St Catherine's School

Waverley, Sydney

An Anglican Day and Boarding School for Girls, Kindergarten to Year 12. Founded in 1856.

Student Number:...

2007
Year 12
Assessment Task 3

Mathematics Extension II

Time allowed: 55 mins

Course weighting: 15%

General Instructions

- Attempt ALL questions
- Write your Student NUMBER at the top of this page and on the writing paper used

Questions	Marks
1	/10
2	/16
3	/13
Total	/39

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1},\quad n \neq 1;\ x \neq 0,\ \text{if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x,\quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax},\quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax,\quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax,\quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax,\quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax,\quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a},\quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a},\quad a > 0,\ -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right)\quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE: $\ln x = \log_e x, \quad x > 0$

Marks

Question 1 (10 marks)

a) (i) Find real numbers a and b and c such that **2**

$$\frac{1}{(x+1)(x^2+2)} = \frac{a}{x+1} + \frac{bx+c}{x^2+2}$$

(ii) Hence find $\int \frac{dx}{(x+1)(x^2+2)}$ **2**

b) Find $\int \frac{3x+5}{x^2+2x+4}dx$ **3**

c) Use the substitution $t = \tan\frac{x}{2}$ to evaluate $\int_0^{\frac{\pi}{2}} \frac{dx}{2+\cos x}$ **3**

Marks

Question 2 (16 marks)

a) Use integration by parts to show that: $\int_0^1 \tan^{-1} x\,dx = \frac{\pi}{4} - \frac{1}{2}\log_e 2$ **3**

b) Use the substitution $x = 4\sin^2\theta$ or otherwise to show that **4**

$$\int_0^2 \sqrt{x(4-x)}\,dx = \pi$$

c) Let n be a positive integer, and let **4**

$$I_n = \int_1^2 (\log_e x)^n\,dx$$

prove that $I_n = 2(\log_e 2)^n - nI_{n-1}$

Hence evaluate $\int_1^2 (\log_e x)^4\,dx$ as a polynomial in $\log_e 2$.

d) (i) Given that $C_n = \int_0^{\frac{\pi}{2}} \cos^n x\;dx$, prove that $C_n = \frac{n-1}{n}C_{n-2}$ where $n = 2, 3, 4, \ldots.$ **3**

(ii) Hence evaluate $\int_0^{\frac{\pi}{2}} \cos^4 x\;dx$ **2**

Marks

Question 3 (13 marks)

a) If $a>0,\ b>0,\ c>0$

(i) Prove that $a^2+b^2>2ab$ 2

(ii) Hence prove that $a^2+b^2+c^2 \geq ab+bc+ca$ 2

(iii) Prove $a^3+b^3+c^3 \geq 3abc$ 2

b) Prove that $\cos x > 1-\frac{x^2}{2}$ for $x \neq 0$ 3

c) If $U_1=2,\quad U_2=3$ and $U_{n+2}=3U_{n+1}-2U_n$ for $n \geq 3$ prove by the method of mathematical induction that $U_n=2^{n-1}+1$ for $n \geq 1$ 4

Student Number: SOLUTIONS

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Top 35.5
2) 32.5

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Qn	Solutions	Marks	Comments+Criteria

1.a) (i)

$$\frac{1}{(x+1)(x^2+2)} = \frac{a(x^2+2)+(bx+c)(x+1)}{(x+1)(x^2+2)}$$

$$\therefore 1 = ax^2+2a+bx^2+bx+cx+c$$

$$= (a+b)x^2+(b+c)x+(2a+c)$$

$$\therefore a+b=0 \quad —①$$

$$b+c=0 \quad —②$$

$$2a+c=1 \quad —③$$

①−② $\quad a-c=0 \quad —④$

③+④ $\quad 3a=1$

$$a=\tfrac{1}{3} \quad b=-\frac{1}{3} \quad c=\frac{1}{3}$$

(ii)

$$\int\frac{dx}{(x+1)(x^2+2)} = \int\frac{\frac{1}{3}}{x+1}dx + \int\frac{-\frac{1}{3}x+\frac{1}{3}}{x^2+2}dx$$

$$= \frac{1}{3}\ln(x+1) - \frac{1}{3}\int\frac{x}{x^2+2}dx + \frac{1}{3}\int\frac{1}{x^2+2}dx$$

$$= \frac{1}{3}\ln(x+1) - \frac{1}{6}\ln(x^2+2) + \frac{1}{3\sqrt{2}}\tan^{-1}\frac{x}{\sqrt{2}}+C$$

$$= \frac{1}{3}\ln\frac{x+1}{\sqrt{x^2+2}} + \frac{1}{3\sqrt{2}}\tan^{-1}\frac{x}{\sqrt{2}} + C$$

b)

$$\int\frac{3x+5}{x^2+2x+4}dx = \frac{3}{2}\int\frac{2x+2}{x^2+2x+4}dx + \int\frac{2}{x^2+2x+4}dx$$

$$= \frac{3}{2}\ln(x^2+2x+4) + 2\int\frac{1}{(x+1)^2+3}dx$$

$$= \frac{3}{2}\ln(x^2+2x+4) + \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{x+1}{\sqrt{3}}\right)+C$$

c)

$$\int_0^{\frac{\pi}{2}}\frac{dx}{2+\cos x} = \int_0^1\frac{1}{2+\frac{1-t^2}{1+t^2}}\cdot\frac{2dt}{1+t^2}$$

$$\boxed{t=\tan\frac{x}{2} \quad x=0\ t=0 \quad x=\frac{\pi}{2}\ t=1}$$

$$= \int_0^1\frac{\cancel{1+t^2}}{2+2t^2+1-t^2}\cdot\frac{2dt}{\cancel{1+t^2}}$$

$$= \int_0^1\frac{2}{t^2+3}dt$$

$$= \left[\frac{2}{\sqrt{3}}\tan^{-1}\frac{t}{\sqrt{3}}\right]_0^1$$

$$= \frac{2}{\sqrt{3}}\tan^{-1}\frac{1}{\sqrt{3}} - 0$$

$$= \frac{2}{\sqrt{3}}\cdot\frac{\pi}{6} = \frac{\pi}{3\sqrt{3}} \text{ or } \frac{\sqrt{3}\pi}{9}$$

Qn	Solutions	Marks	Comments+Criteria

2a)

$$\int_0^1\tan^{-1}x\,dx \qquad u=\tan^{-1}x \quad v=x \qquad u'=\frac{1}{1+x^2} \quad v'=1$$

$$= \left[x\tan^{-1}x\right]_0^1 - \int_0^1\frac{x}{1+x^2}dx$$

$$= \frac{\pi}{4} - \left[\frac{1}{2}\ln(1+x^2)\right]_0^1$$

$$= \frac{\pi}{4} - \frac{1}{2}\ln 2$$

b)

$$\int_0^2\sqrt{x(4-x)}\,dx \qquad x=4\sin^2\theta \quad dx=8\sin\theta\cos\theta \quad x=0\ \theta=0 \quad x=2\ \theta=\pi/4$$

$$= \int_0^{\pi/4}\sqrt{4\sin^2\theta(4-4\sin^2\theta)}\cdot 8\sin\theta\cos\theta\,d\theta$$

$$= \int_0^{\pi/4}\sqrt{4\sin^2\theta\, 4\cos^2\theta}\cdot 8\sin\theta\cos\theta\,d\theta$$

$$= \int_0^{\pi/4}4\sin\theta\cos\theta\cdot 8\sin\theta\cos\theta\,d\theta$$

$$= \int_0^{\pi/4}32\sin^2\theta\cos^2\theta\,d\theta$$

$$= \int_0^{\pi/4}8\cdot 4\sin^2\theta\cos^2\theta\,d\theta \qquad 2\sin^2 2\theta\cos^2_2\theta$$

$$= 8\int_0^{\pi/4}\sin^2 2\theta\,d\theta$$

$$= 4\int_0^{\pi/4}(1-\cos 4\theta)d\theta$$

$$= 4\left[\theta - \frac{\sin 4\theta}{4}\right]_0^{\pi/4}$$

$$= 4\left[\left(\frac{\pi}{4}-0\right)-(0)\right]$$

$$= \pi$$

Qn	Solutions	Marks	Comments+Criteria

2c

$I_n = \int_1^2 (\log_e x)^n \, dx \qquad u = (\log_e x)^n \qquad v = x$

$u' = \dfrac{n(\log_e x)^{n-1}}{x} \qquad v' = 1$

$\therefore I_n = \left[x(\ln x)^n\right]_1^2 - n\int (\ln x)^{n-1} dx$

$= 2(\ln 2)^n - n I_{n-1}$

$I_4 = \int_1^2 (\ln x)^4 dx$

$= 2(\ln 2)^4 - 4I_3$

Now $I_3 = 2(\ln 2)^3 - 3I_2$

$I_2 = 2(\ln 2)^2 - 2I_1$

$I_1 = 2(\ln 2) - I_0$

$I_0 = 1$

$\therefore I_4 = 2(\ln 2)^4 - 4\left[2(\ln 2)^3 - 3\left[2(\ln 2)^2 - 2\left[2\ln 2 - 1\right]\right]\right]$

$= 2(\ln 2)^4 - 8(\ln 2)^3 + 24(\ln 2)^2 - 48\ln 2 + 24$

d) (i) $C_n = \int_0^{\frac{\pi}{2}} \cos^n x \, dx \qquad u = \cos^{n-1} x \qquad v = \sin x$

$u' = -(n-1)\cos^{n-2} x \sin x \qquad v' = \cos x$

$= \int_0^{\frac{\pi}{2}} \cos^{n-1} x . \cos x \, dx$

$= \left[\sin x \cos^{n-1} x\right]_0^{\frac{\pi}{2}} + (n-1)\int_0^{\frac{\pi}{2}} \cos^{n-2} x \sin^2 x \, dx$

$= (n-1)\int_0^{\frac{\pi}{2}} \cos^{n-2} x (1 - \cos^2 x) \, dx$

$= (n-1)\int_0^{\frac{\pi}{2}} \cos^{n-2} x - (n-1)\int_0^{\frac{\pi}{2}} \cos^n x$

$= (n-1) C_{n-2} - (n-1) C_n$

$\therefore C_n + (n-1)C_n = (n-1)C_{n-2}$

$n C_n = (n-1) C_{n-2}$

$C_n = \dfrac{n-1}{n} C_{n-2}$

Qn	Solutions	Marks	Comments+Criteria

2d) (ii) $C_4 = \int_0^{\frac{\pi}{2}} \cos^4 x \, dx$

$= \frac{3}{4} C_2$

$= \frac{3}{4} \cdot \frac{1}{2} C_0 \qquad$ now $C_0 = \int_0^{\frac{\pi}{2}} dx = \frac{\pi}{2}$

$= \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2}$

$= \frac{3\pi}{16}$

Q3a (i) $(a-b)^2 \geq 0$

$\therefore a^2 + b^2 - 2ab \geq 0$

$\therefore a^2 + b^2 \geq 2ab$

(ii) from (i) $a^2 + b^2 \geq 2ab$

$b^2 + c^2 \geq 2bc$

$c^2 + a^2 \geq 2ca$

$\therefore 2(a^2+b^2+c^2) \geq 2(ab+bc+ca)$

$\therefore a^2+b^2+c^2 \geq ab+bc+ca$

(iii) multiply both sides of (ii) by $(a+b+c)$

$LHS = (a^2+b^2+c^2)(a+b+c) = a^3+b^3+c^3+a^2b+a^2c+b^2a+b^2c+c^2a+c^2b$

$RHS = (ab+bc+ca)(a+b+c) = a^2b+ab^2+abc+abc+b^2c+bc^2+ca^2+abc+c^2a$

$\therefore a^3+b^3+c^3 \geq 3abc$

b) let $f(x) = \cos x - 1 + \frac{x^2}{2} = 0$ for $x = 0$

$f'(x) = -\sin x + x = 0$ for $x = 0$ (ONLY)

$f''(x) = 1 - \cos x \geq 0$ for all x

$\therefore f(x)$ is concave upwards for all x with a stationary pt (minimum) at $x = 0$

$\therefore f(x) > 0$ for all $x \neq 0$

$\therefore \cos x - 1 + \frac{x^2}{2} > 0$

$\therefore \cos x > 1 - \frac{x^2}{2}$

Qn	Solutions	Marks	Comments+Criteria
Q3c)	$U_1 = 2 \quad U_2 = 3 \quad U_{n+2} = 3U_{n+1} - 2U_n \quad n \geq 3$ prove $U_n = 2^{n-1}+1 \quad n \geq 1$ $U_1 = 2^{1-1}+1 = 2$ true $U_2 = 2^{2-1}+1 = 3$ true assume true for $n=k \quad k \geq 2$ ie $U_k = 2^{k-1}+1$ aim to prove true for $n=k+1$ i.e. $U_{k+1} = 2^k+1$ Now $U_{k+1} = 3U_k - 2U_{k-1}$ (note $n=k-1$) $= 3(2^{k-1}+1) - 2(2^{k-2}+1)$ $= 3.2^{k-1}+3 - 2^{k-1} - 2$ $= 2.2^{k-1}+1$ $= 2^k+1$ $\therefore$ true for $n=k+1$ if true for $n=k$ Now since true for $n=1$ and $n=2$ then true for $n=3,4,5,\ldots$ $\therefore$ true for all $n \geq 1$		