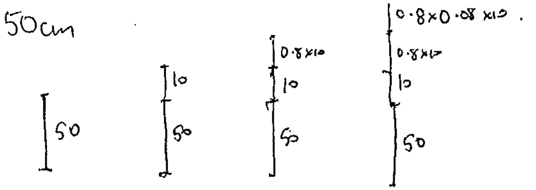
Qn	Solutions	Marks	Comments: Criteria
	Question 1 (16 marks)		
(a)	(i) $\frac{d}{dx}(7-6x)^4 = 4(7-6x)^3 \times -6$ $= -24(7-6x)^3$	1	
	(ii) $\frac{d}{dx}\frac{1}{6\sqrt{x}} = \frac{d}{dx}\frac{1}{6}.x^{-\frac{1}{2}}$ $= \frac{1}{6}.\frac{-1}{2}x^{-3/2}$ $= \frac{-1}{12}x^{-3/2}$	1	
	(iii) $\frac{d}{dx}\frac{x^3}{x-2} = \frac{vu'-uv'}{v^2}$ let $u=x^3$, $v=x-2$ $u'=3x^2$, $v'=1$ $= \frac{(x-2).3x^2 - x^3.1}{(x-2)^2}$ $= \frac{3x^3-6x^2-x^3}{(x-2)^2}$ $= \frac{2x^3-6x^2}{(x-2)^2}$ $= \frac{2x^2(x-3)}{(x-2)^2}$	2	
	(iv) $\frac{d}{dx}x^3(2-x)^4 = uv'+vu'$ Let $u=x^3$, $u'=3x^2$, $v=(2-x)^4$, $v'=-4(2-x)^3$ $= -4x^3(2-x)^3 + 3x^2(2-x)^4$ $= x^2(2-x)^3[-4x+3(2-x)]$ $= x^2(2-x)^3(-4x+6-3x)$ $= x^2(2-x)^3(6-7x)$	3	

Qn	Solutions	Marks	Comments: Criteria
	Question 1 continued.		
(b)	(i) $y = x^3-9x^2+20x-8$ $y' = 3x^2-18x+20$ When $x=1$, gradient of tangent $= 3(1)^2-18(1)+20$ $= 5$	2	
	(ii) $\therefore$ gradient of normal $= -\frac{1}{5}$ (since $m_1m_2=-1$)	1	
	(iii). Using $y-y_1 = m(x-x_1)$ $y-4 = 5(x-1)$ $y-4 = 5x-5$ $y = 5x-1$, is the equation of the tangent.	1	
	(iv). Gradient of line $y=-4x+3$ is -4. $\therefore$ Tangents which are parallel will also have the same gradient. $\therefore 3x^2-18x+20=-4$. $3x^2-18x+24=0$ $\div 3$ $x^2-6x+8=0$ $(x-4)(x-2)=0$ $\therefore x=4, 2$. $\therefore y=-8, 4$. $\therefore$ Pts where the tangents are parallel to the line are at $(4,-8)$ and $(2,4)$.	3	
(c)	$f(x) = \sqrt{8x+1}$ $= (8x+1)^{\frac{1}{2}}$ $\therefore f'(x) = \frac{1}{2}(8x+1)^{-\frac{1}{2}}.8$ $= \frac{4}{\sqrt{8x+1}}$ $\therefore f'(1) = \frac{4}{\sqrt{9}}$ $= \frac{4}{3}$	2	

Qn	Solutions	Marks	Comments: Criteria
	Question 2 (16 marks)		
(a)	(i) $y = x^3 - 3x^2 - 12$ $y' = 3x^2 - 6x$ $y'' = 6x - 6$ For stationary pts let $y' = 0$. ie $3x^2 - 6x = 0$ $3x(x-2) = 0$. $\therefore x = 0, 2$ $y = -12, -16$ $\therefore (0,-12)$ and $(2,-16)$ are stationary points When $x = 0$, $y'' = -6$ < 0 $\therefore (0,-12)$ is a maximum turning point When $x = 2$, $y'' = 6$ > 0 $\therefore (2,-16)$ is a minimum turning point.	3	
	(ii) For possible points of inflexion, let $y'' = 0$. i.e $6x - 6 = 0$. $\therefore x = 1$, $y = -14$. Checking: $\begin{array}{c\|c\|c\|c} x & 0.5 & 1 & 1.5 \\ \hline y'' & <0 & 0 & >0 \end{array}$ $\therefore$ change in concavity $\therefore (1,-14)$ is a point of inflexion.	2	
	(iii). 	2	

Qn	Solutions	Marks	Comments: Criteria
	Question 2 continued		
(b)	$y = 3x^2 + \frac{a}{x^2}$ $y' = 6x - 2ax^{-3}$ If a turning point exists at $x = 3$, then $y' = 0$. $\therefore 0 = 6(3) - 2a(3)^{-3}$ $0 = 18 - \frac{2a}{27}$ $\frac{2a}{27} = 18$ $2a = 486$ $\therefore a = 243$	3	
(c)	(i) $V = x^2 y$ $\therefore 500 = x^2 y$ $y = \frac{500}{x^2}$	1	
	(ii) Area $= x^2 + 4xy$ $= x^2 + 4x\left(\frac{500}{x^2}\right)$ $= x^2 + \frac{2000}{x}$, as required	2	
	(iii) $A = x^2 + 2000x^{-1}$ $\frac{dA}{dx} = 2x - 2000x^{-2}$ $\frac{d^2A}{dx^2} = 2 + 4000x^{-3}$ Let $2x - \frac{2000}{x^2} = 0$ $2x^3 - 2000 = 0$ $x^3 = 1000$ $\therefore x = 10$. When $x = 10$, $\frac{d^2A}{dx^2} = 2 + \frac{4000}{10^3}$ > 0 $\therefore$ minimum area. $\therefore$ When $x = 10$, $A = 10^2 + \frac{2000}{10}$ $A = 300$ $\therefore$ least area of sheet metal is $300\,u^2$.	3	

Qn	Solutions	Marks	Comments: Criteria
	Question 3 (16 marks)		
(a)	$\sum_{r=0}^{4}(2^{r-1}+3r)$. $=(2^{-1}+0)+(2^{0}+3)+(2^{1}+6)+(2^{2}+9)+(2^{3}+12)$ $=\frac{1}{2}+4+8+13+20$ $=45\frac{1}{2}$	2	
(b)	$1, 4, 7, \ldots$ (i) $a=1$, $d=3$ $\quad T_n=a+(n-1)d$ $T_{40}=1+(39)3$ $\therefore T_{40}=118$	2	
	(ii). $S_n=\frac{n}{2}[2a+(n-1)d]$ $\therefore S_{40}=\frac{40}{2}[2+(39)3]$ $=20[119]$ $=2380$	2	
(c)	$9, 6, \ldots$ (i) $a=9$, $r=\frac{2}{3}$ $\quad T_n=ar^{n-1}$ $T_3=9\left(\frac{2}{3}\right)^2$ $\therefore T_3=4$ (more simply, $\frac{2}{3}\times 6=4$)	1	
	(ii) First 5 terms are $9, 6, 4, \frac{8}{3}, \frac{16}{9}$ $\therefore$ Sum of first 5 terms $=9+6+4+\frac{8}{3}+\frac{16}{9}$ $=23\frac{4}{9}$	2	
	Note: Also, $S_5=\frac{9\left(1-\left(\frac{2}{3}\right)^5\right)}{1-\frac{2}{3}}$ $=23\frac{4}{9}$		

Qn	Solutions	Marks	Comments: Criteria
	Question 3 continued		
(d)	$S_7=5T_7$ $\quad T_6+T_7=40$. $\therefore \frac{7}{2}[2a+6d]=5[a+6d]$ $7a+21d=5a+30d$ $2a-9d=0$. (1) $\therefore (a+5d)+(a+6d)=40$ $2a+11d=40$ (2) $2a+11d=40$ (1) $2a-9d=0$ (2) (1)−(2) $20d=40$ $\therefore d=2$ Substitute $d=2$ into (1): $2a-18=0$ $2a=18$ $\therefore a=9$. $\therefore S_{10}=\frac{10}{2}[18+9\times 2]$ $=5(36)$ $\therefore S_{10}=180$	3	• Students who gave equation — $\frac{1}{2}$ mark. • Very poorly done question.
(e)	50cm  $\therefore$ Ultimate height $=50+\underbrace{10+0.8\times 10+0.8^2\times 10+\ldots}_{GP:\ a=10,\ r=0.8}$ cm $=50+\frac{10}{1-0.8}$ $=50+\frac{10}{0.2}$ $=100$cm.	2	• Students did not add initial 50cm to answer. • Use of $S_\infty=\frac{a}{1-r}$, gave $\frac{1}{2}$ mark as many students stated $a=50$, not $a=10$.

Qn	Solutions	Marks	Comments: Criteria
	Question 3 continued		
(f)	If $S_\infty = \frac{a}{1-r}$ $\frac{5}{8} = \frac{2}{1-r}$ $5-5r = 8 \times 2$ $-5r = 11$ $r = -\frac{11}{5}$ $\therefore$ Since, for a limiting sum to exist, $-1 < r < 1$ (i.e. $\lvert r \rvert < 1$), then an infinite geometric series <u>does</u> <u>not</u> exist as $r = -\frac{11}{5}$.	2	• $1\frac{1}{2}$ marks for correct use of limiting sum formula • $\frac{1}{2}$ mark for reason, $-1 < r < 1$.

– End of Paper –