Student Number: ______

Waverley

Mathematics Extension 2

2010
ASSESSMENT TASK 3
(15%)

Total marks – 38

- Attempt Questions 1–4
- Marks for each question are indicated on this page

General Instructions

- Working time – 60 minutes
- Start each question on a **new page** in your answer booklet.
- If any additional booklet is used, please label it clearly and attach it to the appropriate booklet.
- Write using black or blue pen only.
- Board-approved calculators may be used.
- All necessary **working** must be shown.
- Marks may be deducted for careless or badly arranged work.
- Standard Integrals appear on page 5

TEACHER'S USE ONLY

Question 1	/13
Question 2	/9
Question 3	/11
Question 4	/5
Total	/38

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1},\quad n\neq 1;\ x\neq 0,\ \text{if } n<0$$

$$\int \frac{1}{x}\,dx = \ln x,\quad x>0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax},\quad a\neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax,\quad a\neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax,\quad a\neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax,\quad a\neq 0$$

$$\int \sec ax\tan ax\,dx = \frac{1}{a}\sec ax,\quad a\neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a},\quad a\neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a},\quad a>0,\ -a<x<a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right),\quad x>a>0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE: $\ln x = \log_e x, \quad x>0$

Question 1 (13 marks) **Marks**

(a) Find $\int \frac{x^2-1}{x^2+x-6}\,dx$ 3

(b) Find $\int e^{2x}\sin x\ dx$ 3

(c) (i) Use the substitution $x=\frac{4}{5}\sin\theta$ to prove 3

$$\int_0^{\frac{4}{5}}\sqrt{16-25x^2}\,dx=\frac{4\pi}{5}$$

(ii) Hence or otherwise find the area enclosed by the ellipse $25x^2+y^2=16$ 2

(d) Find $\int \sin^2 x\cos^3 x\ dx$ 2

Question 2 (9 marks)

(e) (i) Find constants A, B, C such that 2

$$\frac{18x^2}{9x^2-16}\equiv \overset{2}{A}+\frac{\overset{4}{B}}{3x-4}+\frac{\overset{-4}{C}}{3x+4}$$

(ii) Hence evaluate $\int_2^3 \frac{18x^2}{9x^2-16}dx$ 2

(f) (i) If $I_n=\int_0^{\frac{\pi}{2}}\cos^n x\ dx$ show that for $n>1$, 3

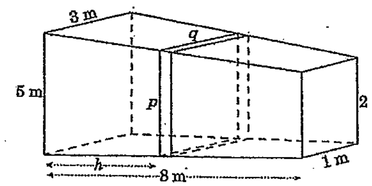
$$I_n=\frac{n-1}{n}I_{n-2}$$

(ii) Hence find the area of the finite region bounded by the curve 2

$y=\cos^5 x$ and the x-axis for $0\le x\le\frac{\pi}{2}$

Question 3 (11 marks) **Marks**

(a)



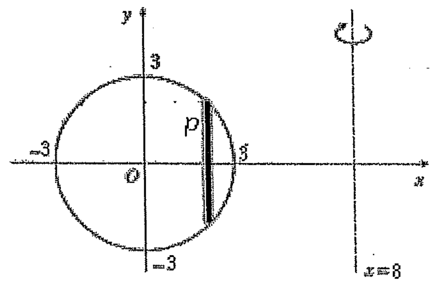
A wooden beam of length 8 metres has plane sides with cross sections parallel to the ends which are rectangular as shown above.

(i) Express p and q in terms of h 2

(ii) Calculate the area of the cross section (typical slice) 2

(iii) Hence calculate the volume of the beam 2

(b) 5



The circle $x^2+y^2=9$ is rotated about the line $x=8$ to form a Torus (doughnut)

When the circle is rotated, the strip p forms a cylindrical shell.

(i) Show that the volume of the typical slice is given by

$$\delta V=4\pi(8-x)\sqrt{9-x^2}\,\delta x$$

(ii) Hence find the volume of Torus.

Question 4 (5 marks) **Marks**

(a) A solid is built on an elliptical base of equation $\frac{x^2}{9}+\frac{y^2}{4}=1$.

Each cross section (perpendicular to the major axis of the ellipse) is an equilateral triangle.

(i) Show that the area of the cross-section is given by $A=y^2\sqrt{3}$. **3**

(ii) Hence or otherwise, find the volume of the solid. **2**

Solutions	Marks/Comments
Question 1	
a) $\int \frac{x^2-1}{x^2+x-6}\,dx = a + \frac{b}{x-2} + \frac{c}{x+3}$	1
$\therefore x^2-1 = A(x-2)(x+3) + b(x+3) + c(x-2)$	
$x=-3 \Rightarrow C = \frac{-8}{5}$	
$x=2 \Rightarrow B = \frac{3}{5}$	1
Also $A = 1$	
$\therefore \int \frac{x^2-1}{x^2+x-6}\,dx = \int 1\,dx + \frac{3}{5}\int \frac{1}{x-2}\,dx - \frac{8}{5}\int \frac{1}{x+3}\,dx$	
$= x + \frac{3}{5}\ln(x-2) - \frac{8}{5}\ln(x+3) + c$	1
b) $\int e^{2x}\sin x\,dx$ $u=e^{2x}$ $v'=\sin x$ $u'=2e^{2x}$ $v=-\cos x$	1
$\therefore \int e^{2x}\sin x\,dx = -e^{2x}\cos x + 2\int e^{2x}\cos x\,dx$	
Now $\int e^{2x}\cos x\,dx$ $u=e^{2x}$ $v'=\cos x$ $u'=2e^{2x}$ $v=\sin x$	1
$\therefore \int e^{2x}\sin x\,dx = -e^{2x}\cos x + 2\left[e^{2x}\sin x - 2\int e^{2x}\sin x\,dx\right]$	
$= -e^{2x}\cos x - 4\int e^{2x}\sin x\,dx + 2e^{2x}\sin x$	
$\therefore 5\int e^{2x}\sin x\,dx = 2e^{2x}\sin x - e^{2x}\cos x$	
$\therefore \int e^{2x}\sin x\,dx = \frac{2}{5}e^{2x}\sin x - \frac{1}{5}e^{2x}\cos x + c$	1

Solutions	Marks/Comments
Question 1	
c) (i) $\int_0^{4/5}\sqrt{16-25x^2}\,dx$ $x=\frac{4}{5}\sin\theta$, $dx=\frac{4}{5}\cos\theta\,d\theta$	1
$x=0$, $\theta=0$; $x=\frac{4}{5}$, $\theta=\frac{\pi}{2}$	
$=\int_0^{\pi/2}\sqrt{16-16\sin^2\theta}\cdot\frac{4}{5}\cos\theta\,d\theta$	
$=\frac{16}{5}\int_0^{\pi/2}\cos^2\theta\,d\theta$	
$=\frac{8}{5}\int_0^{\pi/2}(1+\cos 2\theta)\,d\theta$	1
$=\left[\frac{8}{5}\theta-\frac{4}{5}\sin 2\theta\right]_0^{\pi/2}$	
$=\left(\frac{4\pi}{5}-\frac{4}{5}\sin\pi\right)-\left(0-\frac{4}{5}\sin 0\right)$	1
$=\frac{4\pi}{5}$	
(ii) $25x^2+y^2=16 \Rightarrow \frac{25x^2}{16}+\frac{y^2}{16}=1$	
$a^2=\frac{16}{25}$ $b^2=16$	
$y=\sqrt{16-25x^2}$	
$\therefore$ Area $=4\int_0^{4/5}\sqrt{16-25x^2}\,dx$	1
$=4\times\frac{4\pi}{5}$	
$=\frac{16\pi}{5}$ units^2	1
otherwise $a=\frac{4}{5}$ $b=4$	OR
area $=\pi ab$	1
$=\frac{16\pi}{5}$ units^2	1

Year 12 Extension 2 – Assessment Task 3 - Solutions

Solutions	Marks/Comments
Question 1:	
d) $\int\sin^2x\cos^3x\,dx$	
$=\int\sin^2x\,(1-\sin^2x).\cos x\,dx$	
$=\int(\sin^2x-\sin^4x)\cos x\,dx$	1
let $u=\sin x$, $du=\cos x\,dx$	
$=\int(u^2-u^4)\,du$	
$=\frac{u^3}{3}-\frac{u^5}{5}+C$	
$=\frac{1}{3}\sin^3x-\frac{1}{5}\sin^5x+C$	1
Question 2:	
a) (i) $\frac{18x^2}{9x^2-16}=A+\frac{B}{3x-4}+\frac{C}{3x+4}$	
$18x^2=A(3x-4)(3x+4)+B(3x+4)+C(3x-4)$	1
$18x^2=9Ax^2 \Rightarrow A=2$	
let $x=\frac{4}{3}$ $32=8B \Rightarrow B=4$	1
let $x=-\frac{4}{3}$ $32=-8C \Rightarrow C=-4$	
(ii) $\int_2^3\frac{18x^2}{9x^2-16}\,dx=\int_2^3\left(2+\frac{4}{3x-4}-\frac{4}{3x+4}\right)dx$	
$=\left[2x+\frac{4}{3}\ln\lvert 3x-4\rvert-\frac{4}{3}\ln\lvert 3x+4\rvert\right]_2^3$	1
$=\left(6+\frac{4}{3}\ln 5-\frac{4}{3}\ln 13\right)-\left(4+\frac{4}{3}\ln 2-\frac{4}{3}\ln 10\right)$	
$=2+\frac{4}{3}\ln\frac{5}{2}+\frac{4}{3}\ln\frac{10}{13}$	1
$=2+\frac{4}{3}\ln\frac{25}{13}$	

Solutions	Marks/Comments
Question 2: b)	
(i) $I_n = \int_0^{\pi/2} \cos^n x \, dx$	
$= \int_0^{\pi/2} \cos^{n-1} x . \cos x \, dx$	
let $u = \cos^{n-1} x$ $\quad v' = \cos x$	1
$u' = -(n-1)\cos^{n-2} x . \sin x$ $\quad v = \sin x$	
$\therefore I_n = \left[\sin x \cos^{n-1} x\right]_0^{\pi/2} - \int_0^{\pi/2} \sin x (n-1)\cos^{n-2} x(-\sin x)\,dx$	There must be an effort to establish $\left[\sin x \cos^{n-1} x\right]_0^{\pi/2} = 0$
$= 0 + (n-1)\int_0^{\pi/2} \cos^{n-2} x \sin^2 x \, dx$	
$= (n-1)\int_0^{\pi/2} \cos^{n-2} x\,(1-\cos^2 x)\,dx$	
$= (n-1)\left[\int_0^{\pi/2} \cos^{n-2} x \, dx - \int_0^{\pi/2} \cos^n x \, dx\right]$	1
$\therefore I_n = (n-1)\left[I_{n-2} - I_n\right]$	
$I_n = (n-1)I_{n-2} - (n-1)I_n$	
$I_n(1+n-1) = (n-1)I_{n-2}$	
$nI_n = (n-1)I_{n-2}$	
$\therefore I_n = \frac{n-1}{n} I_{n-2}$	1

Year 12 Extension 2 – Assessment Task 3 - Solutions

Solutions	Marks/Comments
Question 2 b: (ii)	
$I_5 = \frac{4}{5} I_3$	
$= \frac{4}{5}\left(\frac{2}{3} C_1\right)$	1
$= \frac{8}{15}\int_0^{\pi/2} \cos x \, dx$	
$= \frac{8}{15}\left[\sin x\right]_0^{\pi/2}$	
$= \frac{8}{15}$ units^2	1
Question 3: a)	
(i) $p = nh + b$ $\quad q = mh + c$	
$h=0$ $p=5$ $\therefore b=5$ $\quad h=0$ $q=3$ $\therefore c=3$	
$h=8$ $p=2$ $\therefore n=\frac{-3}{8}$ $\quad h=8$ $q=1$ $\therefore m=-\frac{1}{4}$	2
$\therefore p = -\frac{3}{8}h + 5$ $\quad q = -\frac{1}{4}h + 3$	
(ii) $A(h) = pq = \left(5 - \frac{3}{8}h\right)\left(3 - \frac{h}{4}\right)$	2
$= 15 - \frac{19h}{8} + \frac{3h^2}{32}$	
(iii) $\delta V \doteqdot A(h).\delta h$	
$\therefore V = \int_0^8 \left(15 - \frac{19h}{8} + \frac{3h^2}{32}\right) dh$	
$= \left[15h - \frac{19h^2}{16} + \frac{h^3}{32}\right]_0^8$	
$= 120 - 76 + 16$	
$= 60$ units^3	2

Solutions | Marks/Comments

Question 3 b:

(i) circumference of shell $= 2\pi(8-x)$

height of shell $= 2y$ — 1

now $y = \sqrt{9-x^2}$

$\therefore \ \delta V = 2\pi(8-x)\cdot 2y\,\delta x$

$= 4\pi(8-x)\cdot\sqrt{9-x^2}\,dx$ — 1

(ii) now $V = \lim\limits_{\delta x \to 0} \sum\limits_{-3}^{3} \left(32\pi\sqrt{9-x^2} - 4\pi x\sqrt{9-x^2}\right)\delta x$

$$= \int_{-3}^{3} 32\pi\sqrt{9-x^2}\,dx - \int_{-3}^{3} 4\pi x\sqrt{9-x^2}\,dx$$ — 1

Note: $\int_{-3}^{3}\sqrt{9-x^2}$ is a semicircle of radius 3

also $4\pi x\sqrt{9-x^2}$ is an odd function — 1

So $\int_{-3}^{3} 4\pi x\sqrt{9-x^2}\,dx = 0$

$\therefore \ V = \int_{-3}^{3} 32\pi\sqrt{9-x^2}\,dx$ $\qquad A = \frac{\pi r^2}{2} = \frac{9\pi}{2}$

$= 32\pi \times \frac{9\pi}{2}$

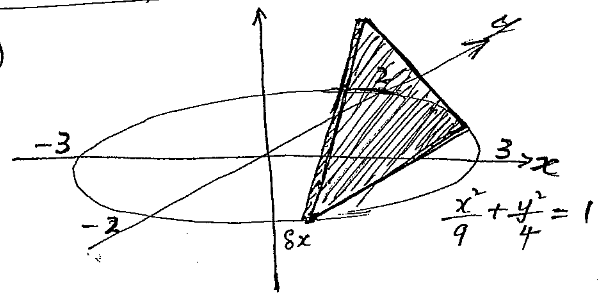
$= 144\pi^2$ units^3 — 1

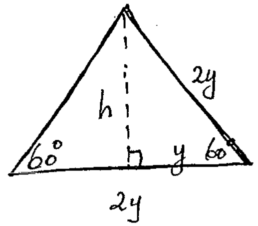
Year 12 Extension 2 – Assessment Task 3 - Solutions

Solutions | Marks/Comments

Question 4 a)

(i)





$h^2 = 4y^2 - y^2$

$h^2 = 3y^2$

$\therefore \ h = \sqrt{3}\,y$ — 1

$\therefore$ Area of cross section $= y \times \sqrt{3}\,y$

$\therefore \ A = y^2\sqrt{3}$ — 1

(ii) $\delta V \doteqdot A\,\delta x = y^2\sqrt{3}\,\delta x$ — 1

$V = \lim\limits_{\delta x \to 0} \sum\limits_{x=0}^{3} 2y^2\sqrt{3}\,\delta x$

$= 2\sqrt{3}\int_0^3 y^2\,dx$

$= 2\sqrt{3}\int_0^3 4\left(1 - \frac{x^2}{9}\right)dx$ — 1

$= 8\sqrt{3}\int_0^3 \left(1 - \frac{x^2}{9}\right)dx$

$= 8\sqrt{3}\left[x - \frac{x^3}{27}\right]_0^3$

$= 8\sqrt{3}\,[2 - 0]$

$= 16\sqrt{3}$ units^3 — 1