Student Number:

St. Catherine's School
Waverley

2011

HIGHER SCHOOL CERTIFICATE

ASSESSMENT TASK 1 – 15%

Tuesday 22nd FEBRUARY 2011

Extension 1 Mathematics

General Instructions

- Working time – 50 minutes
- Write using black or blue pen
- Board approved calculators may be used
- All necessary working should be shown
- Start each question on a new booklet.

Total marks - 47

Attempt questions 1-2
The value of all parts is indicated

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1},\ n\neq -1;\ x\neq 0,\ \text{if } n<0$$

$$\int \frac{1}{x}\,dx = \ln x,\ x>0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax},\ a\neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax,\ a\neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax,\ a\neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax,\ a\neq 0$$

$$\int \sec ax\tan ax\,dx = \frac{1}{a}\sec ax,\ a\neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a},\ a\neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a},\ a>0,\ -a<x<a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right),\ x>a>0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE: $\ln x = \log_e x,\ x>0$

Question 1 (Marks 25)

(a) (i) Show using the factor theorem that $x+2$ is a factor of $P(x)=x^3-x^2-10x-8$ 1

(ii) Hence express $P(x)=x^3-x^2-10x-8$ as a product of 3 linear factors. 2

(b) (i) Show that the equation $x^3-2x-5=0$ has a root α such that $2<\alpha<3$ 2

(ii) Use one application of Newton's method and an initial approximation of 2 to find the next approximation of α 2

(c) $P(x)=(x^2-1)Q(x)$

(i) If $P(x)=x^3-ax^2+bx-2$ find the values of a and b. 2

(ii) Find $Q(x)$ 2

(iii) Solve $P(x)=0$ and sketch $P(x)$ clearly indicating the roots and the y-intercept. 2

(d) If p, q, and r are the roots of $x^3-4x^2+2x-1=0$, evaluate:

(i) $p+q+r$ 1

(ii) $pq+qr+rp$ 1

(iii) $(p-q)^2+(q-r)^2+(r-p)^2$ 3

(e) The polynomial $P(x)=ax^7+bx^6+1$ has a double zero and a turning point at $x=1$. Find the coefficients a and b. 3

(f) Use mathematical induction to show that $5^n+2(11^n)$ is a multiple of 3 for all positive integers n. 4

Question 2 (Marks 22)

(a) Simplify $\sin 5A\cos 3A+\cos 5A\sin 3A$ 1

(b) If $\cos\alpha=\frac{3}{4}$ for $0<\alpha<\frac{\pi}{2}$, and $\cos\beta=\frac{-2}{3}$ for $\frac{\pi}{2}<\beta<\pi$, 3

Find the value of $\tan(\alpha+\beta)$ to 2 decimal places.

(c) Solve the following and give the answer as a general solution. 2

$$\sqrt{2}\cos 2x=1$$

(d) (i) Show that $2\cos\theta+2\cos\left(\theta+\frac{\pi}{3}\right)=3\cos\theta-\sqrt{3}\sin\theta$ 2

(ii) Use part 'a' to express $2\cos\theta+2\cos\left(\theta+\frac{\pi}{3}\right)$ in the form $R\cos(\theta+\alpha)$ 2

(iii) Hence or otherwise solve $2\cos\theta+2\cos\left(\theta+\frac{\pi}{3}\right)=3$ for $0<\theta<2\pi$ 2

(e) (i) Show that 2

$$\tan 2x+\tan x=\frac{\sin 3x}{\cos 2x\cos x}$$

(ii) Hence solve the equation $\tan 2x+\tan x=0$ in the domain $0<x<\frac{\pi}{2}$ 1

(f) Use $t=\tan\frac{\theta}{2}$ or otherwise, to solve $2\sin\theta+\cos\theta=1$ in the domain $0<\theta<360°$ 4

(g) Prove the following. 3

$$\frac{2-\operatorname{cosec}^2\theta}{\operatorname{cosec}^2\theta+2\cot\theta}=\frac{\sin\theta-\cos\theta}{\sin\theta+\cos\theta}$$

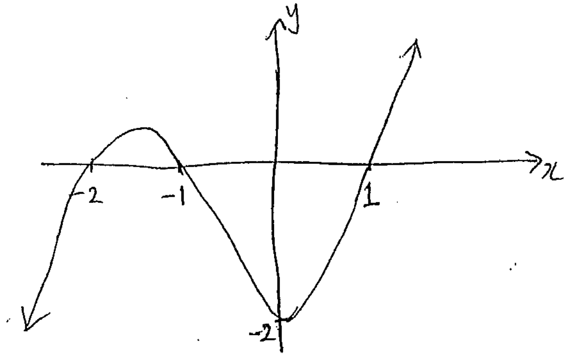
End of Paper

Year 12 Ext 1 Assessment Task 1. 2011 Meydene Ung.

Qn	Solutions	Marks	Comments: Criteria
1.	a.) (i) If $x+2$ is a factor of $P(x)$ then $P(-2) = 0$.		
	$P(-2) = (-2)^3-(-2)^2-10(-2)-8$	(1)	
	$= 0$		
	$\therefore$ $(x+2)$ is a factor		
	(ii) Method 1		
	test factors of the constant term 8.		
	test $x=-1$		
	$P(-1) = (-1)^3-(-1)^2-10(-1)-8$		
	$= 0$	(1)	
	$\therefore$ $(x+1)$ is a factor		
	test $x=4$		
	$P(4) = 4^3-4^2-10(4)-8$		
	$= 0$		
	$\therefore$ $(x-4)$ is a factor	(1)	
	$\therefore P(x) = (x+2)(x-4)(x+1)$		
	Method 2	OR	
	$x+2\,\overline{)\,x^3-x^2-10x-8\,}$ quotient x^2-3x-4		
	x^3+2x^2		
	$-3x^2-10x$		
	$-3x^2-6x$		
	$-4x-8$		
	$-4x-8$		
	0		
	$P(x) = (x+2)(x^2-3x-4)$	(1)	
	$\therefore P(x) = (x+2)(x+1)(x-4)$	(1)	

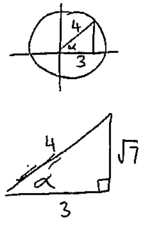
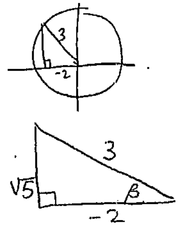
On	Solutions	Marks	Comments: Criteria
b.)	(i) $f(x) = x^3-2x-5$		
	$f(2) = 2^3-4-5$		
	$= -1$	(1)	
	< 0		
	$f(3) = 3^3-6-5$		
	$= 16$		
	> 0	(1)	
	$\therefore$ root α lies between 2 and 3 since $f(2)<0$ and $f(3)>0$.		
	(ii) $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$		
	$x_1 = 2$ $f'(x) = 3x^2-2$	$(\frac{1}{2})$	$(\frac{1}{2})$ to differentiate
	$x_2 = 2 - \frac{-1}{3(2)^2-2}$	(1)	1 mark to plug $x_1=2$ into formula.
	≈ 2.1	$(\frac{1}{2})$	$(\frac{1}{2})$ for answer.
c.)	(i) $P(x) = (x^2-1)Q(x)$		
	~~$x=\pm1$~~ $= (x-1)(x+1)\,Q(x)$		
	$x = \pm 1$ are zeros of the polynomial.		
	$P(1) = 0$		
	$P(-1) = 0$		
	$P(1) = 1^3 - a(1)^2 + b(1) - 2 = 0$		
	$-a + b = 1$ ——— (1)	$(\frac{1}{2})$	
	$P(-1) = (-1)^3 - a(-1)^2 + b(-1) - 2 = 0$		
	$-a - b = 3$ ——— (2)	$(\frac{1}{2})$	

Qn	Solutions	Marks	Comments: Criteria
	(1) − (2) $2b = -2$ $\boxed{\therefore b = -1}$ Sub $b = -1$ into (1) $-a - 1 = 1$ $\boxed{\therefore a = -2}$	(1)	1 mark for ~~doing~~ solving simultaneously & finding a & b.
	(ii) $P(x) = x^3 + 2x^2 - x - 2$ test factors of −2. $P(-2) = -8 + 2(-2)^2 - (-2) - 2$ $= 0$ $\therefore (x+2)$ is a factor of $P(x)$	(1)	OR 1 mark for long division.
	$P(x) = (x^2-1)\,Q(x)$ $\boxed{\therefore Q(x) = (x+2)}$	(1)	
	(iii) $P(x) = (x^2-1)(x+2) = 0$ $(x+1)(x-1)(x+2) = 0$ $\therefore x = \pm 1, -2$ y-int when $x = 0$ $P(0) = -2$	(1)	

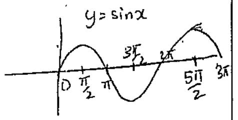
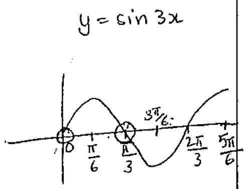
Qn	Solutions	Marks	Comments: Criteria
		(2)	(1) mark for roots, (1) mark for y-intercept.
	d.) (i) $x^3 - 4x^2 + 2x - 1 = 0$ $p + q + r = -\frac{b}{a}$ $p + q + r = \underline{\underline{4}}$	(1)	
	(ii) $pq + qr + rp = \frac{c}{a}$ $pq + qr + rp = \underline{\underline{2}}$	(1)	
	(iii) $(p-q)^2 + (q-r)^2 + (r-p)^2$ $= p^2 - 2pq + q^2 + q^2 - 2qr + r^2 + r^2 - 2pr + p^2$ $= 2(p^2+q^2+r^2) - 2(pq+qr+rp)$ (1) $= 2[(p+q+r)^2 - 2(pq+rp+qr)] - 2(pq+qr+rp)$ (1) $= 2[4^2 - 2(2)] - 2(2)$ $= 20.$	(1)	

Qn	Solutions	Marks	Comments: Criteria
	e) $P(x) = ax^7+bx^6+1$ $P'(x) = 7ax^6+6bx^5$	$(\frac{1}{2})$	
	$P(1)=0$ $\therefore \quad a+b+1=0$ $a+b=-1$ ——(1)	$(\frac{1}{2})$	
	$P'(1)=0$ $\therefore \quad 7a+6b=0$ ——(2)	$(\frac{1}{2})$	
	(1) × 6 $6a+6b=-6$ ——(3)	$(\frac{1}{2})$	→ for ~~some~~ any method of solving simultaneous equations,
	(2) − (3) $\boxed{a=6}$	$(\frac{1}{2})$	
	sub $a=6$ into (1) $6+b=-1$ $\boxed{\therefore b=-7}$	$(\frac{1}{2})$	

Qn	Solutions	Marks	Comments: Criteria
	f.) Show $5^n+2(11^n)$ is a multiple of 3 for all positive integers n.		
	Step1. Show true for $n=1$ $5^1+2(11^1)=27$ which is divisible by 3. $\therefore$ true for $n=1$.	$(\frac{1}{2})$	
	Step2. Assume true for $n=k$ ie $5^k+2(11^k)=3P$ for any integer P.	(1)	
	Step3. Prove true for $n=k+1$ i.e prove $5^{k+1}+2(11^{k+1})=3Q$ for any integer Q LHS: $5^{k+1}+2(11^{k+1}) = 5^k.5+2(11^k.11)$ $=5[3P-2(11^k)]+22.11^k$ (from ~~ste~~ step2) $=15P-10(11^k)+22(11^k)$ $=15P-11^k(10-22)$ $=15P+12(11k)$ $=3[5P+4(11^k)]$ $=3Q$ (P and k are integers) $=$ RHS $\therefore$ true for $n=k+1$ if true for $n=k$.	(2)	
	Step4. Since true for $n=1$, then $5^n+2(11^n)$ is divisible by 3 for $n=2,3,4.....$ and all positive integers n.	$(\frac{1}{2})$	

Qn	Solutions	Marks	Comments: Criteria
2.	a.) $\sin 5A \cos 3A + \cos 5A \sin 3A = \sin(5A+3A)$		
	$= \sin 8A$	(1)	
	b.) α is in 1st quad β is in 2nd quad		
	 	(1)	
	$\tan(\alpha+\beta) = \dfrac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$	(1)	
	$= \dfrac{\frac{\sqrt{7}}{3} + \frac{\sqrt{5}}{-2}}{1 - \left(\frac{\sqrt{7}}{3}\right)\left(-\frac{\sqrt{5}}{2}\right)}$		
	$= -0.12$ (to 2 dec places)	(1)	½ if decimal is wrong but 2 d.p.
	c.) $\sqrt{2}\cos 2x = 1$		
	$\cos 2x = \frac{1}{\sqrt{2}}$	$(\frac{1}{2})$	
	$2x = 2n\pi \pm \cos^{-1}\frac{1}{\sqrt{2}}$		
	$= 2n\pi \pm \frac{\pi}{4}$ ✓	(1)	
	$\therefore x = \frac{1}{2}\left(2n\pi \pm \frac{\pi}{4}\right)$		
	$\therefore x = n\pi \pm \frac{\pi}{8}$ $n = 0, 1, 2, \cdots$	$(\frac{1}{2})$	

Qn	Solutions	Marks	Comments: Criteria
	d.) (i) $2\cos\theta + 2\cos\left(\theta + \frac{\pi}{3}\right)$		
	$= 2\cos\theta + 2\left[\cos\theta\cos\frac{\pi}{3} - \sin\theta\sin\frac{\pi}{3}\right]$	(1)	
	$= 2\cos\theta + 2\left(\frac{1}{2}\cos\theta - \frac{\sqrt{3}}{2}\sin\theta\right)$		
	$= 2\cos\theta + \cos\theta - \sqrt{3}\sin\theta$		
	$= 3\cos\theta - \sqrt{3}\sin\theta$	(1)	
	$= RHS$		
	(ii) $3\cos\theta - \sqrt{3}\sin\theta \longrightarrow R\cos(\theta + \alpha)$		
	$R = \sqrt{3^2 + (\sqrt{3})^2}$ $\tan\alpha = \frac{\sqrt{3}}{3}$		1 mark for R
	$= \sqrt{12}$ $\alpha = \frac{\pi}{6}$	(2)	1 mark for α.
	$= 2\sqrt{3}$		
	$\therefore 3\cos\theta - \sqrt{3}\sin\theta = 2\sqrt{3}\cos\left(\theta + \frac{\pi}{6}\right)$		
	(iii) $2\sqrt{3}\cos\left(\theta + \frac{\pi}{6}\right) = 3$ for $0 < \theta < 2\pi$		
	$\cos\left(\theta + \frac{\pi}{6}\right) = \frac{3}{2\sqrt{3}}$	$(\frac{1}{2})$	
	$\theta + \frac{\pi}{6} = \cos^{-1}\frac{2}{2\sqrt{3}}$	$(\frac{1}{2})$	
	$= \frac{\pi}{6}, 2\pi - \frac{\pi}{6}, 2\pi + \frac{\pi}{6}$	$(\frac{1}{2})$	
	$\therefore \theta = \frac{\pi}{6} - \frac{\pi}{6}, \frac{11\pi}{6} - \frac{\pi}{6}, \frac{13\pi}{6} - \frac{\pi}{6}$	$(\frac{1}{2})$	
	$\therefore \theta = \frac{10\pi}{6}$		

Qn	Solutions	Marks	Comments: Criteria
	e.)(i) $\tan 2x + \tan x = \frac{\sin 2x}{\cos 2x} + \frac{\sin x}{\cos x}$		
	$= \frac{\cos x \sin 2x + \sin x \cos 2x}{\cos 2x \cos x}$	(1)	
	Numerator......use $\sin(x+y) = \sin x \cos y + \cos x \sin y$		
	$= \frac{\sin(x+2x)}{\cos 2x \cos x}$	(1)	
	$= \frac{\sin 3x}{\cos 2x \cos x}$		
	$= RHS$		
	(ii) $\frac{\sin 3x}{\cos 2x \cos x} = 0$ for $0 < x < \frac{\pi}{2}$		
	$\sin 3x = 0$		
	$3x = \sin^{-1} 0$ 		
	$= 0, \pi$		
	$\boxed{\therefore x = \frac{\pi}{3}}$	1	
			

Qn	Solutions	Marks	Comments: Criteria
	f.) $\sin\theta = \frac{2t}{1+t^2}$ $\cos\theta = \frac{1-t^2}{1+t^2}$		
	$2\sin\theta + \cos\theta = 1$		
	$2\left(\frac{2t}{1+t^2}\right) + \frac{1-t^2}{1+t^2} = 1$	(1)	
	$\frac{4t + 1 - t^2}{1+t^2} = 1$		
	$4t + 1 - t^2 = 1 + t^2$		
	$0 = 2t^2 - 4t$	(1)	
	$0 = 2t(t-2)$		
	$\therefore t = 0$ or $t = 2$		
	but $t = \tan\frac{\theta}{2}$		
	$\tan\frac{\theta}{2} = 0$ or $\tan\frac{\theta}{2} = 2$	(1)	
	$\frac{\theta}{2} = 0,$ $\frac{\theta}{2} = 63.4°, 180 + 63.4°$		
	$\therefore \theta = 0, 360°$ $\theta = 126.8°, 486.8°$	(1)	
	$\therefore \theta = 126.8°$		
	(domain $0 < \theta < 360°$)		

Qn	Solutions	Marks	Comments: Criteria
	9.) LHS: $= \frac{2-\operatorname{cosec}^2\theta}{\operatorname{cosec}^2\theta+2\cot\theta}$ $\cos^2\theta+\sin^2\theta=1$ $1+\tan^2\theta=\sec^2\theta$ $\cot^2\theta+1=\operatorname{cosec}^2\theta$		
	$= \frac{2-(\cot^2\theta+1)}{\cot^2\theta+1+2\cot\theta}$ OR $= \frac{2-\frac{1}{\sin^2\theta}}{\frac{1}{\sin^2\theta}+\frac{2\cos\theta}{\sin\theta}}$	(1)	1 mark for substitution of identities
	$= \frac{2-\cot^2\theta-1}{\cot^2\theta+2\cot\theta+1}$ $= \frac{2\sin^2\theta-1}{1+2\sin\theta\cos\theta}$ $= \frac{2\sin^2\theta-(\sin^2\theta+\cos^2\theta)}{\sin^2\theta+2\sin\theta\cos\theta+\cos^2\theta}$		Note $1=\sin^2\theta+\cos^2\theta$
	$= \frac{1-\cot^2\theta}{(\cot\theta+1)^2}$ $= \frac{\sin^2\theta-\cos^2\theta}{(\sin\theta+\cos\theta)^2}$	(1)	
	$= \frac{\sin\theta-\cos\theta}{\sin\theta+\cos\theta}$ $= \frac{(1-\cot\theta)\cancel{(1+\cot\theta)}}{(\cot\theta+1)^{\cancel{2}}}$ $=$ RHS.		
	$= \frac{1-\cot\theta}{1+\cot\theta}$		
	$= \frac{1-\frac{\cos\theta}{\sin\theta}}{1+\frac{\cos\theta}{\sin\theta}}$		
	$= \frac{\sin\theta-\cos\theta}{\cancel{\sin\theta}} \times \frac{\cancel{\sin\theta}}{\sin\theta+\cos\theta}$	(1)	
	$= \frac{\sin\theta-\cos\theta}{\sin\theta+\cos\theta}$		
	$=$ RHS.		