Student Number: 9

St. Catherine's School Waverley

Mathematics Extension 2

2012
ASSESSMENT TASK 1
(15%)

FEBRUARY 27, 2012

Total marks – 44

- Attempt Questions 1–2
- Marks for each question are indicated on this page

General Instructions

- Working time – 60 minutes
- Start each question on a **new page** in your answer booklet.
- Write using black or blue pen only.
- Board-approved calculators may be used.
- All necessary **working** must be shown.
- Marks may be deducted for careless or badly arranged work.

TEACHER'S USE ONLY

Question 1	/21
Question 2	/23
Total	/44

STUDENT NUMBER/NAME: ....

STANDARD INTEGRALS

$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n<0$

$\int \frac{1}{x}\,dx = \ln x, \quad x>0$

$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$

$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$

$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$

$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$

$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$

$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$

$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a>0, \ -a<x<a$

$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x>a>0$

$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$

NOTE: $\ln x = \log_e x, \quad x>0$

Question 1 (21 marks) **Marks**

(a) Let $z=\dfrac{3+4i}{1+2i}$. Express z in the form $a+ib$ where a and b are real **2**

(b) Let $w=1-i\sqrt{3}$

(i) Express w in modulus-argument form. **2**

(ii) Hence, or otherwise, write the exact value of w^5 in the form $a+ib$ where a and b are real. **2**

(c) (i) Show that $(1-2i)^2=-3-4i$ **1**

(ii) Hence solve the equation $z^2-5z+(7+i)=0$ **2**

(d) On an Argand Diagram sketch *neatly* and shade the region where both **2**

$|z-(1+i)|\le\sqrt{2}$ and $0\le \arg z\le\dfrac{\pi}{2}$ hold simultaneously.

(e) (i) Find all solutions to the equation $z^3-1=0$ in modulus-argument form. **2**

(ii) If w is a complex root of $z^3-1=0$, show that w^2 is also a complex root. **1**

(iii) Hence prove that $(1+w)^3=-1$ **2**

(f) If z represents a complex number, find the equation, in Cartesian form, of the locus of the point z if, **3**

$$\mathrm{Re}\left(\frac{z-4}{z}\right)=0$$

(g) *Describe the locus* of Z on the Argand diagram if **2**

$\arg(z-1)-\arg(z+1)=\dfrac{\pi}{2}$ and give its Cartesian equation

Question 2 (23 marks)

(a) Given that $z=\cos\theta+i\sin\theta$

(i) Show that $z^n+\dfrac{1}{z^n}=2\cos n\theta$ **2**

(ii) Hence show that $\cos^4\theta=\dfrac{1}{8}(\cos4\theta+4\cos2\theta+3)$ **3**

(b) (i) Show that $\dfrac{x^3-4x-10}{x^2-x-6}=x+1+\dfrac{3x-4}{x^2-x-6}$ **2**

(ii) Hence express $\dfrac{x^3-4x-10}{x^2-x-6}$ as the sum of partial fractions. **2**

(c) The roots of $x^3+5x^2+11=0$ are α,β and γ

(i) Find the polynomial equation with roots $\dfrac{1}{\alpha},\dfrac{1}{\beta}$ and $\dfrac{1}{\beta}$ **2**

(ii) Find the value of $\alpha^3+\beta^3+\gamma^3$ **2**

(d) When the polynomial is $P(x)$ is divided by $(x-2)$ and $(x-3)$ the respective remainders are 4 and 9. **3**

Determine the remainder when $P(x)$ is divided by $(x-2)(x-3)$

(e) (i) Determine the complex roots of $z^6=1$ in the form $\cos\theta+i\sin\theta$ **3**

(ii) Hence factorise $z^6-1=0$ over the complex field **1**

(iii) Factorise $z^6-1=0$ over real field using linear and quadratic factors **3**

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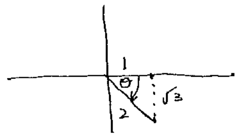
Marking Scheme for Task: TASK 1 Academic Year: 2011-12

Q	Solutions	Marks	Comments
1. a)	$\frac{3+4i}{1+2i} \times \frac{1-2i}{1-2i} = \frac{3-2i+8}{5}$ $= \frac{11}{5} - \frac{2}{5}i$	2	
b) (i)	$\omega = 1-\sqrt{3}i$ modulus $= 2$ argument $= -\frac{\pi}{3}$  $\tan(-\theta) = \sqrt{3}$ $\therefore \theta = -\frac{\pi}{3}$ $\therefore \omega = 2\,\text{Cis}\left(-\frac{\pi}{3}\right)$	1 1	
(ii)	$\omega^5 = 2^5 \text{cis}\left(-\frac{5\pi}{3}\right)$ by De Moivres' $= 32\left[\cos\left(-\frac{5\pi}{3}\right) + i\sin\left(-\frac{5\pi}{3}\right)\right]$ $= 32\left[\frac{1}{2} + \frac{\sqrt{3}i}{2}\right]$ $= 16 + 16\sqrt{3}i$ or $16(1+\sqrt{3}i)$	1 1	
c) (i)	$(1-2i)^2 = 1-4i-4$ $= -3-4i$	1	
(ii)	$z^2 - 5z + (7+i) = 0$ $\therefore z = \frac{5 \pm \sqrt{25-4(7+i)}}{2}$ $= \frac{5 \pm \sqrt{-3-4i}}{2}$ $= \frac{5 \pm (1-2i)}{2}$ $= \frac{6-2i}{2}, \frac{4+2i}{2}$ $= 3-i, 2+i$ c/. sum of roots $3-i+2+i = 5$ ✓ product of roots $(3-i)(2+i)$ $= 6+i+1$ $= 7+i$ ✓	1 1	

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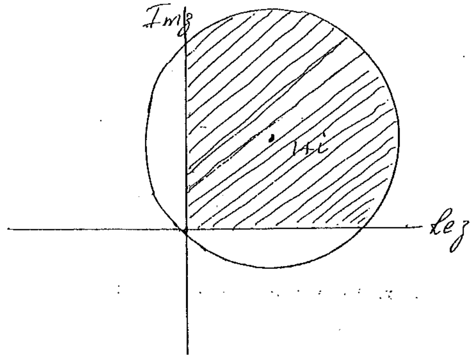
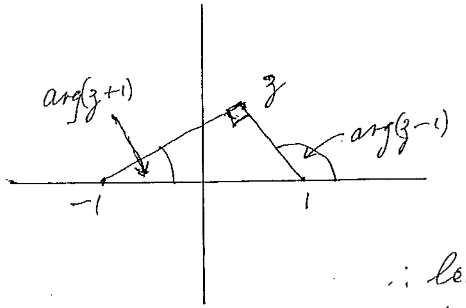
Marking Scheme for Task: Academic Year: 2011-12

Q	Solutions	Marks	Comments
Q1 d)		2	
e) (i)	$z^3 - 1 = 0$ ie $z^3 = 1$ let $z = r\text{cis}\theta$ $\therefore z^3 = r^3 \text{cis}3\theta$ and $1 = 1\text{Cis}0°$ $\therefore r^3 = 1$ $\therefore r = 1$ (r real) $3\theta = 0, 2\pi, 4\pi, \ldots$ $\therefore \theta = 0, \frac{2}{3}\pi, \frac{4}{3}\pi\left(-\frac{2\pi}{3}\right)$. $\therefore$ roots are $\text{cis}0°$, $\text{cis}\frac{2\pi}{3}$, $\text{cis}-\frac{2\pi}{3}$	1 1	
(ii)	let $\omega = \text{cis}\frac{2\pi}{3}$ $\therefore \omega^2 = \text{cis}\left(\frac{4\pi}{3}\right)$ $= \text{cis}\left(-\frac{2\pi}{3}\right)$	1	
(iii)	LHS $= (1+\omega)^3$ $= 1 + 3\omega + 3\omega^2 + \omega^3$ $= 3(1+\omega+\omega^2) - 2 + 1$ [note $1+\omega+\omega^2 = 0$ and $\omega^3 = 1$] $= -1$	1 1	

Q	Solutions	Marks	Comments
Q1f)	$Re\left(\frac{z-4}{z}\right)=0$		
	let $z=x+iy$	1	
	$\therefore \frac{z-4}{z} = \frac{(x-4)+iy}{x+iy} \times \frac{x-iy}{x-iy}$		
	$= \frac{x^2-4x+ixy-ixy+4iy+y^2}{x^2+y^2}$		
	$= \frac{x^2-4x+y^2}{x^2+y^2} + \frac{4iy}{x^2+y^2}$	1	
	$\therefore \frac{x^2-4x+y^2}{x^2+y^2}=0$		
	$\therefore x^2-4x+y^2=0$		
	$x^2-4x+4+y^2=4$		
	$(x-2)^2+y^2=4$ is the equation of the Locus.	1	
Q1g)	$arg(z-1)-arg(z+1)=\frac{\pi}{2}$		
		1	
	by exterior angle rule angle at z is $\frac{\pi}{2}$		
	$\therefore$ locus is a semi circle centre: origin radius: 1 equation $y=\sqrt{1-x^2}$	1	

Q	Solutions	Marks	Comments
Q2a)	(i) $z=\cos\theta+i\sin\theta$		
	$z^n=\cos n\theta+i\sin n\theta$		
	$\frac{1}{z^n}=\frac{\cos n\theta-i\sin n\theta}{\cos^2 n\theta+i\sin^2\theta}$		
	$=\cos n\theta-i\sin n\theta$		
	$\therefore z^n+\frac{1}{z^n}=2\cos n\theta$	1	
	(ii) from (i) $z+\frac{1}{z}=2\cos\theta$	1	
	$\therefore (2\cos\theta)^4=\left(z+\frac{1}{z}\right)^4$		
	$\therefore 16\cos^4\theta = z^4+4.z^3.\frac{1}{z}+6z^2.\frac{1}{z^2}+4.z.\frac{1}{z^3}+\frac{1}{z^4}$		
	$=\left(z^4+\frac{1}{z^4}\right)+4\left(z^2+\frac{1}{z^2}\right)+6.$	1	
	$\therefore 16\cos^4\theta = 2\cos4\theta+8\cos2\theta+6.$		
	$\therefore \cos^4\theta=\frac{1}{8}\cos4\theta+\frac{1}{2}\cos2\theta+\frac{3}{8}$		
	$=\frac{1}{8}(\cos4\theta+4\cos2\theta+3).$	1	
b)	(i) RHS $= x+1+\frac{3x-4}{x^2-x-6}$	1	
	$=\frac{x^3-\cancel{x^2}-6x+\cancel{x^2}-x-6+3x-4}{x^2-x-6}$		
	$=\frac{x^3-4x-10}{x^2-x-6}.$		
	$=$ RHS $\therefore$ shown.	1	

Q	Solutions	Marks	Comments
Q2b	ii) now $\frac{3x-4}{x^2-x-6} = \frac{A}{x-3} + \frac{B}{x+2}$		
	$\therefore\ 3x-4 \equiv Ax+2A+Bx-3B$		
	$3x-4 = (A+B)x + (2A-3B)$		
	$\therefore\ A+B = 3$ —— ①	1	
	$2A-3B = -4$ —— ②		
	①×2 − ② $5B = 10 \Longrightarrow B = 2$		
	sub in ① $A = 1$		
	$\therefore\ \frac{x^3-4x-10}{x^2-x-6} = x+1 + \frac{1}{x-3} + \frac{2}{x+2}$	1	
c)	(i) $P(x) = x^3+5x^2+11=0$ roots α, β, γ		
	$\therefore$ Polynomial with roots $\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}$		
	is $P(\frac{1}{x}) = 0$	1	
	$\therefore\ (\frac{1}{x})^3 + 5(\frac{1}{x})^2 + 11 = 0$		
	$(\times x^3)$ $1 + 5x + 11x^3 = 0$	1	
	(ii) $P(\alpha)$: $\alpha^3+5\alpha^2+11 = 0$ —— ①		
	$P(\beta)$: $\beta^3+5\beta^2+11 = 0$ —— ②	1	
	$P(\gamma)$: $\gamma^3+5\gamma^2+11 = 0$ —— ③		
	①+②+③ $\alpha^3+\beta^3+\gamma^3 + 5(\alpha^2+\beta^2+\gamma^2) + 33 = 0$		
	now $\alpha^2+\beta^2+\gamma^2 = (\alpha+\beta+\gamma)^2 - 2(\alpha\beta+\alpha\gamma+\beta\gamma)$		
	$= (-5)^2 - 2(0)$		
	$= 25$		
	$\therefore\ \alpha^3+\beta^3+\gamma^3 = -5(25)-33$	1	
	$= -158$		

Q	Solutions	Marks	Comments
Q2d)	Given $P(2) = 4$ and $P(3) = 9$		
	when $P(x)$ is divided by $(x-2)(x-3)$ which is		
	quadratic, the remainder will be linear	1	
	ie $P(x) = (x-2)(x-3)Qx + (ax+b)$		
	$\therefore\ P(2) = 2a+b = 4$ —— ①		
	$P(3) = 3a+b = 9$ —— ②		
	② − ① $a = 5$		
	$\therefore\ b = -6$		
	$\therefore$ remainder when $P(x)$ is divided by		
	$(x-2)(x-3)$ is $5x-6$	1	
2e)	(i) $z^6 = 1$		
	let $z = r\operatorname{cis}\theta$ $\lvert z\rvert = r$ $\arg z = \theta$		
	now $z^6 = 1$		
	$(r\operatorname{cis}\theta)^6 = 1$		
	$r^6(\cos 6\theta + i\sin 6\theta) = 1 + 0i$	1	
	$\therefore\ r = 1$		
	$\cos 6\theta = 1$		
	$\therefore\ 6\theta = 0, 2\pi, 4\pi, 6\pi, 8\pi, 10\pi$		
	$\therefore\ \theta = 0, \frac{\pi}{3}, \frac{2\pi}{3}, \pi, \frac{4\pi}{3}, \frac{5\pi}{3}$	1	
	$\therefore\ z^6 = 1$ has roots		
	$\operatorname{cis}0 = 1$, $\operatorname{cis}\frac{\pi}{3}$, $\operatorname{cis}\frac{2\pi}{3}$, $\operatorname{cis}\pi = -1$, $\operatorname{cis}\frac{4\pi}{3}$, $\operatorname{cis}\frac{5\pi}{3}$		
	but note $\operatorname{cis}\frac{4\pi}{3} = \operatorname{cis}(-\frac{2\pi}{3}) = \overline{\operatorname{cis}\frac{2\pi}{3}}$	1	
	and $\operatorname{cis}\frac{5\pi}{3} = \operatorname{cis}(-\frac{\pi}{3}) = \overline{\operatorname{cis}\frac{\pi}{3}}$		

Q	Solutions	Marks	Comments
(ii)	$\therefore$ factors over $\mathbb{C}$ are		
	$(z-1)(z+1)(z-\text{cis}\frac{\pi}{3})(z-\overline{\text{cis}\frac{\pi}{3}})(z-\text{cis}\frac{2\pi}{3})(z-\overline{\text{cis}\frac{2\pi}{3}})$	1	
(iii)	Now $(z-\text{cis}\frac{\pi}{3})(z-\overline{\text{cis}\frac{\pi}{3}})$		
	$= z^2 - z(\text{cis}\frac{\pi}{3}+\overline{\text{cis}\frac{\pi}{3}}) + \text{cis}\frac{\pi}{3}.\overline{\text{cis}\frac{\pi}{3}}$	1	
	Note: $z+\bar{z} = 2\text{Re}\,z \quad z\bar{z}=1$		
	$= z^2 - 2z\cos\frac{\pi}{3} + 1 = (z^2 - z + 1)$		
	also then $(z-\text{cis}\frac{2\pi}{3})(z-\overline{\text{cis}\frac{2\pi}{3}})$	1	
	$= z^2 - 2z\cos\frac{2\pi}{3} + 1 = (z^2+z+1)$		
	$\therefore z^6 - 1 = (z-1)(z+1)(z^2-z+1)(z^2+z+1)$	1	
	End of Paper		