Student Number:__________

St. Catherine's School Waverley

3rd June 2010

HSC ASSESSMENT TASK 3
15%

Extension I Mathematics

Time allowed: *55 minutes*
Total marks: *37*

INSTRUCTIONS

- Marks for each part of a question are indicated
- All questions should be attempted on the separate paper provided
- Each question should start on a new page.
- Solutions to all questions should be in one booklet.
- All necessary working should be shown
- Start each question on a new page
- Approved scientific calculators and drawing templates may be used badly arranged work.

Question 1 **Marks**

A spherical balloon is being blown up so that its surface area is increasing at the constant rate of $6\,cm^2/s$.

(i) Find the rate of increase of the radius when the radius is $5cm$. **3**

(ii) Hence, find the rate at which the volume is then increasing. **3**

Question 2 ***(Start a new page)***

In a group of 1000 computers linked to each other via the internet, the number N infected with a virus at time t years is given by $N = \dfrac{1000}{1+ce^{-1000t}}$, where c is a constant.

(i) Show that, eventually, all the computers will be infected with the virus. **1**

(ii) Suppose that when $t=0$ only one computer was infected with the virus. After how many days will 50% of the computers be infected? **3**

(iii) Show that $\dfrac{dN}{dt} = N(1000-N)$. **2**

Question 3 ***(Start a new page)***

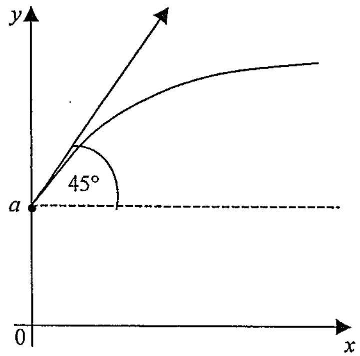
The acceleration of a particle moving in a straight line is given by $\ddot{x} = 3x-2$, where x is the displacement, in metres, from the origin O and t is the time in seconds. The particle is initially at rest at $x=2$.

(i) Show that $v^2 = 3x^2-4x-4$, where v is the velocity of the particle is in m/s. **3**

(ii) Determine if the particle passes through the origin. Justify your answer. **2**

(iii) Determine the displacement when $v = 2\sqrt{7}$. **2**

Question 4 *(Start a new page)* **Marks**



A particle is projected from the point $(0,a)$ with a velocity of $V\,m/s$. and at an angle of 45° to the horizontal.

The equations of motion for the particle are given by $\ddot{x}=0$ and $\ddot{y}=-g$.

(i) By means of calculus derive expressions for the horizontal and vertical positions of the particle at time t. **4**

(ii) Using the expressions from part (i), show that the path of the particle is given by the Cartesian equation $y=a+x-\dfrac{gx^2}{V^2}$. **2**

A tennis player returns a ball served to him by hitting it back at an angle of projection of 45° and an initial speed $V\ m/s$.

At that instant the ball is 1 metre above the ground and its horizontal distance from the net is 9.3 metres. The ball clears the net at a height of 2.3 metres.

(iii) Show that the ball is initially hit with a speed of approximately 10.3 m/s^2. (Assume $g=9.8m/s^2$.) **1**

(iv) How far from the net does the ball land? **2**

Question 5 *(Start a new page)*

A particle moves in straight line and its position at time t is given by

$$x=2+\sin 2t+\cos 2t$$

(i) Show that the particle is undergoing simple harmonic motion about $x=2$. **3**

(ii) Find the amplitude of the motion. **3**

(iii) Find when the particle first reaches maximum speed after $t=0$. **3**

End of Task

YR12 Ext 1 – (Tsk 3)
JUNE 2010

Q1 (i) $A = 4\pi r^2$ $\frac{dA}{dt} = 6$ $\frac{dA}{dr} = 8\pi r$ (1)

$\frac{dr}{dt} = \frac{dA}{dt} \times \frac{dr}{dA}$ (1) $\therefore \frac{dr}{dA} = \frac{1}{8\pi r}$

$= 6 \times \frac{1}{8\pi r}$

$= \frac{3}{4\pi r}$

when $r=5$ $\frac{dr}{dt} = \frac{3}{20\pi}$ (1)

(ii) $V = \frac{4}{3}\pi r^3$ $\frac{dV}{dr} = 4\pi r^2$ (1)

$\frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dt}$ (1)

$= 4\pi r^2 \times \frac{3}{20\pi}$

$= 15 \ m^3/s$ (1)

Q2 $N = \frac{1000}{1+ce^{-1000t}}$

(i) as $t \to \infty$ $e^{-1000t} \to 0$ $\therefore N \to 1000$

$\therefore$ eventually all computers will be infected (1)

(ii) when $t=0$ $N=1$ find t when $N=50$

$1 = \frac{1000}{1+c} \Rightarrow 1+c = 1000 \Rightarrow c = 999$ (1)

$\therefore N = \frac{1000}{1+999e^{-1000t}}$

when $N = 500$

$500 = \frac{100}{1+999e^{-1000t}}$ (1)

$500(1+999e^{-1000t}) = 1000$

$1+999e^{-1000t} = 2$

$999e^{-1000t} = 1$

$e^{-1000t} = \frac{1}{999}$

$-1000t = \ln\frac{1}{999}$ (1)

$= 0.006907$ years

$= 2.52$ days (2 d p)*

*(No deduction for not in days

(iii) $N = \frac{1000}{1+ce^{-1000t}} = 1000(1+ce^{-1000t})^{-1}$

$= -1000(1+ce^{-1000t})^{-2} \, ce^{-1000t} \cdot -1000$ (1)

$= \frac{1000^2}{(1+ce^{-1000t})^2} \cdot ce^{-1000t}$

$= N^2 \cdot ce^{-1000t}$

$= N(Nce^{-1000t})$

$= N(1000-N)$* (1)

*Note: $N = \frac{1000}{1+ce^{-1000t}}$

$N + Nce^{-1000t} = 1000$

$\therefore Nce^{-1000t} = 1000 - N$

Q3 $\ddot{x} = 3x - 2$ when $t=0$ $v=0$ $x=2$ (1)

(i) $\frac{d}{dx}\left(\frac{1}{2}v^2\right) = 3x - 2$ (1)

$\frac{1}{2}v^2 = \frac{3x^2}{2} - 2x + C$

$x=2$ $v=0$ $0 = 6 - 4 + c$

$-2 = C$

$\therefore \frac{1}{2}v^2 = \frac{3x^2}{2} - 2x - 2$

$v^2 = 3x^2 - 4x - 4$ —— (1)

(ii) If particle passes through origin $x = 0$ (1)

$\therefore v^2 = -4$

this is not possible $\therefore$ particle does not pass through the origin (1)

(iii) Sub $2\sqrt{7}$ into (1) $v = 2\sqrt{7}$

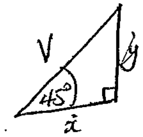
$28 = 3x^2 - 4x - 4$

$3x^2 - 4x - 32 = 0$

$(3x+8)(x-4) = 0$

$\therefore x = -\frac{8}{3}, 4$ (1)

Since particle starts at $x=2$ and does (1) not pass through the origin the displacement when $v = 2\sqrt{7}$ is $\underline{x=4}$

Qn	Solutions	Marks	Comments: Criteria
4	(i) $\frac{\dot{x}}{V} = \cos 45°$ $\dot{x} = V\cos 45°$ $\dot{x} = V \cdot \frac{1}{\sqrt{2}}$ $= \frac{\sqrt{2}V}{2}$  $\frac{\dot{y}}{V} = \sin 45°$ $\dot{y} = V\sin 45°$ $= V \cdot \frac{1}{\sqrt{2}}$ $= \frac{\sqrt{2}V}{2}$ Horizontal displacement $\dot{x} = \frac{\sqrt{2}V}{2}$ $\therefore x = \int \frac{\sqrt{2}V}{2}\, dt$ $= \frac{\sqrt{2}Vt}{2} + C$ At $x=0$, $t=0$ $\therefore 0 = 0 + C$ $C = 0$ $\therefore x = \frac{\sqrt{2}Vt}{2}$ Vertical Disp. $\ddot{y} = -g$ $\dot{y} = \int g\, dt$ $= -gt + D$ At $t=0$, $\dot{y} = \frac{\sqrt{2}V}{2}$ $\frac{\sqrt{2}V}{2} = -g(0) + D$ $D = \frac{\sqrt{2}V}{2}$ $\therefore \dot{y} = -gt + \frac{\sqrt{2}V}{2}$ $y = \int -gt + \frac{\sqrt{2}V}{2}\, dt$ $= -g\frac{t^2}{2} + \frac{\sqrt{2}Vt}{2} + E$ At $t=0$, $y = a$ $a = -g\frac{(0)^2}{2} + \frac{\sqrt{2}V(0)}{2} + E$ $E = a$ $\therefore y = -\frac{1}{2}gt^2 + \frac{\sqrt{2}Vt}{2} + a$	4	'✗' means 0.5 mark '✓' mean 1 mark

Qn	Solutions	Marks	Comments: Criteria

4(ii) From part (i)

$x = \frac{\sqrt{2}Vt}{2}$ and $y = a + \frac{\sqrt{2}Vt}{2} - \frac{1}{2}gt^2$ — 2

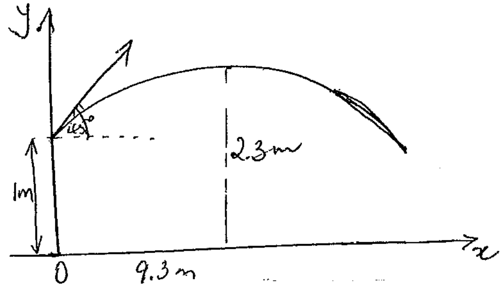
$\therefore t = \frac{2x}{\sqrt{2}V}$ ✓ and then

$y = a + \frac{\sqrt{2}V}{2}\left(\frac{2x}{\sqrt{2}V}\right) - \frac{1}{2}g\left(\frac{2x}{\sqrt{2}V}\right)^2$ ✓

$= a + x - \frac{1}{2}g\frac{4x^2}{2V^2}$

i.e. $y = a + x - \frac{gx^2}{V^2}$ ✓

(iii)



$y = a + x - \frac{gx^2}{V^2}$

$2.3 = 1 + 9.3 - \frac{9.8(9.3)^2}{V^2}$ ✓

$\therefore \frac{9.8(9.3)^2}{V^2} = 10.3 - 2.3 = 8$ — 1

$8V^2 = 9.8(9.3)^2$

$V^2 = \frac{9.8(9.3)^2}{8}$ ✓

$V = \pm\sqrt{\frac{9.8(9.3)^2}{8}} = \pm 10.3 \text{ m/s}$

$\therefore$ The ball is initially hit with a speed of approx 10.3 m/s.

Comments: $-\frac{1}{2}$ if $\pm$ is not shown (to prove that speed must be +ve)

Qn	Solutions	Marks	Comments: Criteria

5(ii) The particle reaches the endpoints of its motion when $\dot{x} = 0$. ✓

$\therefore 0 = 2\cos 2t - 2\sin 2t$

$\frac{2\sin 2t}{2\cos 2t} = \frac{2\cos 2t}{2\cos 2t}$

$\tan 2t = 1$ ✓

$2t = \frac{\pi}{4}, \frac{5\pi}{4}$ ✓

$\therefore x = 2 + \sin\frac{\pi}{4} + \cos\frac{\pi}{4}$ (for $2t = \frac{\pi}{4}$)

$= 2 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}$

$= 2 + \frac{2}{\sqrt{2}}$ ✓ $\left(\frac{2}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} = \frac{2\sqrt{2}}{2} = \sqrt{2}\right)$

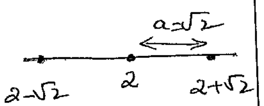
$= 2 + \sqrt{2}$

and $x = 2 + \sin\frac{5\pi}{4} + \cos\frac{5\pi}{4}$ (for $2t = \frac{5\pi}{4}$)

$= 2 - \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}$

$= 2 - \frac{2}{\sqrt{2}}$

$= 2 - \sqrt{2}$ ✓



Since the centre of the motion is at $x = 2$, the amplitude is $\sqrt{2}$, ✓ the distance between the centre and either endpoints

Qn	Solutions	Marks	Comments: Criteria
4 (iii)	The ball hits the ground at $y=0$. $\therefore 0 = 1 + x - \frac{9.8x^2}{10.3^2}$ ✓ $= 10.3^2 + 10.3^2x - 9.8x^2$ i.e. $9.8x^2 - 10.3^2x - 10.3^2 = 0$ $x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$ $= \frac{-(-10.3^2) \pm \sqrt{(-10.3^2)^2 - 4(9.8)(10.3)^2}}{2(9.8)}$ ✓ $=$ $= 11.78$ m, since $x > 0$ $\therefore$ Distance beyond the net is $11.78 - 9.3$ ✓ $= 2.48$ metres.	2	
5 (i)	$x = 2 + \sin 2t + \cos 2t$ $\dot{x} = 2\cos 2t - 2\sin 2t$ ✓ $\ddot{x} = -4\sin 2t - 4\cos 2t$ ✓ $= -4(\sin 2t + \cos 2t)$ $= -4(x-2)$, ✓ from $x = 2 + \sin 2t + \cos 2t$, $x - 2 = \sin 2t + \cos 2t$ ✓ $= -n^2X$ ✓ where $X = x - 2$, $n = 2$ ✓ Since the acceleration is proportional to the displacement, the particle undergoes SHM	3	

Qn	Solutions	Marks	Comments: Criteria
5(ii)	The maximum speed occurs when the particle passes through the centre of motion, i.e. at $x=2$ ✓ $2 = 2 + \sin 2t + \cos 2t$ $\sin 2t + \cos 2t = 0$ ✓ * $\frac{\sin 2t}{\cos 2t} = \frac{-\cos 2t}{\cos 2t}$ ✓ $\tan 2t = -1$ ✓ $2t = \frac{3\pi}{4}, \frac{7\pi}{4}, \ldots$ ✓ $t = \frac{3\pi}{8}$ seconds after $t=0$ or when $a=0$ i.e. $-4\sin 2t - 4\cos 2t = 0$ $\sin 2t + \cos 2t = 0$ etc. (as above *)	3	