Student Number:________________

Mathematics Extension 2

St. Catherine's School Waverley

2011

ASSESSMENT TASK 1
(15%)

Total marks – 46

- Attempt Questions 1–2
- Marks for each question are indicated on this page

General Instructions

- Working time – 60 minutes
- Start each question on a **new page** in your answer booklet.
- Write using black or blue pen only.
- Board-approved calculators may be used.
- All necessary **working** must be shown.
- Marks may be deducted for careless or badly arranged work.

TEACHER'S USE ONLY

Question 1	/24
Question 2	/22
Total	/46

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1;\ x \neq 0, \text{ if } n<0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x>0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a>0,\ -a<x<a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x>a>0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE: $\ln x = \log_e x, \quad x>0$

Question 2 **(22 marks)**

(a) When $P(x)=x^4+ax^2+bx+2$ is divided by (x^2+1) the remainder is $-x+1$. **3**

By expressing $P(x)=D(x).Q(x)+R(x)$, find the values of a and b.

(b) (i) Show that $\dfrac{18x^2}{9x^2-16}\equiv 2+\dfrac{32}{9x^2-16}$ **1**

(ii) Hence, find the constants A, B, C such that **2**

$$\frac{18x^2}{9x^2-16}\equiv A+\frac{B}{3x-4}+\frac{C}{3x+4}$$

(c) The cubic equation $x^3-5x^2+5=0$ has roots α,β and γ

(i) Find the equation that has roots $\alpha-1$, $\beta-1$ and $\gamma-1$ **2**

(ii) By using $\alpha^2+\beta^2+\gamma^2=(\alpha+\beta+\gamma)^2-2(\alpha\beta+\beta\gamma+\alpha\gamma)$ **3**

Find the value of $\alpha^3+\beta^3+\gamma^3$

(d) Given that $P(x)=x^3+3px+q$ has a double root at $x=k$

(i) Show that $p=-k^2$ **2**

(ii) By also finding q in terms of k, show that $4p^3+q^2=0$ **2**

(e) Given that $z=\cos\theta+i\sin\theta$

(i) Show that $z^n+\dfrac{1}{z^n}=2\cos n\theta$ **1**

(ii) Hence, using the expansion of $\left(z+\dfrac{1}{z}\right)^4$ show that **3**

$$\cos^4\theta=\frac{1}{8}(\cos 4\theta+4\cos 2\theta+3)$$

(f) If $P(x)=x^4-2x^3-x^2+6x-6$ has a zero $(1-i)$ find all the zeros of $P(x)$ and factorise $P(x)$ fully over the real numbers. **3**

Question 1 **(24 marks)** **Marks**

(a) Two complex numbers are given by :

$$z=3-4i \quad\text{and}\quad w=2-2i$$

(i) Find the value of the product $\bar{z}w$ **2**

(ii) Find the two square roots of z **2**

(b) Calculate the modulus and argument of the product of the roots of the equation $(5+3i)z^2-(1+4i)z+(8-2i)=0$ **3**

(c) (i) If $w=\dfrac{1+i\sqrt{3}}{2}$ show that $w^3=-1$ **2**

(ii) Hence, or otherwise, evaluate w^{10} **2**

(d) If z moves in the Argand Diagram such that $|z-i|=\sqrt{2}|z+i|$ **2**

Find the equation of the locus of the complex number z

(e) If $z=2-i$ find real numbers p and q such that $pz+\dfrac{q}{z}=1$ **2**

(f) On the Argand diagram, sketch *neatly* the region where the inequalities **3**

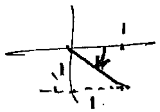
$1\le|z-i|\le 2$ and $\dfrac{\pi}{4}\le\arg z\le\dfrac{\pi}{2}$ are satisfied simultaneously.

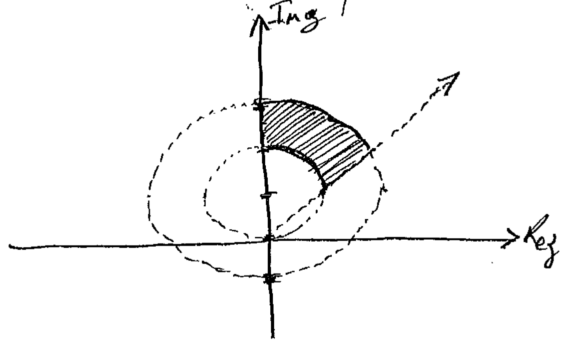
(g) (i) Factorize z^5-1 into real linear and quadratic factors. **3**

(ii) Hence, and noting that $z^5-1=(z-1)(z^4+z^3+z^2+z+1)$, deduce that ;

(a) $\cos\dfrac{2\pi}{5}+\cos\dfrac{4\pi}{5}=-\dfrac{1}{2}$ **2**

(b) $\cos\dfrac{2\pi}{5}.\cos\dfrac{4\pi}{5}=-\dfrac{1}{4}$ **1**

Qn	Solutions	Marks	Comments: Criteria
1.	a) $z = 3-4i \qquad w = 2-2i$		
	(i) $(3+4i)(2-2i) = 6-6i+8i+8$	1	
	$= 14+2i$	1	
	(ii) $\sqrt{3-4i} = a+ib$		
	$3-4i = a^2-b^2+2abi$		
	equating real and imaginary parts		
	$a^2-b^2=3$ —① $\quad 2ab=-4$ —②		
	also $a^2+b^2=5$ —③ | NOTE: $(a^2+b^2)^2=(a^2-b^2)^2+4a^2b^2$	1	
	①+③ $2a^2 = 8$ | $= 9+16$		
	$a = \pm 2$ | $= 25$		
	sub in ② $b = \mp 1$ | $\therefore a^2+b^2=5$	1	
	$\therefore \sqrt{3-4i} = \pm(2-i)$	1	
	b) $\dfrac{8-2i}{5+3i}$ is product of roots		
	$= \dfrac{8-2i}{5+3i} \times \dfrac{5-3i}{5-3i}$		
	$= \dfrac{40-10i-24i-6}{25+9}$		
	$= \dfrac{34-34i}{34}$	1	
	$= 1-i$ 		
	modulus $=\sqrt{2}$ argument $= -\dfrac{\pi}{4}$	1	
	c) (i) $\omega = \dfrac{1+i\sqrt{3}}{2} = \dfrac{1}{2}+\dfrac{\sqrt{3}}{2}i$	1	
	$\lvert\omega\rvert = 1 \qquad \arg\omega = \dfrac{\pi}{3}$		
	$\therefore \omega^3 = \left(\operatorname{cis}\dfrac{\pi}{3}\right)^3$		
	$= \operatorname{cis}\pi$	1	
	$= -1$		
	(ii) $\omega^{10} = (\omega^3)^3 \cdot \omega$	1	
	$= -\omega$		
	$= \dfrac{-1-i\sqrt{3}}{2}$	1	

Qn	Solutions	Marks	Comments: Criteria
1	d) $\lvert z-i\rvert = \sqrt{2}\,\lvert z+i\rvert$ let $z=x+iy$		
	$\lvert x+iy-i\rvert = \sqrt{2}\,\lvert x+iy+i\rvert$		
	$\lvert x+(y-1)i\rvert = \sqrt{2}\,\lvert x+(y+1)i\rvert$	1	
	$\sqrt{x^2+(y-1)^2} = \sqrt{2}\cdot\sqrt{x^2+(y+1)^2}$		
	$x^2+y^2-2y+1 = 2(x^2+y^2+2y+1)$		
	$x^2+y^2-2y+1 = 2x^2+2y^2+4y+2$		
	$\therefore x^2+y^2+6y+1=0$		
	or $x^2+(y+3)^2=8$ better for describing the locus if required	1	
	e) $z=2-i \quad \therefore p(2-i)+\dfrac{q}{2-i}=1$		
	$2p-ip+\dfrac{q(2+i)}{5}=1$		
	$2p-ip+\dfrac{2}{5}q+\dfrac{q}{5}i = 1+0i$	1	
	$\therefore 2p+\dfrac{2}{5}q=1$ —①		
	$-p+\dfrac{q}{5}=0$ —②		
	from ② $p=\dfrac{q}{5}$		
	Sub in ① $\dfrac{2q}{5}+\dfrac{2q}{5}=1$		
	$\dfrac{4q}{5}=1$		
	$q=\dfrac{5}{4}$		
	$\therefore p=\dfrac{1}{4}$	1	
	f). 	3	

Qn	Solutions	Marks	Comments: Criteria
1	g) (i) $z^5-1=0$ let $z=\cos\theta+i\sin\theta$		
	$\cos 5\theta+i\sin 5\theta=\text{cis}(0+2k\pi)$ $k=0,1,2,3,4$		
	$\therefore\ \text{cis}\,5\theta=\text{cis}(0+2k\pi)$ "		
	$5\theta=0,\ 2\pi,\ 4\pi,\ 6\pi,\ 8\pi$		
	$\theta=0,\ \frac{2\pi}{5},\ \frac{4\pi}{5},\ \frac{6\pi}{5},\ \frac{8\pi}{5}$		
	$\therefore$ roots of $z^5-1=0$		
	are $1,\ \text{cis}\frac{2\pi}{5},\ \text{cis}\frac{4\pi}{5},\ \text{cis}\frac{-2\pi}{5},\ \text{cis}\frac{-4\pi}{5}$	1	
	$\therefore z^5-1=(z-1)(z-\text{cis}\frac{2\pi}{5})(z-\text{cis}\frac{-2\pi}{5})(z-\text{cis}\frac{4\pi}{5})(z-\text{cis}\frac{-4\pi}{5})$		
	$=(z-1)(z^2-2\cos\frac{2\pi}{5}z+1)(z^2-2\cos\frac{4\pi}{5}z+1)$	2	
	(ii) (a) now $z^5-1=(z-1)(z^4+z^3+z^2+z+1)$		
	$\therefore z^4+z^3+z^2+z+1=(z^2-2\cos\frac{2\pi}{5}z+1)(z^2-2\cos\frac{4\pi}{5}z+1)$	1	
	equating coefficients of z^3 gives		
	$-2\cos\frac{2\pi}{5}-2\cos\frac{4\pi}{5}=1$		
	$\therefore \cos\frac{2\pi}{5}+\cos\frac{4\pi}{5}=-\frac{1}{2}$	1	
	(b) Equating coefficients of z^2 gives		
	$4\cos\frac{2\pi}{5}\cdot\cos\frac{4\pi}{5}+2=1$		
	$\therefore \cos\frac{2\pi}{5}\cdot\cos\frac{4\pi}{5}=-\frac{1}{4}$	1	

Qn	Solutions	Marks	Comments: Criteria
Q2	a) $P(x)=D(x)Q(x)+R(x)$		
	$x^4+ax^2+bx+2=(x^2+1)Q(x)-x+1$	1	
	let $x=i$ $1-a+bi+2=-i+1$	1	
	$-a+bi+3=1-i$		
	$\therefore -a+3=1$ } equating real		
	$\therefore a=2$ & imaginary	1	
	and $b=-1$ parts		
	b)(i) $\frac{18x^2}{9x^2-16}=\frac{18x^2-32+32}{9x^2-16}$		
	$=2+\frac{32}{9x^2-16}$		
	$=RHS$	1	
	now $\frac{18x^2}{9x^2-16}=A+\frac{B}{3x-4}+\frac{C}{3x+4}$		
	$\therefore A=2$ and $\frac{32}{9x^2-16}=\frac{B}{3x-4}+\frac{C}{3x+4}$		
	$\therefore 32\equiv 3Bx+4B+3Cx-4C$		
	$\therefore 3B+3C=0$ — (1)	1	
	$4B-4C=32$ — (2)		
	from (1) $C=-B$		
	sub in (2) $4B+4B=32$		
	$B=4$	1	
	$C=-4$		
	$\therefore \frac{18x^2}{9x^2-16}=2+\frac{4}{3x-4}-\frac{4}{3x+4}$		

Qn	Solutions	Marks	Comments: Criteria
Q2	c) $x^3-5x^2+5=0 \qquad \alpha,\beta,\gamma$		
	(i) let $x=\alpha-1$	1	
	$\therefore \alpha = x+1$		
	$\therefore (x+1)^3-5(x+1)^2+5=0$		
	$x^3+3x^2+3x+1-5x^2-10x-5+5=0$		
	$\therefore x^3-2x^2-7x+1=0$	1	
	(ii) for $x^3-5x^2+5=0$		
	$\alpha+\beta+\gamma=5$		
	$\alpha\beta+\alpha\gamma+\beta\gamma=0$		
	$\alpha\beta\gamma=-5$		
	now $\alpha^2+\beta^2+\gamma^2=(\alpha+\beta+\gamma)^2-2(\alpha\beta+\alpha\gamma+\beta\gamma)$		
	$=25-2(0)$		
	$=25$		
	if α is a root $\alpha^3-5\alpha^2+5=0$ — ①		
	$\therefore \beta$ " " " $\beta^3-5\beta^2+5=0$ — ②		
	$\therefore \gamma$ " " " $\gamma^3-5\gamma^2+5=0$ — ③		
	①+②+③ $\alpha^3+\beta^3+\gamma^3-5(\alpha^2+\beta^2+\gamma^2)+15=0$		
	$\therefore \alpha^3+\beta^3+\gamma^3=5(\alpha^2+\beta^2+\gamma^2)-15$		
	$=125-15$		
	$=110$		
	d) $P(x)=x^3+3px+q=0$ double root at $x=k$		
	(i) $P'(x)=3x^2+3p$		
	now $P'(k)=0$		
	$\therefore 3k^2+3p=0$		
	$\therefore p=-k^2$		

Qn	Solutions	Marks	Comments: Criteria
2	d)(ii) $P(k)=0 \therefore k^3-3k^3+q=0$		
	$\therefore q=2k^3$	1	
	now $4p^3+q^2=4(-k^2)^3+(2k^3)^2$		
	$=-4k^6+4k^6$		
	$=0.$	1	
	e) $z=\cos\theta+i\sin\theta$		
	(i) $z^n=\cos n\theta+i\sin n\theta$ (De Moivres)		
	$\frac{1}{z^n}=\frac{1}{\cos n\theta+i\sin n\theta}\times\frac{\cos n\theta-i\sin n\theta}{\cos n\theta-i\sin n\theta}$		
	$=\cos n\theta-i\sin n\theta$	1	
	now $z^n+\frac{1}{z^n}=2\cos n\theta$		
	(ii) $\left(z+\frac{1}{z}\right)^4=z^4+4z^3\cdot\frac{1}{z}+6z^2\cdot\frac{1}{z^2}+4z\cdot\frac{1}{z^3}+\frac{1}{z^4}$		
	$=\left(z^4+\frac{1}{z^4}\right)+4\left(z^2+\frac{1}{z^2}\right)+6.$	1	
	$\therefore (2\cos\theta)^4=2\cos4\theta+8\cos2\theta+6$	1	
	$16\cos^4\theta=2\cos4\theta+8\cos2\theta+6$		
	$\cos^4\theta=\frac{1}{8}\cos4\theta+\frac{1}{2}\cos2\theta+\frac{3}{8}$		
	$=\frac{1}{8}\left[\cos4\theta+4\cos2\theta+3\right]$	1	

Qn	Solutions	Marks	Comments: Criteria
2	f) $P(x) = x^4 - 2x^3 - x^2 + 6x - 6$		
	$P(1-i) = 0 \quad \therefore P(1+i) = 0$	1	
	real coefficients $\Rightarrow$ conjugate root thm		
	$\therefore [x-(1-i)][x-(1+i)]$ is a factor of $P(x)$.		
	$\therefore x^2 - (1-i)x - (1+i)x + 2$ is a factor		
	$\therefore (x^2 - 2x + 2)$ is a factor.	1	
	$\therefore x^4 - 2x^3 - x^2 + 6x - 6 = (x^2 - 2x + 2)(x^2 - 3)$		
	$\therefore$ factors are $(x^2 - 2x + 2)(x - \sqrt{3})(x + \sqrt{3})$	1	