St Catherine's
School
Waverley, Sydney

Student Number:..

Year 12
Assessment Task 1
2010

Mathematics Extension II

Time allowed: 55 minutes

Course weighting: 15%

General Instructions

- Attempt ALL questions
- Write your Student NUMBER at the top of this page and on each writing booklet used

Sections **Marks**

Total marks

TABLE OF STANDARD INTEGRALS

$$\int x^n dx = \frac{1}{n+1}x^{n+1}; \quad n \neq -1$$

$$\int \frac{1}{x} dx = \ln x, \quad x > 0$$

$$\int e^{ax} dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

Note: $\ln x = \log_e x, \quad x > 0$

Question 1.

a) Let $z = 1 + 2i$ and $w = 1 + i$ find in the form $x + iy$

(i) $z\overline{w}$ 1m

(ii) $\frac{1}{w}$ 1m

(iii) Find the Argument of $\boxed{\frac{1}{w}}$ 1m

b) If w is a complex root of the equation $z^3 = 1$, **note that** w^2 is the other complex root. Note also that. $1 + w + w^2 = 0$ (No need to prove this)

Form a quadratic equation whose roots are $3 + w$ *and* $3 + w^2$ 3m

Question: 2

a) Given that $1 + 2i$ is a root of the polynomial $P(z) = z^4 - 2z^3 + 6z^2 - 2z + 5$, factorise P(Z) in the Complex Number system. 4m

b) Given that $Q(x) = x^4 + x^3 - 9x^2 + 11x - 4$, has a root of multiplicity 3, factorise $Q(x)$. 4m

c) Express $\frac{x^3 + 2}{x^2 - 4}$ as a sum of partial fractions. 3m

Question 3

Sketch the locus of z: in each of the following: Clearly state the feature of each locus.

(i) $|z - i| = |z - 2|$. 2m

(ii) $\frac{z - 4i}{z - 1}$ is purely imaginary Let $z = x + iy$ 3m

Question 4

$\alpha,\ \beta, \gamma$ are the roots of the polynomial equation $P(x) = 0$,
where $P(x) = x^3 - 3x + 1$,

write down the polynomial equation, whose roots are
$\alpha^2 + 2,\ \beta^2 + 2,\ \gamma^2 + 2$ 3m

Question 5

(i) Solve for z: $z^7 = 1$, in the Complex Number System, where z is a complex number.

2m

(ii) Show that if w is a complex root of $z^7 = 1$,
w^2, w^3, w^4, w^5, w^6 are the other complex roots.
Also Identify the complex conjugates in these complex roots

2m

(iii) Factorise fully $z^7 - 1$ in the field of Real Numbers.

2m

(iv) Explain why $w, w^2, w^3, w^4, w^5, w^6$ are the roots of the equation
$z^6 + z^5 + z^4 + z^3 + z^2 + z + 1 = 0$

1m

(v) Show that $\cos\frac{2\pi}{7} + \cos\frac{4\pi}{7} + \cos\frac{6\pi}{7} = -\frac{1}{2}$

2m

END OF PAPER

Q.1

$z = 1+2i$; $w = 1+i$

(i) $z\bar{w} = (1+2i)(1-i)$

$= 1 - 2i^2 - i + 2i$

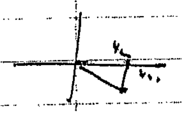
$= 3 + i$

(ii) $\frac{1}{w} = \frac{1}{1+i} \cdot \frac{1-i}{1-i} = \frac{1-i}{1-i^2}$

$= \frac{1}{2} - \frac{1}{2}i$

(iii) Arg of $\frac{1}{w}$ = Arg $\frac{1}{2} - \frac{1}{2}i$

$= -\pi/4$.



b) Sum of roots is $3+\omega + 3+\omega^2$

$= 6 + \omega + \omega^2$

$= 6 - 1 = 5$ $(1+\omega+\omega^2 = 0$
$\omega+\omega^2 = -1)$

Product of roots is $(3+\omega)(3+\omega^2)$

$= 9 + 3\omega + 3\omega^2 + \omega^3$

$= 10 + 3(\omega+\omega^2)$

$= 10 - 3$.

$= 7$

∴ The equation is

$x^2 - 5x + 7 = 0$.

Q.2 $1+2i$ is a root; coeffs of $P(z)$ are real

∴ $1-2i$ is a root. (1M)

∴ $(z-(1+2i))(z-(1-2i))$ is a factor.

$z^2 - (2)z + (1-4i^2)$ " "

$z^2 - 2z + 5$ is a factor. (1M)

$$\begin{array}{r|l} & z^2 + 1 \\ \hline z^2-2z+5 & z^4 - 2z^3 + 6z^2 - 2z + 5 \\ & z^4 - 2z^3 + 5z^2 \\ \hline & z^2 - 2z + 5 \\ & z^2 - 2z + 5 \\ \hline \end{array}$$ (1M)

∴ $P(z) = (z-(1+2i))(z-(1-2i))(z-i)(z+i)$

(M)

(b) $Q(z) = z^4 + z^3 - 9z^2 + 11z - 4$ has a ($\frac{1}{2}$)

if α is a root of multiplicity 3 for $Q(z)$

" " " " 2 for $Q'(z)$

" " " " 1 for $Q''(z)$

$Q'(z) = 4z^3 + 3z^2 - 18z + 11$. ($\frac{1}{2}$)

$Q''(z) = 12z^2 + 6z - 18$

$= 6(2z^2 + z - 3)$

$= 6(2z+3)(z-1)$ (1)

α could be 1 or $-\frac{3}{2}$ $Q'(1) = 4 + 3 - 18 + 11$

$= 0$ $\frac{1}{2}$

$Q(1) = 1 + 1 - 9 + 11 - 4$ $\frac{1}{2}$

$= 0$

$\therefore \alpha = 1$

Q

$Q(z) = (z-1)^3 \ (z+4)$ Observation

c) $$\begin{array}{r} x \\ x^2-4 \overline{) x^3+2} \\ \underline{x^3-4x} \\ 4x+2 \end{array}$$

$$\therefore \frac{x^3+2}{x^2-4} = x + \frac{4x+2}{x^2-4}$$

$$= x + \frac{A}{x-2} + \frac{B}{x+2}$$

$\therefore 4x+2 = A(x+2) + B(x-2)$

$x=2$ $\quad 10 = 4A$; $A = \frac{5}{2}$

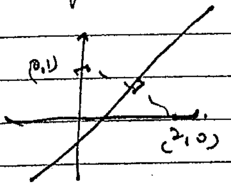
$x=-2$ $\quad -6 = -4B$ $\quad B = \frac{3}{2}$

$$\therefore \frac{x^3+2}{x^2-4} = x + \frac{5}{2(x-2)} + \frac{3}{2(x+2)}$$

Q.3

$|z-i| = |z-2)$

dist. from $(0,1)$ = dist. from $(2,0$



Perpendicular bisector of the interval joining $(0,1)$ and $(2,0)$

(ii) $$\frac{z-4i}{z-1} = \frac{x+i(y-4)}{(x-1)+iy} \cdot \frac{(x-1)-iy}{(x-1)-iy}$$

$$= \frac{x(x-1) + y(y-4) + i\left((x-1)(y-4) - xy\right)}{(x-1)^2+y^2}$$

Purely Imaginary $\therefore$ Real part = 0

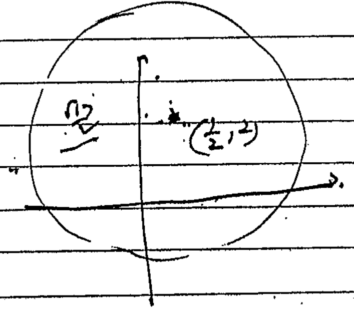
$x(x-1) + y(y-4) = 0$

$x^2 - x + y^2 - 4y = 0$

$(x-\frac{1}{2})^2 + (y-2)^2 = 4 + \frac{1}{4}$

– Circle : Centre $(\frac{1}{2}, 2)$

$r = \frac{\sqrt{17}}{2}$.



Q. 4

$P(x) = x^3 - 3x + 1$

$y = x^2 + 2 \quad ; \quad x = \sqrt{y-2}$

The required polynomial is

$(\sqrt{y-2})^3 - 3\sqrt{y-2} + 1 = 0$

$\sqrt{y-2}\,(y-2-3) = -1$

$(y-2)\,(y-5)^2 = 1$ ③

or $(x-2)\,(x-5)^2 = 1.$

(or find $Q(x)$ with roots $\alpha^2, \beta^2, \gamma^2$
& then find the polynomial with roots $\alpha^2+2, \beta^2+2, \gamma^2+2$)

Q. 5

$z^7 = 1.$

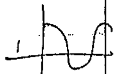
Let $z = r\,\mathrm{cis}\,\theta$

$|z| = r$

$|z^7| = 1$

$|z|^7 = 1$

$r^7 = 1$



$r = 1$ (r is real

$\therefore z = \mathrm{cis}\,\theta$

$z^7 = 1$

$\mathrm{cis}\,7\theta = 1$ (De Moivre's th.)

$\cos 7\theta = 1 \; ; \; \sin 7\theta = 0$

$7\theta = 0, 2\pi, 4\pi, 6\pi, 8\pi, 10\pi, 12\pi$

$\theta = 0, \frac{2\pi}{7}, \frac{4\pi}{7}, \frac{6\pi}{7}, \frac{8\pi}{7}, \frac{10\pi}{7}, \frac{12\pi}{7}$

$\therefore$ The roots are
$\mathrm{cis}\,0\,;\; \mathrm{cis}\,\frac{2\pi}{7},\; \mathrm{cis}\,\frac{4\pi}{7},\; \mathrm{cis}\,\frac{6\pi}{7},\; \mathrm{cis}\,\frac{8\pi}{7},\; \mathrm{cis}\,\frac{10\pi}{7}.$
& $\mathrm{cis}\,\frac{12\pi}{7}$

②

Let $\omega = \mathrm{cis}\,\frac{2\pi}{7}$

$; \; \mathrm{cis}\,\frac{4\pi}{7} = \left(\mathrm{cis}\,\frac{2\pi}{7}\right)^2$ (De Moivre's th.)

$= \omega^2$

$\mathrm{cis}\,\frac{6\pi}{7} = \left(\mathrm{cis}\,\frac{2\pi}{7}\right)^3$ (")

$= \omega^3$

$\mathrm{cis}\,\frac{8\pi}{7} = \left(\mathrm{cis}\,\frac{2\pi}{7}\right)^4$ (")

$= \omega^4 \quad = \overline{\omega^3}$

$\mathrm{cis}\,\frac{10\pi}{7} = \omega^5 = \overline{\omega^2}$

② $\mathrm{cis}\,\frac{12\pi}{7} = \omega^6. = \overline{\omega}$

(III) $z^7 - 1 = (z - \mathrm{cis}\,0)(z-\omega)(z-\overline{\omega})(z-\omega^2)(z-\overline{\omega^2})$
$(z-\omega^3)(z-\overline{\omega^3})$

note: $(z-\omega)(z-\overline{\omega}) = z^2 - (\omega+\overline{\omega}) + \omega\overline{\omega}$

$= (z-1)\left(z^2 - 2z\cos\frac{2\pi}{7} + 1\right)\left(z^2 - 2z\cos\frac{4\pi}{7} + 1\right)$

② $\left(z^2 - 2z\cos\frac{6\pi}{7} + 1\right)$

(IV) $(z^7 - 1) = (z-1)(z^6 + z^5 + z^4 + z^3 + z^2 + z + 1)$

The roots are $1, \omega, \omega^2, \omega^3, \omega^4, \omega^5, \omega^6$

① $\therefore \omega, \omega^2, \omega^3, \omega^4, \omega^5, \omega^6$ are the roots of
$z^6 + z^5 + z^4 + z^3 + z^2 + z + 1 = 0$

(V) $\therefore \omega + \omega^2 + \omega^3 + \omega^4 + \omega^5 + \omega^6 = \frac{-1}{1}$

② or $\omega + \omega^2 + \omega^3 + \overline{\omega^3} + \overline{\omega^2} + \overline{\omega} = -1$

$$(\omega + \bar{\omega}) + (\omega^2 + \overline{\omega^2}) + (\omega^3 + \overline{\omega^3}) = -1$$

$$2\cos\frac{2\pi}{7} + 2\cos\frac{4\pi}{7} + 2\cos\frac{6\pi}{7} = -1$$

$$\therefore \cos\frac{2\pi}{7} + \cos\frac{4\pi}{7} + \cos\frac{6\pi}{7} = \frac{1}{2}$$