Student Number:

St. Catherine's School
Waverley

April 2008
HSC ASSESSMENT TASK
MID COURSE EXAMINATION

Extension II Mathematics

Time allowed: 2 hours

Reading Time: 5 mins

INSTRUCTIONS

- All questions should be attempted on the separate paper provided
- All necessary working should be shown
- Marks for each part of a question are indicated
- Start each question on a new page
- Approved scientific calculators and drawing templates may be used
- Marks may be deducted for untidy or poorly arranged work.
- Standard Integrals are provided at the end of the paper

QUESTION 1 (15 marks) Start a *new* page. **Marks**

(a) Let $z=\dfrac{3+4i}{1+2i}$. Express z in the form $a+ib$, where a and b are real. **2**

(b) Let $\beta=1-i\sqrt{3}$.

(i) Express β in modulus-argument form. **2**

(ii) Hence write the exact value of β^{20} in the form $a+ib$, where a and b are real. **2**

(c) (i) On an Argand diagram shade the region where both $|z-(1+i)|\le\sqrt{2}$ and $0\le\arg z\le\dfrac{\pi}{2}$ hold. **2**

(ii) Find the exact perimeter of the shaded region. **2**

(d) $z=1+i$ is a root of the equation $z^3+az^2+bz+6=0$, where a and b are real numbers. **3**

Find the values of a and b and hence find all the roots of the equation.

(e) Draw a neat sketch of the locus specified by $z^2+(\overline{z})^2=0$. **2**

QUESTION 2 (15 marks) Start a *new* page. **Marks**

(a) Consider the ellipse E with equation $\dfrac{x^2}{25}+\dfrac{y^2}{9}=1$

(i) Calculate the eccentricity of E. **1**

(ii) Find the coordinates of the foci of E. and the equations of the directrices of E **2**

(iii) Sketch the graph of the ellipse E **1**

(iv) Show the point $(5\cos 60°, 3\sin 60°)$ on E **1**

(b) Show that the equation of the normal to the ellipse $x=3\cos\theta$, $y=4\sin\theta$ at the point where $\theta=\dfrac{\pi}{6}$ is $6x-8\sqrt{3}y+7\sqrt{3}=0$. **3**

(c) If $ax^3+cx+d=0$ has a double root, show that $27ad^2+4c^3=0$ **3**

(d) If $z=r(\cos\theta+i\sin\theta)$

(i) Prove that $\arg\left[\dfrac{z^2(3+3i)}{\left(1-\sqrt{3}\,i\right)z}\right]=\theta+\dfrac{7\pi}{12}$. **2**

(ii) Find the modulus of $\left[\dfrac{z^2(3+3i)}{\left(1-\sqrt{3}\,i\right)z}\right]$ in terms of r **2**

QUESTION 3 (15 marks) Start a *new* page. **Marks**

(a) You are given the equation of the normal to the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1 \quad (a^2>b^2)$ at the point $P(x_1,y_1)$ is $a^2y_1x-b^2x_1y=(a^2-b^2)x_1y_1$. This normal meets the major axis of the ellipse at G.

(i) Find the coordinates of G. **1**

(ii) Show that $GS=e.PS$, where e is the eccentricity of the ellipse. **3**

(b) If $z=\cos\theta+i\sin\theta$ **3**

By using the expansion of $\left(z+\dfrac{1}{z}\right)^4$, write $\cos^4\theta$ in terms of $\cos n\theta$

(c) Solve the equation $x^4-6x^3+12x^2-10x+3=0$, given that it has a root of multiplicity three. **3**

(d) For the ellipse $\dfrac{x^2}{16}+\dfrac{y^2}{9}=1$

(i) Show that the equation of the tangent at (x_1,y_1) is $\dfrac{xx_1}{16}+\dfrac{yy_1}{9}=1$ **2**

(ii) Hence find the equation of the chord of contact from $(5,1)$ to the ellipse. **3**

QUESTION 4 (15 marks) Start a *new* page. **Marks**

(a) When a polynomial $P(x)$ is divided by $(x-2)$ and $(x-3)$ the respective remainders are 4 and 9. Realising that the remainder is of the form $ax+b$ when dividing by a quadratic, determine what the remainder must be when $P(x)$ is divided by $(x-2)(x-3)$ **3**

(b) The equation $x^3-5x^2+5=0$ has roots α, β, γ.

(i) Find a polynomial equation with integer coefficients whose roots are $\alpha-1, \beta-1$, and $\gamma-1$ **2**

(ii) Find a polynomial equation with integer coefficients whose roots are $\alpha^2, \beta^2, \gamma^2$ **2**

(iii) Evaluate $\alpha^3+\beta^3+\gamma^3$. **2**

(c) (i) Using $\cos 2\alpha = 2\cos^2\alpha - 1$ to show that $\cos 4\theta = 8\cos^4\theta - 8\cos^2\theta + 1$ **2**

(ii) Hence solve the equation $8x^4-8x^2+1=0$ **2**

(iii) Hence deduce the exact values of $\cos\frac{\pi}{8}$ and $\cos\frac{5\pi}{8}$ **2**

QUESTION 5 (15 marks) (Start a *new* page.) **Marks**

(a) Given that $P(x)=x^3+3px+q$ has a double root at $x=k$

(i) Show that $p=-k^2$ **2**

(ii) Find q in terms of k. **2**

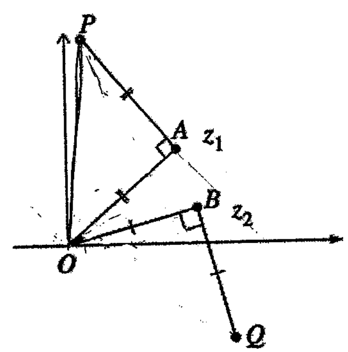
(iii) Hence, verify that $4p^3+q^2=0$ **1**

(b) Given that $\sin^{-1}x, \cos^{-1}x$ and $\sin^{-1}(1-x)$ are all acute angles

(i) Show that $\sin^{-1}(\sin^{-1}x - \cos^{-1}x) = 2x^2-1$ **3**

(ii) Solve the equation $\sin^{-1}x - \cos^{-1}x = \sin^{-1}(1-x)$ **2**

(c)

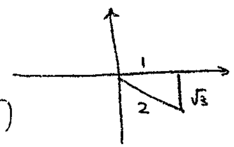


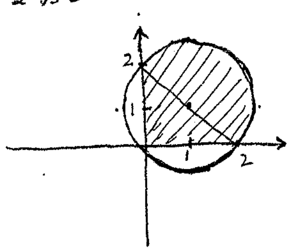
The points A and B in the complex plane correspond to the complex numbers z_1 and z_2 respectively. Both triangles OAP and OBQ are right-angled isosceles triangles.

(i) Explain why P corresponds to the complex number $(1+i)z_1$ and Q corresponds to $(1-i)z_2$ **2**

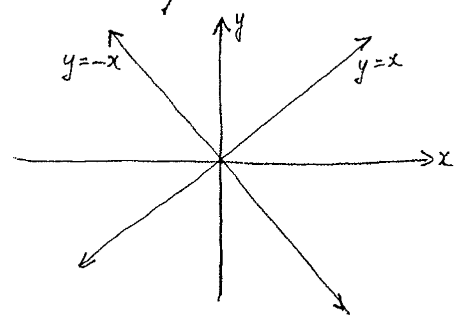
(ii) Let M be the midpoint of PQ. What complex number corresponds to M. **2**

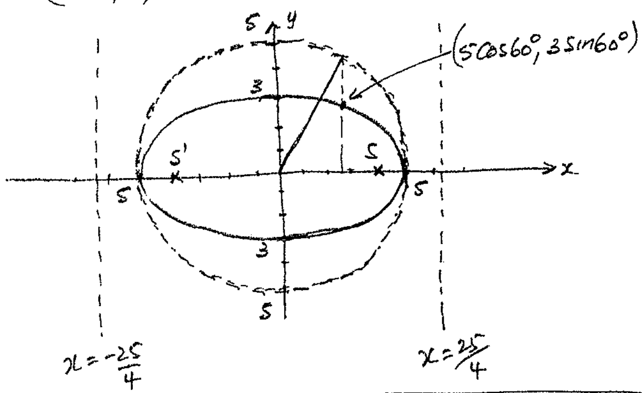
Solutions	Marks	Comments
Question 1		
a) $z = \frac{3+4i}{1+2i} \times \frac{1-2i}{1-2i}$	1	
$= \frac{3-6i+4i+8}{1+4}$		
$= \frac{11}{5} - \frac{2i}{5}$	1	
b) i) $\beta = 1 - i\sqrt{3}$		
$\therefore \beta = 2\,\text{cis}\left(-\frac{\pi}{3}\right)$		
$= 2\left(\cos\frac{\pi}{3} - i\sin\frac{\pi}{3}\right)$	2	
(ii) $\beta^{20} = \left[2\,\text{cis}\left(\frac{-\pi}{3}\right)\right]^{20}$		
$= 2^{20}\,\text{cis}\left(\frac{-20\pi}{3}\right)$	1	
$= 2^{20}\left(\cos\frac{-20\pi}{3} + i\sin\frac{-20\pi}{3}\right)$		
$= 2^{20}\left(-\frac{1}{2} - \frac{\sqrt{3}i}{2}\right)$		
$= -2^{19} - 2^{19}\sqrt{3}\,i$	1	
c) (i) $(x-1)^2 + (y-1)^2 = 2$	2	
(ii) P = half circle + intercepts		
$P = \sqrt{2}\pi + 4$	2	
[A = Area Circle − 2 segments]		
area one segment $= \frac{2\pi - 4}{4}$ $\left[\frac{\text{area of circle} - \text{square}}{4}\right]$		





Solutions	Marks	Comments
Question 1 d) $z = 1+i$ $\quad z^3 + az^2 + bz + 6 = 0$		
Coefficients real $z_1 = 1+i$ and $z_2 = 1-i$ are roots.		
let the other root be α		
$\therefore (1+i)(1-i)\alpha = -6$ (product of roots)		
$2\alpha = -6$		
$\alpha = -3$ $\quad\therefore$ roots are $z = 1+i$, $z = 1-i$, $z = -3$		
now $(1+i) + (1-i) + (-3) = -a$ (sum of roots)	3	
$\therefore a = 1$		
also $(1+i)(1-i) + (1-i)\alpha + (1+i)\alpha = b$ $(\Sigma\alpha\beta)$		
$2 + 2\alpha = b$		
$2 - 6 = b$		
$b = -4$		
e) $z^2 + (\bar{z})^2$		
let $z = x+iy$		
$\therefore (x+iy)^2 + (x-iy)^2 = 0$		
$x^2 - y^2 + 2ixy + x^2 - 2ixy - y^2 = 0$	1	
$\therefore 2x^2 - 2y^2 = 0$		
$y^2 = x^2$		
	1	



Solutions	Marks	Comments
Question 2: a) $\frac{x^2}{25}+\frac{y^2}{9}=1$		
i) $a=5$ $b=3$ $b^2=a^2(1-e^2)$ $9=25(1-e^2)$ $9=25-25e^2$ $\therefore 25e^2=16$ $e^2=\frac{16}{25}$ $e=\frac{4}{5}$	1	
(ii) foci $(\pm ae,0)$ directrices $y=\pm\frac{a}{e}$ $(\pm 4,0)$ $y=\pm\frac{25}{4}$	2	
(iii) (iv) 	1 1	
b) $x=3\cos\theta$ $y=4\sin\theta$ $\Rightarrow$ $\frac{dx}{d\theta}=-3\sin\theta$ $\frac{dy}{d\theta}=4\cos\theta$ $\therefore \frac{dy}{dx}=\frac{dy}{d\theta}\times\frac{d\theta}{dx}=-\frac{4\cos\theta}{3\sin\theta}$	1	
at $\theta=\frac{\pi}{6}$ $\frac{dy}{dx}=-\frac{4\sqrt3}{3}$ $x=\frac{3\sqrt3}{2}$ $y=2$ gradient of normal $=\frac{3}{4\sqrt3}=\frac{\sqrt3}{4}$	1	
$\therefore y-2=\frac{\sqrt3}{4}(x-\frac{3\sqrt3}{2})$ $4y-8=\sqrt3x-\frac{9}{2}$ $\Rightarrow$ $8y-16=2\sqrt3x-9$ $8\sqrt3y-16\sqrt3=6x-9\sqrt3$	1	

$\therefore 6x-8\sqrt3y+7\sqrt3=0$

Solutions	Marks	Comments
Question 2 c) $ax^3+cx+d=0$ double root at (say) $x=k$		
If $P(x)=ax^3+cx+d$ $P'(x)=3ax^2+c=0$ for $x=k$ $\therefore 3ak^2+c=0$ $\therefore k^2=\frac{-c}{3a}$ & $k=\sqrt{\frac{-c}{3a}}$	1	
also $P(k)=0$ $\therefore ak^3+ck+d=0$ $\therefore a\sqrt{\frac{-c}{3a}}\cdot\frac{-c}{3a}+c\sqrt{\frac{-c}{3a}}+d=0$ $\therefore \sqrt{\frac{-c}{3a}}\left[\frac{-c}{3}+c\right]=-d.$ Square both sides $\frac{-c}{3a}\cdot\frac{4c^2}{9}=d^2$ $\frac{-4c^3}{27a}=d^2$ $\therefore 27ad^2+4c^3=0$	2	
(d) $z=r(\cos\theta+i\sin\theta)$		
(i) $\arg\left[\frac{z^2(3+3i)}{(1-\sqrt3i)z}\right]$ $=\left(2\theta+\frac{\pi}{4}\right)-\left(-\frac{\pi}{3}+\theta\right)$ $=\theta+\frac{\pi}{4}+\frac{\pi}{3}$ $=\theta+\frac{7\pi}{12}$	2	
(ii) $\left\|\frac{z^2(3+3i)}{(1-\sqrt3i)z}\right\|=\frac{\|z^2\|\times\|3+3i\|}{\|1-\sqrt3i\|\times\|z\|}$ $=\frac{r^2\times3\sqrt2}{2r}$ $=\frac{3\sqrt2 r}{2}$	2	

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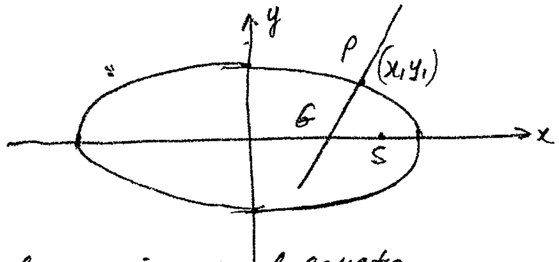
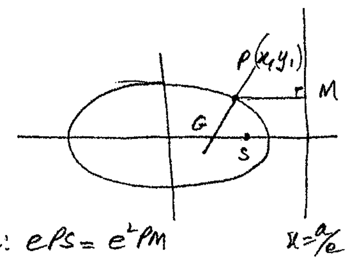
Course:

Marking Scheme for Task:

Academic Year: 2007-8

Solutions	Marks	Comments
Question 3: a) $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ $\quad (a^2>b^2)$		
normal $a^2y_1x-b^2x_1y=(a^2-b^2)x_1y_1$		
		
(i) let $y=0$ in normal equation		
$\therefore x=\frac{(a^2-b^2)x_1\cancel{y_1}}{a^2\cancel{y_1}}$		
$\therefore x=\frac{(a^2-b^2)x_1}{a^2}$ $\quad \therefore G\left(\frac{(a^2-b^2)x_1}{a^2},0\right)$		
but $b^2=a^2(1-e^2)$		
$\therefore G(e^2x_1,0)$	1	
(ii) 		
$PS=ePM$ $\quad \therefore ePS=e^2PM$		
$GS=ae-e^2x_1$	1	
$PM=\frac{a}{e}-x_1$	1	
$\therefore e^2PM=e^2\left(\frac{a}{e}-x_1\right)$		
$=ae-e^2x_1$		
$\therefore GS=ePS$	1	

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Course:

Marking Scheme for Task:

Academic Year: 2007-8

Solutions	Marks	Comments
Question 3 b). $z=\cos\theta+i\sin\theta$ $\quad z^n=\cos n\theta+i\sin n\theta$		
$\frac{1}{z}=\cos\theta-i\sin\theta$ $\quad \frac{1}{z^n}=\cos n\theta-i\sin n\theta$		
$\therefore z+\frac{1}{z}=2\cos\theta$ $\quad z^n+\frac{1}{z^n}=2\cos n\theta$		
now $\left(z+\frac{1}{z}\right)^4=16\cos^4\theta$		
and $\left(z+\frac{1}{z}\right)^4=z^4+4z^2+6+\frac{4}{z^2}+\frac{1}{z^4}$		
$=\left(z^4+\frac{1}{z^4}\right)+4\left(z^2+\frac{1}{z^2}\right)+6$	1	
$\therefore 16\cos^4\theta=2\cos4\theta+4(2\cos2\theta)+6$		
$\therefore \cos^4\theta=\frac{1}{8}\cos4\theta+\frac{1}{2}\cos2\theta+\frac{3}{8}$	1	
(c) $P(x)=x^4-6x^3+12x^2-10x+3=0$		
$P'(x)=4x^3-18x^2+24x-10$		
$P''(x)=12x^2-36x+24=0$ (triple root)	1	
$x^2-3x+2=0$		
$(x-2)(x-1)=0$		
$\therefore x=1,2$		
now $P'(1)=0$ $\quad P(1)=0$ $\quad \therefore x=1$ is triple root	1	
$\therefore P(x)=(x-1)^3(x-\alpha)$		
product of roots $=3$ $\quad \therefore \alpha=3$	1	
$\therefore x=1,1,1,3$		

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Course:

Marking Scheme for Task:

Academic Year: 2007-8

Solutions	Marks	Comments
Question 3 d) i) $\frac{x^2}{16}+\frac{y^2}{9}=1$		
$\frac{2x}{16}+\frac{2y}{9}\cdot\frac{dy}{dx}=0 \Rightarrow \frac{dy}{dx}=\frac{-9x}{16y}$		
at (x_1,y_1) $\frac{dy}{dx}=-\frac{9x_1}{16y_1}$	1	
equation of tangent: $y-y_1=\frac{-9x_1}{16y_1}(x-x_1)$		
$16yy_1-16y_1^2=-9xx_1+9x_1^2$		
$9xx_1+16yy_1=9x_1^2+16y_1^2$		
$\div 9.16.$ $\frac{xx_1}{16}+\frac{yy_1}{9}=\frac{x_1^2}{16}+\frac{y_1^2}{9}$		
but $\frac{x_1^2}{16}+\frac{y_1^2}{9}=1$		
$\therefore$ tangent is: $\frac{xx_1}{16}+\frac{yy_1}{9}=1$	1	
(ii) tangent at $P(x_1,y_1)$: $\frac{xx_1}{16}+\frac{yy_1}{9}=1$		
tangent at $Q(x_2,y_2)$: $\frac{xx_2}{16}+\frac{yy_1}{9}=1$		
If $(x_0\ y_0)$ is point of intersection of tangents then x_0y_0 satisfies both equations		
ie. $\frac{x_0x_1}{16}+\frac{y_0y_1}{9}=1$ and $\frac{x_0x_2}{16}+\frac{y_0y_2}{9}=1$	1	
$\therefore$ $(x_1\ y_1)$ & $(x_2\ y_2)$ satisfies the equation $\frac{xx_0}{16}+\frac{yy_0}{9}=1$	1	
$\therefore$ equation of chord of contact is		
$\frac{xx_0}{16}+\frac{yy_0}{9}=1$		
$\therefore$ required equation is $\frac{5x}{16}+\frac{y}{9}=1$	1	
ie. $45x+16y-144=0$		

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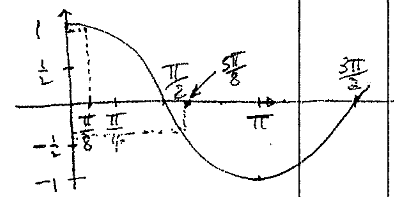
Course:

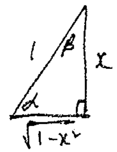
Marking Scheme for Task:

Academic Year: 2007-8

Solutions	Marks	Comments
Question 4 a) $P(x)=(x-2)Q(x)+4$		
$P(x)=(x-3)Q(x)+9$		
Now $P(x)=(x-2)(x-3)Qx+(ax+b)$	1	
Note: remainder when divisor is a quadratic is of the form $ax+b$		
$P(2)=4$ $\therefore$ $4=2a+b$ —①	1	
$P(3)=9$ $\quad 9=3a+b$ —②		
②−① $a=5$	1	
$\therefore$ $b=-6$		
b) $x^3-5x^2+5=0$		
(i) $y=x-1 \quad x=y+1$		
$(y+1)^3-5(y+1)^2+5=0$	1	
$y^3+3y^2+3y+1-5y^2-10y-5+5=0$		
$y^3-2y^2-7y+1=0$		
$\therefore$ polynomial with roots $\alpha-1, \beta-1, \gamma-1$		
$x^3-2x^2-7x+1=0$	1	
(ii) $y=x^2$ $\therefore$ $x=\sqrt{y}$		
$(\sqrt{y})^3-5(\sqrt{y})^2+5=0$		
$y\sqrt{y}-5y+5=0$	1	
$y\sqrt{y}=5y-5$		
(Squaring) $y^3=25y^2-50y+25$		
$\therefore$ polynomial with roots $\alpha^2\beta^2\gamma^2$ is		
$x^3-25x^2+50x-25=0$	1	

Solutions	Marks	Comments
Question 4 b)(iii). If α, β, γ are roots then $\alpha^3 - 5\alpha^2 + 5 = 0$ $\beta^3 - 5\beta^2 + 5 = 0$ $\gamma^3 - 5\gamma^2 + 5 = 0$ adding $\therefore (\alpha^3+\beta^3+\gamma^3) - 5(\alpha^2+\beta^2+\gamma^2) + 15 = 0$ $\therefore \alpha^3+\beta^3+\gamma^3 = 5(\alpha^2+\beta^2+\gamma^2) - 15$ $= 5(25) - 15$ (from part ii) $= 110$	2	
c). (i) $\cos 4\theta = \cos 2(2\theta) = 2\cos^2 2\theta - 1$ $= 2(2\cos^2\theta - 1)^2 - 1$ $= 2(4\cos^4\theta - 4\cos^2\theta + 1) - 1$ $= 8\cos^4\theta - 8\cos^2\theta + 1$	1	
(ii) In $8x^4 - 8x^2 + 1 = 0$ let $x = \cos\theta$ $\therefore 8\cos^4\theta - 8\cos^2\theta + 1 = 0$ $\therefore \cos 4\theta = 0$ (from part (i)) $4\theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}$ $\theta = \frac{\pi}{8}, \frac{3\pi}{8}, \frac{5\pi}{8}, \frac{7\pi}{8}$ $\therefore$ roots are $\cos\frac{\pi}{8}, \cos\frac{3\pi}{8}, \cos\frac{5\pi}{8}, \cos\frac{7\pi}{8}$ note. $\cos\frac{7\pi}{8} = -\cos\frac{\pi}{8}$ $\cos\frac{5\pi}{8} = -\cos\frac{3\pi}{8}$ $\therefore$ roots are $\pm\cos\frac{\pi}{8}, \pm\cos\frac{3\pi}{8}$	2	

Solutions	Marks	Comments
(iii) now $8x^4 - 8x^2 + 1 = 0$ $8(x^2)^2 - 8x^2 + 1 = 0$ $\therefore x^2 = \frac{8 \pm \sqrt{32}}{16}$ $x^2 = \frac{2 \pm \sqrt{2}}{4}$ $\therefore x = \pm\frac{\sqrt{2 \pm \sqrt{2}}}{2}$  Use calculator or graph to determine $-\frac{1}{2} < \cos\frac{5\pi}{8} < 0 \quad \therefore \cos\frac{5\pi}{8} = -\frac{\sqrt{2-\sqrt{2}}}{2}$ $\cos\frac{\pi}{8} > \quad \therefore \cos\frac{\pi}{8} = \frac{\sqrt{2+\sqrt{2}}}{2}$	2	
Question 5 a) $p(x) = x^3 + 3px + q$ double root at $x = k$ (i) $p'(x) = 3x^2 + 3p = 0$ when $x = k$ $\therefore 3k^2 + 3p = 0$ $\therefore p = -k^2$	2	
(ii) $p(k) = 0. \therefore k^3 + 3pk + q = 0$ but $p = -k^2$ $\therefore k^3 - 3k^3 + q = 0$ $\therefore q = 2k^3$	2	
(iii) $4p^3 + q^2 = 4(-k^2)^3 + (2k^3)^2$ $= -4k^6 + 4k^6$ $= 0$	1	

Solutions	Marks	Comments
Questions b)		
(i) $\sin(\sin^{-1}x - \cos^{-1}x)$ $= \sin(\alpha - \beta)$ $= \sin\alpha\cos\beta - \cos\alpha\sin\beta$ $= \frac{x}{1}\cdot\frac{x}{1} - \sqrt{1-x^2}\cdot\sqrt{1-x^2}$ $= x^2 - (1-x^2)$ $= 2x^2 - 1$  let $\sin^{-1}x = \alpha$ $\cos^{-1}x = \beta$	3	
(ii) $\sin^{-1}x - \cos^{-1}x = \sin^{-1}(1-x)$ take sin of both sides $\sin(\sin^{-1}x - \cos^{-1}x) = \sin(\sin^{-1}(1-x))$ from (i) $2x^2 - 1 = 1-x$ $2x^2 + x - 2 = 0$ $x = \frac{-1 \pm \sqrt{1+16}}{4}$ $x = \frac{-1 \pm \sqrt{17}}{4}$	2	
(c) (i) $(1+i)z_1 = z_1 + iz_1$ iz_1 anticlockwise rotation of z_1 thru $\frac{\pi}{2}$ $\therefore P = z_1 + iz_1 = (1+i)z_1$	1	
Similarly $(1-i)z_2 = z_2 + (-iz_2)$ $\therefore Q = z_1 - iz_2$	1	

Solutions	Marks	Comments
Questions c) ii) $\overrightarrow{OP} = (1+i)z_1$ $\overrightarrow{OQ} = (1-i)z_2$ $\overrightarrow{OP} + \overrightarrow{OQ} = OR$ (say) now M is the mid point of PQ and OR (OPRQ a parallelogram).	1	
$\therefore$ M is $\overrightarrow{OM} = \frac{\overrightarrow{OP} + \overrightarrow{OQ}}{2}$ $= \frac{z_1(1+i) + z_2(1-i)}{2}$	1	