St Catherine's School

Year: 12
Subject: Extension II Mathematics
Time Allowed: 55 minutes
Assessment task 3
16/6/2004
Task weight 15%

Exam number: ____________

Directions to candidates:

- All questions are to be attempted..
- All necessary **working** must be shown in every question.
- Full marks may not be awarded for careless or badly arranged work.
- Approved calculators may be used.
- Hand in your work in **1 bundle**:

Attach the question paper to your answer sheets

Assessment task 3 Extension II 16/6/2004

Q.1 Integrate the following:

(a) $\int \frac{2x+5}{x+3} dx$ (2m)

(b) $\int \frac{2x+3}{x^2+2x+5} dx$ (4m)

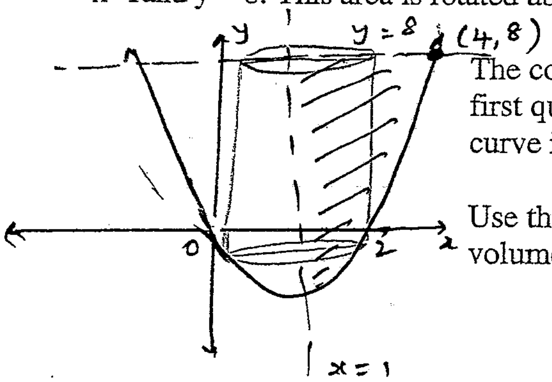
(c) $\int_0^2 \frac{x^2}{\sqrt{4-x^2}} dx$ (4m)

(d) $\int \sin^4 x \ \cos^3 x dx$ (3m)

(e) Use integration by parts to find $\int e^{2x} \cos \ x \ dx$ (4m)

(f) If $u_n = \int \sin^n x dx$, show that $u_n = -\frac{1}{n}\sin^{n-1} x \cos x + \frac{n-1}{n} u_{n-2}$ (4m)

Q.2 The shaded area shows the area bounded by the curve y = x (x-2) and the lines x=1 and y = 8. This area is rotated about the line x = 1.



The coordinates of the point of intersection in the first quadrant between the line y=8 and the given curve is (4,8).

Use the method of cylindrical shells to find the volume generated. **(5m)**

Q.3 The area bounded by the curve y = x (4-x) and the x axis is rotated around the y axis. Take a slice perpendicular to the axis of rotation and use the slice method to find the volume generated.

(5m)

Q.4 The base of a particular solid is the circle $x^2 + y^2 = 1$.
Find the volume of the solid if every cross-section perpendicular to the x-axis is an equilateral triangle by firstly showing that (a) the area of the cross-section is $\sqrt{3}\,y^2$ (b) the volume of the solid is $\frac{4\sqrt{3}}{3}$ units^3.

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Question 1.

(a) $\int \frac{2x+5}{x+3} dx$.

$= \int 2 - \frac{1}{x+3} dx$.

$= 2x - \ln|x+3| + c$.

$$\begin{array}{r} 2 \\ x+3 \overline{)\,2x+5} \\ -\;2x+6 \\ \hline -1 \end{array}$$

(b) $\int \frac{2x+3}{x^2+2x+5} dx$

~~$= \int \frac{2x+3}{(x+1)^2+4} dx$~~.

~~$= \int \frac{2x}{(x+1)^2+4}$~~

$= \int \frac{2x+2+1}{x^2+2x+5} dx$.

$= \int \frac{2x+2}{x^2+2x+5} + \frac{1}{x^2+2x+5} dx$.

$= \ln|x^2+2x+5| + \int \frac{1}{(x+1)^2+4} dx$.

$= \ln|x^2+2x+5| + \frac{1}{2}\tan^{-1}\left(\frac{x+1}{2}\right) + c$.

(c) $\int_0^2 \frac{x^2}{\sqrt{4-x^2}} dx$.

Let $x = 2\sin\theta$.

$dx = 2\cos\theta d\theta$.

$x = 2, \quad \theta = \frac{\pi}{2}$.

$x = 0, \quad \theta = 0$.

$= \int_0^{\frac{\pi}{2}} \frac{4\sin^2\theta}{\sqrt{4-4\sin^2\theta}} \cdot 2\cos\theta d\theta$.

$= \int_0^{\frac{\pi}{2}} \frac{4\sin^2\theta}{\sqrt{4(1-\sin^2\theta)}} \cdot 2\cos\theta d\theta$.

$= \int_0^{\frac{\pi}{2}} \frac{4\sin^2\theta}{2\cos\theta} \cdot 2\cos\theta d\theta$.

$= 4\int_0^{\frac{\pi}{2}} \sin^2\theta d\theta$.

$= 2\int_0^{\frac{\pi}{2}} 1 - \cos 2\theta d\theta$.

$= 2\left[\theta - \frac{1}{2}\sin 2\theta\right]_0^{\frac{\pi}{2}}$.

$= 2\left(\frac{\pi}{2}\right)$

$= \pi$

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(d) $\int \sin^4\theta \cos^3\theta \, d\theta$

$= \int \sin^4\theta (1-\sin^2\theta)\cos\theta \, d\theta$

$= \int u^4 (1-u^2) \, du$

$= \int u^4 - u^6 \, du$

$= \frac{1}{5}u^5 - \frac{1}{7}u^7 + c$

$= \frac{1}{5}\sin^5\theta - \frac{1}{7}\sin^7\theta + c$ ✓

$u = \sin\theta$

~~$d\theta$~~ $du = \cos\theta d\theta$

(e) $\int e^{2x}\cos x dx = e^{2x}\sin x - \int 2e^{2x}\sin x$

$= e^{2x}\sin x - 2\int e^{2x}\sin x$

$= e^{2x}\sin x - 2\left[-e^{2x}\cos x + \int 2e^{2x}\cos x dx\right]$

$= e^{2x}\sin x + 2e^{2x}\cos x - 4\int e^{2x}\cos x dx$

$5\int e^{2x}\cos x dx = e^{2x}\sin x + 2e^{2x}\cos x$

$\int e^{2x}\cos x dx = \frac{1}{5}(e^{2x}\sin x + 2e^{2x}\cos x) + C$ ✓

Let:

$u = e^{2x}$ $v' = \cos x$

$u' = 2e^{2x}$ $v = \sin x$

$u = e^{2x}$ $v' = \sin x$

$u' = 2e^{2x}$ $v = -\cos x$

(f) $U_n = \int \sin^n x dx$

$= \int \sin^{n-1}x \sin x dx$

$= -\sin^{n-1}x\cos x + (n-1)\int \sin^{n-2}x\cos^2 x dx$

$= -\sin^{n-1}x\cos x + (n-1)\int \sin^{n-2}x(1-\sin^2 x) dx$

$= -\sin^{n-1}x\cos x + (n-1)\int \sin^{n-2}x - \sin^n x dx$

$= -\sin^{n-1}x\cos x + (n-1)\int \sin^{n-2}x dx - (n-1)\int \sin^n x dx$

$= -\sin^{n-1}x\cos x + (n-1)U_{n-2} - (n-1)U_n$

$U_n(1+n-1) = -\sin^{n-1}x\cos x + (n-1)U_{n-2}$

$U_n = -\frac{1}{n}\sin^{n-1}x\cos x + \left(\frac{n-1}{n}\right)U_{n-2}$ ✓

$u = \sin^{n-1}x$ $v' = \sin x$

~~$u' = (n-1)$~~ $v = -\cos x$

$u' = (n-1)\sin^{n-2}x\cos x$

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Question 2.

$\Delta V = 2\pi r h \ \Delta x$

$= 2\pi (x-1)(8-y)\Delta x$ ✓

~~$= 2\pi (8x - xy - 8 + y)\Delta x$~~

$= 2\pi (x-1)(8-(x^2-2x))\Delta x$

$= 2\pi (x-1)(8+2x-x^2)\Delta x$

$= 2\pi (8x+2x^2-x^3-8-2x+x^2)\Delta x$

$= 2\pi (3x^2-x^3+6x-8)\Delta x$

$V = 2\pi\int_1^4 -x^3+3x^2+6x-8 \ dx$

$= 2\pi \left[-\frac{1}{4}x^4+x^3+3x^2-8x\right]_1^4$

$= 2\pi \left[16+4\frac{1}{4}\right]$

$= 40\frac{1}{2}\pi$ units^3 ✓

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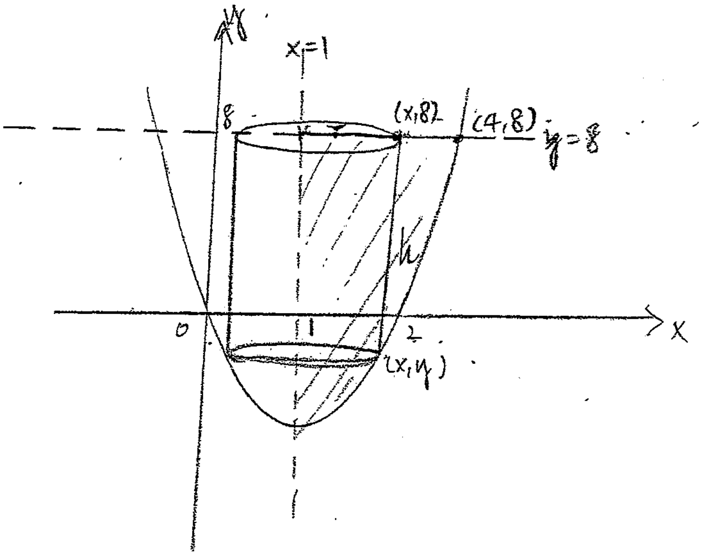

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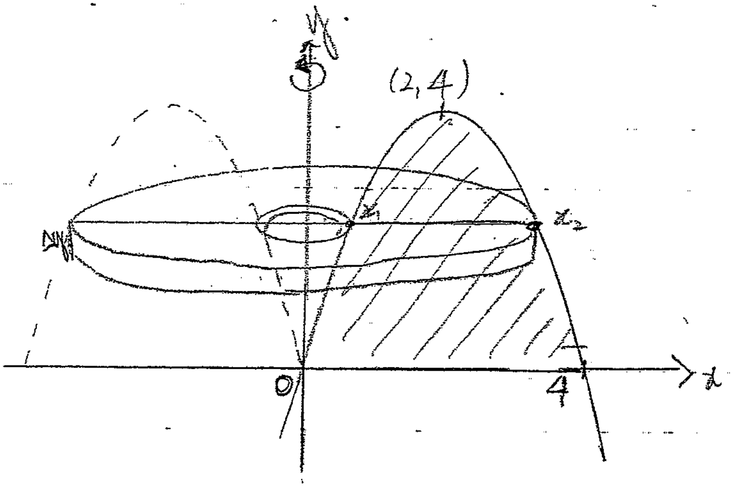
Question 3



$y = x(4-x)$.

$y = 4x - x^2$.

$x^2 - 4x + y = 0$.

$x_1 + x_2 = -(\frac{-4}{1}) = 4$.

$x_1 x_2 = y$.

$(x_2 - x_1)^2 = (x_2 + x_1)^2 - 4x_1x_2$

$= 16 - 4y$.

$x_2 - x_1 = 2\sqrt{4-y}$.

$\Delta V = \pi(x_2^2 - x_1^2)\,\Delta y$.

$= \pi(x_2 + x_1)(x_2 - x_1)\,\Delta y$.

$= \pi(4)(2\sqrt{4-y})\,\Delta y$.

$V = 8\pi \int_0^4 \sqrt{4-y}\,dy$.

$= 8\pi \int_0^4 (4-y)^{\frac{1}{2}}\,dy$.

$= 8\pi\left[-\frac{2}{3}(4-y)^{\frac{3}{2}}\right]_0^4$

$= 8\pi\left[\frac{16}{3}\right]$.

$= \frac{128}{3}\pi$ units^3

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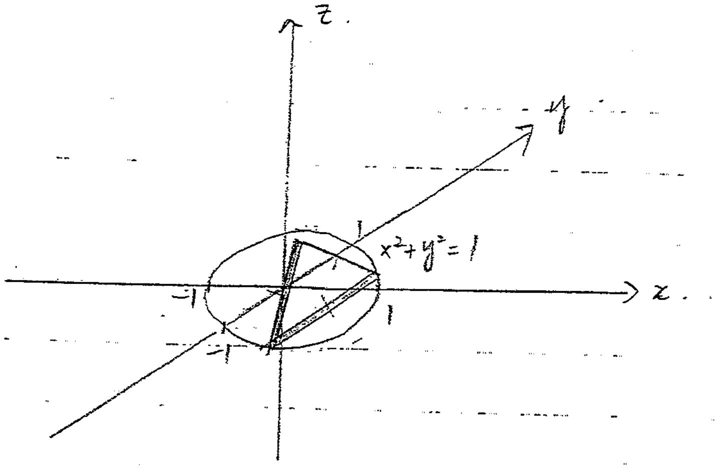
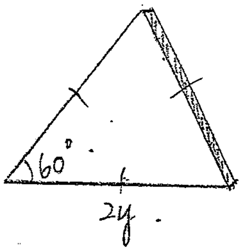
Question 4.



Typical slice =



(a). $\Delta V = (\text{Area of triangle})\,\Delta x$

$= \sqrt{3}\,y^2\,\Delta x$

✓

Area of triangle =

$\frac{1}{2}ab\sin C$

$= \frac{1}{2}(2y)(2y)\sin 60°$

$= \frac{1}{2}(4y^2)\frac{\sqrt{3}}{2}$

$= \sqrt{3}\,y^2$

(b) $V = \sqrt{3}\int_{-1}^{1} y^2\,dx$

~~$= 2\sqrt{3}\int_0^1 1 - x^2\,dx$~~

$= \sqrt{3}\left[x - \frac{1}{3}x^3\right]_{-1}^{1}$

$= \cancel{\frac{4\sqrt{3}}{4}}\cdot\sqrt{3}\left[\frac{4}{3}\right]$

✓

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$= \frac{4\sqrt{3}}{3}$ units^3