St. Catherine's School
Waverley

18th September 2009

Preliminary Extension I Mathematics

Time allowed: 2 hours *Preliminary Assessment Weighting: 45%*

INSTRUCTIONS

- Write your STUDENT NUMBER on each page
- Marks for each part of a question are indicated
- All questions should be attempted on your own paper
- All necessary working should be shown
- Start each question on a NEW page
- Approved scientific calculators and drawing templates may be used

Question 1 **(15 marks)** **START A NEW BOOKLET** **Marks**

(a) Solve for x: $\dfrac{x+4}{x-2} \geq 3$ **3**

(b) Evaluate:

(i) $\lim\limits_{x \to 4} \dfrac{3x-12}{2x^2-32}$ **2**

(ii) $\lim\limits_{x \to \infty} \dfrac{2x^3-1}{4-5x^3}$ **2**

(c) The acute angle between the line $x-3y+7=0$ and the line $y=kx$ is 45^0.

(i) Show that $\left|\dfrac{1-3k}{k+3}\right|=1$. **3**

(ii) Find the two possible values of k. **2**

(d) Solve $\sin 2\theta = \sin\theta$ for $0° \leq \theta \leq 360°$ **3**

Question 2 (15 marks) **START A NEW BOOKLET** — **Marks**

(a) Solve $\sin\theta\cot\theta - \sin\theta = 0$ for $0^\circ \le \theta \le 360^\circ$ — **3**

(b) R is the point $(-2,-1)$ and B is the point $(1, 5)$.
Find the co-ordinates of the point Q which divides the interval RB *externally* in the ratio 5:3. — **2**

(c) If $\sin\alpha = \frac{3}{4}$, $0^0 < \alpha < 90^0$ and $\sin\beta = \frac{2}{3}$, $90^0 < \beta < 180^0$, find the **exact** value of:

(i) $\tan\alpha$ — **1**

(ii) $\cos(\alpha - \beta)$. — **2**

(d) (i) Express $\cos x + \sqrt{3}\sin x$ in the form $R\cos(x-\alpha)$, where $0^\circ < \alpha < 90^\circ$ and $R > 0$. — **1**

(ii) Hence, or otherwise solve — **3**

$$\cos x + \sqrt{3}\sin x = 1$$

for $0^\circ \le x \le 360^\circ$

(e) If $\tan\theta = \frac{2t}{1-t^2}$, prove $\frac{1+\sin\theta-\cos\theta}{1+\sin\theta+\cos\theta} = t$ — **3**

Question 3 (15 marks) **START A NEW BOOKLET** — **Marks**

(a) Consider the graph of the function $f(x) = \frac{x}{9-x^2}$.

(i) For what values of x is the function undefined? — **1**

(ii) Show that $f(x)$ is an odd function. — **1**

(iii) Find the horizontal asymptote — **1**

(iv) Show that $f'(x) > 0$ for all values of x for which the function is defined. — **2**

(v) Find the values of f(x) at $x=1$ and $x=-1$. — **1**

(vi) Neatly sketch the graph of $y = \frac{x}{9-x^2}$. — **2**

(b) (i) Show that $\frac{d}{dx}6x\sqrt{x^2+4} = \frac{12(x^2+2)}{\sqrt{x^2+4}}$. — **3**

(ii) Hence find the gradient of the tangent to the curve $y = 6x\sqrt{x^2+4}$ at the origin. — **1**

(c) Find the values of a, b, c for which — **3**

$$3x^2 - 2x - 7 \equiv a(x+2)^2 + b(x+2) + c.$$

Question 4 **(15 marks)** **START A NEW BOOKLET** **Marks**

(a) For the circle $x^2 + y^2 + 6x - 2y = 15$,

(i) find the centre and radius. **3**

(ii) and sketch the graph **2**

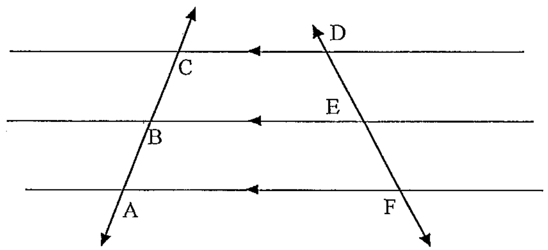
(b) Using the substitution $t = \tan\frac{\theta}{2}$ or otherwise, prove that: **3**

$$\frac{\tan\theta - \tan\frac{\theta}{2}}{\cot\frac{\theta}{2} + \tan\theta} = \tan^2\frac{\theta}{2}$$

(c) Use mathematical induction to prove that for all positive integers *n*, **4**

$2^{n+2} + 3^{3n}$ is divisible by 5

(d)



The diagram shows 3 parallel lines; CD || BE || AF and 2 transversals AC and DF

(i) Copy the diagram onto your answer booklet and include the parallel to AC through E. **1**

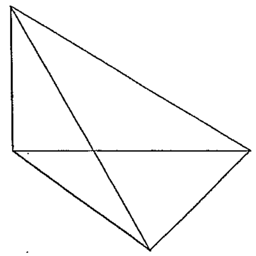
(ii) Prove $\frac{BC}{BA} = \frac{DE}{EF}$ **2**

(You may assume that the opposite sides of a parallelogram are equal)

Question 5 **(15 marks)** **START A NEW BOOKLET** **Marks**

(a) Find the **exact** value of $\sin 75^\circ$ **2**

(b) The elevation of a hill XY of height h metres from a point P due east of it is 44°, and at a point Q due south of P, the elevation is 38°. The distance from P to Q is 1200 metres. The base of the hill is X.



(i) *Neatly* copy the diagram to your answer paper and label it with the given information. (large diagram please) **1**

(ii) Show that $XP = h\tan 46^\circ$ **1**

(iii) Show that $XQ = h\tan 52^\circ$ **1**

(iv) Show that $h^2 = \frac{1200^2}{\tan^2 52 - \tan^2 46}$ **3**

(v) Hence find h to the nearest metre. **1**

(c) The points $P(2ap, ap^2)$ and $Q(2aq, aq^2)$ lie on the parabola $x^2 = 4ay$

(i) Find the gradient of the interval joining P and Q in simplest form. **2**

(ii) Find the equation of the tangent at P in general form. **2**

(iii) Find the equation of the normal at P in general form. **2**

END OF PAPER

Qn	Solutions	Marks	Comments+Criteria

1 (a) $\dfrac{x+4}{x-2} \geq 3 \quad , \quad x \neq 2$

$(x-2)^2 \times \dfrac{x+4}{x-2} \geq 3(x-2)^2$ ½✓

$(x-2)(x+4) \geq 3(x^2-4x+4)$ ½✓

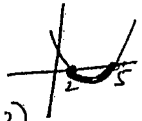
$x^2+2x-8 \geq 3x^2-12x+12$

$0 \geq 2x^2-14x+20$ ½✓

$0 \geq x^2-7x+10$

$0 \geq (x-5)(x-2)$ ½✓

$2 < x \leq 5$ ½✓ ½✓



(note: $x \neq 2$)

Marks: 3

Comments: '✓' means 1 mark. '½✓' means 0.5 mark

(b)(i) $\lim\limits_{x\to 4} \dfrac{3(\cancel{x-4})}{2(\cancel{x-4})(x+4)}$

$= \lim\limits_{x\to 4} \dfrac{3}{2(x+4)}$ ✓

$= \dfrac{3}{16}$ ✓

Marks: 2

(ii) $\lim\limits_{x\to\infty} \dfrac{2x^3-1}{4-5x^3}$

$= \lim\limits_{x\to\infty} \dfrac{\frac{2x^3}{x^3}-\frac{1}{x^3}}{\frac{4}{x^3}-\frac{5x^3}{x^3}}$

$= \lim\limits_{x\to\infty} \dfrac{2-\cancel{\frac{1}{x^3}}\to 0}{\cancel{\frac{4}{x^3}}\to 0 - 5}$ ✓

$= -\dfrac{2}{5}$ ✓

Marks: 2

Comments: $\left(\frac{2}{5}\right) \to$ 0.5 mark

Qn	Solutions	Marks	Comments+Criteria

1 (c)(i) $x-3y+7=0 \qquad m_1 = \frac{1}{3}$ ½✓

$y = kx \qquad m_2 = k$ ½✓

acute angle given by

$\tan\theta = \left|\dfrac{m_1-m_2}{1+m_1m_2}\right|$ ½✓

$\therefore \tan 45° = \left|\dfrac{\frac{1}{3}-k}{1+\frac{k}{3}}\right|$ ½✓

$\tan 45° = \left|\dfrac{1-3k}{3+k}\right|$ ½✓

$\therefore \left|\dfrac{1-3k}{3+k}\right| = 1$ ½✓

Marks: 3

(ii) $\left|\dfrac{1-3k}{k+3}\right| = 1$

$\therefore \dfrac{1-3k}{k+3} = \pm 1$

$\dfrac{1-3k}{k+3} = 1 \qquad$ or $\qquad \dfrac{1-3k}{k+3} = -1$

$1-3k = k+3$ ½✓ $\qquad 1-3k = -k-3$ ½✓

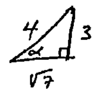
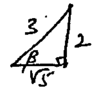
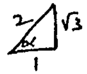
$-4k = 2 \qquad -2k = -4$

$k = \dfrac{2}{-4} \qquad k = 2$ ½✓

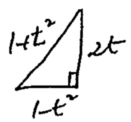
$k = -\frac{1}{2}$ ½✓

Marks: 2

Qn	Solutions	Marks	Comments+Criteria
1	(d) $\sin 2\theta = \sin\theta \quad 0^\circ \leq \theta \leq 360^\circ$ $2\sin\theta\cos\theta - \sin\theta = 0$ $\sin\theta\,(2\cos\theta - 1) = 0$ ✓ $\therefore \sin\theta = 0$ OR $2\cos\theta - 1 = 0$, $\cos\theta = \frac{1}{2}$ ✓ $\therefore \theta = 0^\circ, 60^\circ, 180^\circ, 300^\circ, 360^\circ$ ✓	3	$\frac{1}{2}$ for some answers.
2	(a) $\sin\theta\cot\theta - \sin\theta = 0 \quad 0^\circ \leq \theta \leq 360^\circ$ $\sin\theta\,(\cot\theta - 1) = 0$ ✓ $\therefore \sin\theta = 0$ ✓ or $\cot\theta = 1$ ✓ $\therefore \theta = 0^\circ, 45^\circ, 180^\circ, 225^\circ, 360^\circ$ ✓	3	't' results (some answers) $2\frac{1}{3}$
	(b) $R\overset{x_1\ y_1}{(-2,-1)}$ $\quad B\overset{x_2\ y_2}{(1,5)}$ $\quad \times Q(\ \ )$ $m : n$ $6 : -3$ $Q\left(\frac{mx_2+nx_1}{m+n}, \frac{my_2+ny_1}{m+n}\right)$ $Q\left(\frac{5+6}{2}, \frac{25+3}{2}\right)$ ✓ $Q\left(\frac{11}{2}, 14\right)$ ✓		$-\frac{1}{2}$ for correct formula, minor error

Qn	Solutions	Marks	Comments+Criteria
2	(c) $\sin\alpha = \frac{3}{4} \quad 0^\circ \leq \alpha \leq 90^\circ$  $\therefore \cos\alpha = \frac{\sqrt{7}}{4}$ and $\sin\beta = \frac{2}{3} \quad 90^\circ \leq \beta \leq 180^\circ$  $\therefore \cos\beta = \frac{-\sqrt{5}}{3}$ ✓		
	(i) $\tan\alpha = \frac{3}{\sqrt{7}}$ ✓	1	
	(ii) $\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$ ✓ $= \frac{\sqrt{7}}{4} \cdot \frac{-\sqrt{5}}{3} + \frac{3}{4} \cdot \frac{2}{3}$ ✓ $= \frac{-\sqrt{35}}{12} + \frac{1}{2}$ ✓ $= \frac{6 - \sqrt{35}}{12}$	2	
	(d)(i) $\cos x + \sqrt{3}\sin x = 2\left(\frac{1}{2}\cos x + \frac{\sqrt{3}}{2}\sin x\right)$ $= 2\cos(x - \alpha)$ $r = \sqrt{1^2 + \sqrt{3}^2}$ ✓ $r = 2$  now $\tan\alpha = \sqrt{3}$ ✓ $\therefore \alpha = 60^\circ$ $\therefore \cos x + \sqrt{3}\sin x = 2\cos(x - 60^\circ)$	1	
	(ii) Now $\cos x + \sqrt{3}\sin x = 1$ ✓ $2\cos(x - 60^\circ) = 1$ ✓ $\therefore \cos(x - 60^\circ) = \frac{1}{2}$ $x - 60^\circ = -60^\circ, 60^\circ, 300^\circ$ ✓ $\therefore x = 0^\circ, 120^\circ, 360^\circ$ ✓	3	1 for one answer 2½ for leaving off one 2 for giving more

Qn	Solutions	Marks	Comments+Criteria
2 (e)	If $\tan\theta = \dfrac{2t}{1-t^2}$		



$$h^2 = (1-t^2)^2 + (2t)^2$$
$$= 1 - 2t^2 + t^4 + 4t^2$$
$$= 1 + 2t^2 + t^4$$
$$= (1+t^2)^2$$
$$\therefore h = 1+t^2$$

Now $\dfrac{1+\sin\theta-\cos\theta}{1+\sin\theta+\cos\theta} = \dfrac{1+\frac{2t}{1+t^2}-\frac{1-t^2}{1+t^2}}{1+\frac{2t}{1+t^2}+\frac{1-t^2}{1+t^2}}$ ✓

$$= \frac{1+t^2+2t-1+t^2}{1+t^2+2t+1-t^2} \checkmark$$

$-\frac{1}{2}$ for some other substitution

$$= \frac{2t^2+2t}{2+2t}$$

3

$$= \frac{2t(t+1)}{2(t+1)}$$
$$= t \checkmark$$

Qn	Solutions	Marks	Comments+Criteria
3 (a)	$f(x) = \dfrac{x}{9-x^2}$		

$$= \frac{x}{(3-x)(3+x)}$$

(i) undefined for $x = 3$ ✓, $x = -3$ ✓

as $9-x^2 \neq 0$

1

(ii) $f(-x) = \dfrac{-x}{9-(-x)^2}$ ✓

$$= \frac{-x}{9-x^2}$$

1

$$= -\left(\frac{x}{9-x^2}\right) \checkmark$$
$$= -f(x)$$

$\therefore$ function is odd.

(iii) $\displaystyle\lim_{x\to\infty} \frac{x}{9-x^2} = \lim_{x\to\infty} \frac{x/x^2}{9/x^2 - x^2/x^2}$ ✓

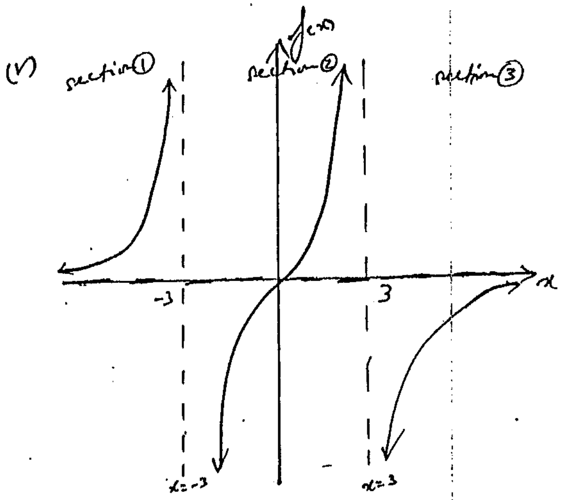
1

$$= 0 \checkmark$$

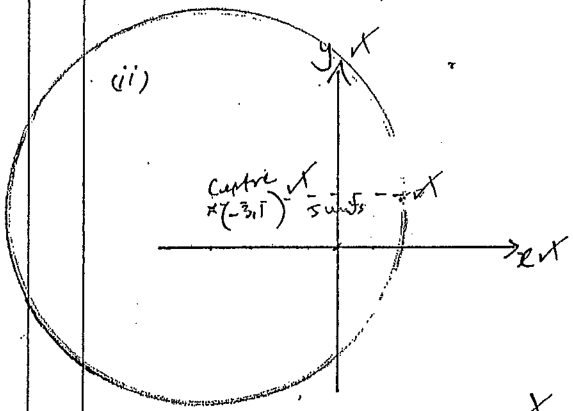
$\therefore$ horizontal asymptote is $y = 0$ (the x-axis)

(iv) $f(x) = \dfrac{x}{9-x^2}$

$$f'(x) = \frac{(9-x^2)1 - x.-2x}{(9-x^2)^2} \checkmark$$
$$= \frac{9-x^2+2x^2}{(9-x^2)^2} \checkmark$$

Qn	Solutions	Marks	Comments+Criteria
3	(a) (iv) $= \dfrac{9+x^2}{(9-x^2)^2}$ >0 for all values of x expect for $x=\pm 3$ as x is squared. ✓ $\therefore f'(x)>0$ for all values for which function is defined	2	
	(a) (iv) When $x=1$ $f(x)=\frac{1}{8}$ ✓ $x=-1$ $f(x)=-\frac{1}{8}$ ✓		
	(v) 		½ for showing vertical asymptotes ½ for showing horizontal asymptote −½ for each incorrect section

Qn	Solutions	Marks	Comments+Criteria
3	(b) (i) $\frac{d}{dx}\, 6x\sqrt{x^2+4} = \sqrt{x^2+4}\cdot 6 + 6x\cdot\frac{1}{2}(x^2+4)^{-\frac{1}{2}}\cdot 2x$ ✓ $= 6\sqrt{x^2+4} + \dfrac{6x^2}{\sqrt{x^2+4}}$ ✓ $= \dfrac{6(x^2+4)+6x^2}{\sqrt{x^2+4}}$ ✓ $= \dfrac{12x^2+24}{\sqrt{x^2+4}}$ ✓ $= \dfrac{12(x^2+2)}{\sqrt{x^2+4}}$ ✓	3	
	(ii) point $(0,0)$ at $x=0$, $f'(x)=\frac{24}{2}$ $=12$ ✓ $\therefore$ gradient of tangent at origin is 12.	1	
	(c) $3x^2-2x-7 = a(x+2)^2+b(x+2)+c$ $= a(x^2+4x+4)+b(x+2)+c$ $= ax^2+4ax+4a+bx+2b+c$ $= ax^2+4ax+bx+4a+2b+c$ $= ax^2+(4a+b)x+4a+2b+c$ $\boxed{a=3}$ ✓ $4a+b=-2$ $4(3)+b=-2$ $12+b=-2$ ✓ $\boxed{b=-14}$ $4a+2b+c=-7$ $4(3)+2(-14)+c=-7$ $12-28+c=-7$ $-16+c=-7$ ✓ $\boxed{c=9}$	3	

Qn	Solutions	Marks	Comments+Criteria
4	(a) $x^2+y^2+6x-2y=15$		
	(i) $x^2+6x+9+y^2-2y+1=15+9+1$ ✓		
	$(x+3)^2+(y-1)^2=25$ ✓		
	$\therefore$ Centre $(-3,1)$ ✓	3	
	Radius $=5$ units ✓		
	(ii) 	2	
	(b) $t=\tan\frac{\theta}{2}\quad\therefore\cot\frac{\theta}{2}=\frac{1}{t}$ ✓		
	$\tan\theta=\frac{2t}{1-t^2}$		
	LHS $=\dfrac{\tan\theta-\tan\frac{\theta}{2}}{\cot\frac{\theta}{2}+\tan\theta}$		
	$=\dfrac{\frac{2t}{1-t^2}-t}{\frac{1}{t}+\frac{2t}{1-t^2}}$ ✓		
	$=\dfrac{2t-t+t^3}{1-t^2}\div\dfrac{1-t^2+2t^2}{t(1-t^2)}$		
	$=\dfrac{t+t^3}{1-t^2}\times\dfrac{t(1-t^2)}{1+t^2}$ ✓		

Qn	Solutions	Marks	Comments+Criteria
4	(b) $=\dfrac{t(1+t^2)}{1-t^2}\times\dfrac{t(1-t^2)}{(1+t^2)}$		
	$=t^2$	4	
	$=\tan^2\frac{\theta}{2}$ ✓		
	(c) Prove $2^{n+2}+3^{3n}=5X$		
	Step 1: Prove true for $n=1$		
	LHS $=2^3+3^3$		
	$=8+27$		
	$=35$		
	$=5\times7\quad\therefore$ true ✓		
	Step 2: Assume true for $n=k$	4	
	i.e. assume $2^{k+2}+3^{3k}=5P$ ✓		
	Step 3: Aim to prove true for $n=k+1$		
	i.e. $2^{k+3}+3^{3k+3}=5M$ ✓		
	LHS $=2.2^{k+2}+27.3^{3k}$		
	$=2(2^{k+2}+3^{3k})+25.3^{3k}$		
	$=2.(5P)+25.3^{3k}$ ✓ ✓		
	$=10P+25.3^{3k}$		
	$=5(2P+5.3^{3k})$		

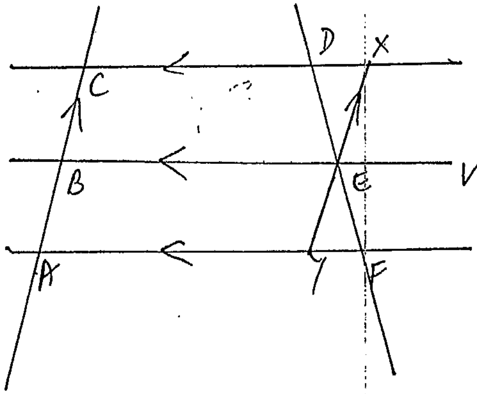
Qn	Solutions	Marks	Comments+Criteria

4 Step 4: Therefore true for $n=k+1$ if true for $n=k$

Since true for $n=1$ then true for $n=2, 3, ----$

$\therefore$ by mathematical induction $2^{n+2}+3^{3n}$ is divisible by 5 for all $n \geq 1$

4 (d)(i)



(ii) in ΔDXE & ΔFYE

$\angle XDE = \angle EFY$ (alt $\angle$s $DC \parallel AF$)

$\angle DXE = \angle EYF$ (" " " ")

$\therefore \Delta DXE \ ||| \ \Delta FYE$ (equiangular)

$\therefore \dfrac{DE}{EF} = \dfrac{XE}{EY}$

but $XE = BC$ (opp. sides of parml. XCBE)

$EY = BA$ (" " ")

$\therefore \dfrac{DE}{EF} = \dfrac{BC}{BA}$

Qn	Solutions	Marks	Comments+Criteria

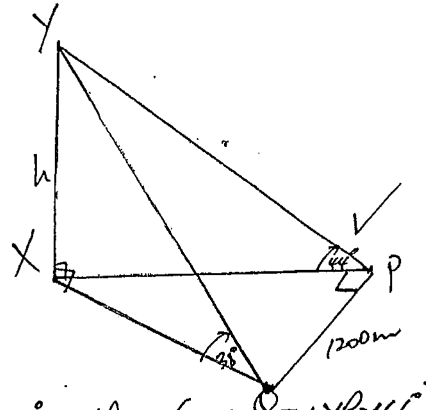
5 (a) $\sin 75° = \sin(45°+30°)$

$= \sin 45° \cos 30° + \cos 45° \sin 30°$

$= \dfrac{1}{\sqrt{2}} \cdot \dfrac{\sqrt{3}}{2} + \dfrac{1}{\sqrt{2}} \cdot \dfrac{1}{2}$

$= \dfrac{\sqrt{3}+1}{2\sqrt{2}} \times \dfrac{\sqrt{2}}{\sqrt{2}}$ OR $\dfrac{\sqrt{6}+\sqrt{2}}{4}$

(b) (i)



(ii) $\tan 46° = \dfrac{XP}{h}$ (note $\angle XYP = 46°$)

$\therefore XP = h \tan 46°$

(iii) $\tan 52° = \dfrac{XQ}{h}$

$\therefore XQ = h \tan 52°$

(iv) in right angled ΔXPQ

$XQ^2 = 1200^2 + XP^2$

$\therefore h^2 \tan^2 52° = 1200^2 + h^2 \tan^2 46°$

$\therefore h^2(\tan^2 52° - \tan^2 46°) = 1200^2$

$\therefore h^2 = \dfrac{1200^2}{\tan^2 52° - \tan^2 46°}$

(v) $h = 1595$ m

Qn	Solutions	Marks	Comments+Criteria
5	(c) $P(2ap, ap^2)$ $\quad x^2 = 4ay$ $Q(2aq, aq^2)$ (i) gradient $= \frac{y_2 - y_1}{x_2 - x_1}$ ✓ $= \frac{ap^2 - aq^2}{2ap - 2aq}$ ✓ $= \frac{a(p^2-q^2)}{2a(p-q)}$ $= \frac{a\cancel{(p-q)}(p+q)}{2a\cancel{(p-q)}}$ $= \frac{p+q}{2}$ ✓ (ii) $y = \frac{x^2}{4a}$ $y' = \frac{2x}{4a}$ $y' = \frac{x}{2a}$ ✓ at P, $_{x=2ap}$, $y' = \frac{2ap}{2a}$ $= p$ $\therefore$ equation of tangent is $y - ap^2 = p(x - 2ap)$ ✓		

Qn	Solutions	Marks	Comments+Criteria
5	(c) (ii) $y - ap^2 = px - 2ap^2$ $px - y - ap^2 = 0$ ✓ (iii) gradient of normal at P is $-\frac{1}{p}$ ✓ $\therefore$ equation of normal is $y - ap^2 = -\frac{1}{p}(x - 2ap)$ ✓ $py - ap^3 = -x + 2ap$ $x + py - 2ap - ap^3 = 0$		